Canad. J. Math. 2024, pp. 1–33
http://dx.doi.org/10.4153/S0008414X24000051
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Canadian Mathematical Society

Faltings' main p-adic comparison theorems for non-smooth schemes

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Abstract. To understand the p-adic étale cohomology of a proper smooth variety over a p-adic field, Faltings compared it to the cohomology of his ringed topos, by the so-called Faltings' main p-adic comparison theorem, and then deduced various comparisons with p-adic cohomologies originating from differential forms. In this article, we generalize the former to any proper and finitely presented morphism of coherent schemes over an absolute integral closure of $\mathbb{Z}_p$ (without any smoothness assumption) for torsion abelian étale sheaves (not necessarily finite locally constant). Our proof relies on our cohomological descent for Faltings' ringed topos, using a variant of de Jong's alteration theorem for morphisms of schemes due to Gabber–Illusie–Temkin to reduce to the relative case of proper log-smooth morphisms of log-smooth schemes over a complete discrete valuation ring proved by Abbes–Gros. A by-product of our cohomological descent is a new construction of Faltings' comparison morphism, which does not use Achinger's results on $K(\pi,1)$-schemes.

1 Introduction

1.1 To understand the p-adic étale cohomology of a p-adic variety, Faltings [Fal88] introduced a new ringed topos as a bridge linking the p-adic étale cohomology with various p-adic cohomologies originating from differential forms. He proved that the cohomology of his ringed topos is (almost) isomorphic to the p-adic étale cohomology of the variety. This result is known as *Faltings' main p-adic comparison theorem*. The local nature of this ringed topos allows to compute its cohomology by Galois cohomology and hence to relate it to differential forms.

1.2 More precisely, let $\mathcal{O}_K$ be a complete discrete valuation ring extension of $\mathbb{Z}_p$, let K be its fraction field, let $\overline{K}$ be an algebraic closure of K, let X be a proper smooth $\mathcal{O}_K$-scheme, and let $Y = X_{\overline{K}}$ be the geometric generic fiber of X. Faltings introduced a ringed site $(\mathbf{E}^{\text{ét}}_{Y\to X}, \overline{\mathscr{B}})$ which admits natural morphisms of sites

$$Y_{\text{ét}} \xrightarrow{\psi} \mathbf{E}^{\text{ét}}_{Y\to X} \xrightarrow{\sigma} X_{\text{ét}}, \tag{1.2.1}$$

where $X_{\text{ét}}$ denotes the étale site of X. The underlying category of $\mathbf{E}^{\text{ét}}_{Y\to X}$ is a fibered category over $X_{\text{ét}}$, whose fiber over an étale X-scheme U is the finite étale site of $U_{\overline{K}}$ (see 3.1 for a precise definition). The sheaves $\mathrm{R}^q\sigma_*(\overline{\mathscr{B}}/p^n\overline{\mathscr{B}})$ $(q, n \in \mathbb{N})$ are computed by Galois cohomology and hence can be related to differential forms of X over $\mathcal{O}_K$.

Received by the editors July 10, 2023; revised December 13, 2023; accepted January 8, 2024.
Published online on Cambridge Core January 12, 2024.
AMS subject classification: **14F30.**
Keywords: Cohomological descent, Faltings topos, v-topology, p-adic Hodge theory, comparison.

https://doi.org/10.4153/S0008414X24000051 Published online by Cambridge University Press

On the other hand, Faltings proved that for any finite locally constant abelian sheaf $\mathbb{F}$ on $Y_{\text{ét}}$ and any integer $q \geq 0$, there is a canonical morphism,

$$H^q(Y_{\text{ét}}, \mathbb{F}) \otimes_{\mathbb{Z}} \mathcal{O}_{\overline{K}} \longrightarrow H^q(\mathbf{E}^{\text{ét}}_{Y\to X}, \psi_* \mathbb{F} \otimes_{\mathbb{Z}} \overline{\mathscr{B}}), \tag{1.2.2}$$

which is an almost isomorphism, that is, its kernel and cokernel are killed by p^r for any rational number $r > 0$. The right-hand side of (1.2.2) can be related to various p-adic cohomologies of X, leading to various p-adic comparison theorems. For instance, the Hodge–Tate decomposition theorem (i.e., the comparison with the Hodge cohomologies) is obtained from the Cartan–Leray spectral sequence for the composed functor $\mathrm{R}\Gamma(X_{\text{ét}}, -) \circ \mathrm{R}\sigma_*$ taking for $\mathbb{F}$ some constant torsion sheaves, see [AG20]. Faltings pushed further this strategy by developing a p-adic Simpson correspondence that compares the p-adic étale cohomology of a p-adic local system with the Dolbeault cohomology of the associated Higgs bundle [Fal05] (cf. [AGT16]).

1.3 Faltings formulated also a relative version of his main comparison theorem relating the relative étale cohomology and the relative cohomology of his ringed topos for a proper log-smooth morphism of log-smooth $\mathcal{O}_K$-schemes. He only roughly sketched the proof in [Fal02], but a complete proof was provided recently by Abbes and Gros [AG20]. They used this relative version in their construction of the relative Hodge–Tate spectral sequence and more generally in their study of the functoriality of the p-adic Simpson correspondence by higher direct images [AG22]. As we have seen above, Faltings topos builds on an integral model of the p-adic variety, whose (logarithmic) smoothness seems necessary for good properties of Faltings topos and thus for the proofs of the comparison theorems. The goal of this article is to get rid of the smoothness assumptions in Faltings' main comparison theorems, not only on the integral models but also on the generic fibers.

1.4 In fact, we generalize Faltings' main comparison theorem to any proper and finitely presented morphism of coherent schemes (i.e., quasi-compact and quasi-separated schemes) over an absolute integral closure of $\mathbb{Z}_p$ (without any further assumption on smoothness or finiteness) for torsion abelian étale sheaves (not necessarily finite locally constant). This generalization takes place in a variant of Faltings site with v-topology, that we introduced and called the *v-site of integrally closed schemes*. We have shown in [He23] that both the étale cohomology and the cohomology of Faltings ringed topos can be computed by this v-site. The latter implies a *cohomological descent for Faltings ringed topos* along proper hypercoverings, which allows us to reduce the proof of our generalization to Faltings' main comparison theorem for proper log-smooth morphisms of log-smooth $\mathcal{O}_K$-schemes using a variant of de Jong's alteration theorem for morphisms of schemes due to Gabber–Illusie–Temkin. Moreover, it allows us to deduce generalizations of Faltings' main comparison theorems for the original Faltings site both in the absolute and the relative cases. We remark that Scholze has generalized Faltings' main comparison theorem to proper smooth morphisms of rigid analytic varieties for finite locally constant abelian sheaves [Sch13a, 5.11], and to proper morphisms of algebraic varieties for torsion abelian sheaves [Sch13b, 3.13]. Compared to his results, our arguments are purely scheme theoretic and our generalization for torsion abelian sheaves holds for proper morphisms of more

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general schemes (i.e., less restrictive on finiteness). On the other hand, it is not clear whether our generalization and Scholze's can be directly deduced from each other.

Firstly, we state our generalization of Faltings' main comparison theorem in the absolute case.

Theorem 1.5 (see 5.17) *Let A be a valuation ring extension of $\mathbb{Z}_p$ with algebraically closed fraction field. Consider a Cartesian square of coherent schemes:*

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & X \arrow[d] \\ \mathrm{Spec}(A[\frac{1}{p}]) \arrow[r] & \mathrm{Spec}(A) \end{tikzcd} \tag{1.5.1}$$

Assume that X is proper of finite presentation over A. Then, for any finite locally constant abelian sheaf $\mathbb{F}$ on $Y_{\text{ét}}$, there exists a canonical morphism

$$\mathrm{R}\Gamma(Y_{\text{ét}}, \mathbb{F}) \otimes^{\mathrm{L}}_{\mathbb{Z}} A \longrightarrow \mathrm{R}\Gamma(\mathbf{E}^{\text{ét}}_{Y\to X}, \psi_*\mathbb{F} \otimes_{\mathbb{Z}} \overline{\mathscr{B}}), \tag{1.5.2}$$

which is an almost isomorphism, where $\psi : Y_{\text{ét}} \to \mathbf{E}^{\text{ét}}_{Y\to X}$ is the natural morphism of sites (see 3.2).

We remark that the natural morphism of sites $\psi : Y_{\text{ét}} \to \mathbf{E}^{\text{ét}}_{Y\to X}$ induces an equivalence of the categories of finite locally constant abelian sheaves on $Y_{\text{ét}}$ and $\mathbf{E}^{\text{ét}}_{Y\to X}$ (5.3)

$$\mathbf{LocSys}(Y_{\text{ét}}) \underset{\psi^{-1}}{\overset{\psi_*}{\rightleftarrows}} \mathbf{LocSys}(\mathbf{E}^{\text{ét}}_{Y\to X}). \tag{1.5.3}$$

1.6 One of the key ingredients of the proof of 1.5 is our cohomological descent for Faltings ringed topos [He23]. Roughly speaking, it allows us to descend important results on Faltings topos associated with nice integral models to Faltings topos associated with general integral model. More concretely, we adopt the following strategy:

(1) Firstly, we use de Jong–Gabber–Illusie–Temkin's alteration theorem for morphisms of schemes [ILO14, X.3] to obtain a proper surjective morphism of finite presentation $X' \to X$ such that the morphism $X' \to \mathrm{Spec}(A)$ is the cofiltered limit of a system of "nice" morphisms $X'_\lambda \to T_\lambda$ of "nice" models over $\mathcal{O}_{K_\lambda}$, where K_λ is a finite extension of $\mathbb{Q}_p$ (see 4.11).
(2) Then, we can apply Faltings' main comparison theorem in the relative case to the "nice" morphisms $X'_\lambda \to T_\lambda$ (formulated by Faltings [Fal02, Theorem 6, page 266] and proved by Abbes and Gros [AG20, 5.7.4], see 5.13). By a limit argument, we get the comparison theorem for X'.
(3) Finally, using our cohomological descent for Faltings ringed topos along a proper hypercovering (see 3.10), we deduce the comparison theorem for X.

1.7 In fact, even the construction of Faltings' comparison morphism (1.5.2) is not trivial, even in the smooth case (1.2.2). It relies on the acyclicity of the morphism ψ for any finite locally constant abelian sheaf $\mathbb{F}$, i.e., $\psi_*\mathbb{F} = \mathrm{R}\psi_*\mathbb{F}$. Faltings' comparison morphism is obtained from the canonical morphisms

$$\mathrm{R}\Gamma(Y_{\text{ét}}, \mathbb{F}) \otimes^{\mathrm{L}}_{\mathbb{Z}} A \xleftarrow{\sim} \mathrm{R}\Gamma(\mathbf{E}^{\text{ét}}_{Y\to X}, \psi_*\mathbb{F}) \otimes^{\mathrm{L}}_{\mathbb{Z}} A \longrightarrow \mathrm{R}\Gamma(\mathbf{E}^{\text{ét}}_{Y\to X}, \psi_*\mathbb{F} \otimes_{\mathbb{Z}} \overline{\mathscr{B}}). \tag{1.7.1}$$

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The acyclicity of ψ is a consequence of Achinger's result on $K(\pi,1)$-schemes (see 5.6 and 5.8).

We don't know if ψ is acyclic for more general coefficients. This is also one of the reasons why Faltings consider only local systems in his main comparison theorems. We propose a new way to construct Faltings' comparison morphism in the derived category of almost modules using our cohomological descent result [He23] (see also 3.9), that avoids the acyclicity of ψ and that holds for more general coefficients. Indeed, there is a natural commutative diagram of sites (see 3.6):

(1.7.2)
$$\begin{tikzcd} (\mathbf{Sch}^{\mathrm{coh}}_{/Y})_{\mathrm{v}} \arrow[r, "a"] \arrow[d, "\Psi"'] & Y_{\text{ét}} \arrow[d, "\psi"] \\ \mathbf{I}_{Y\to X^Y} \arrow[r, "\varepsilon"] & \mathbf{E}^{\text{ét}}_{Y\to X} \end{tikzcd}$$

where $(\mathbf{Sch}^{\mathrm{coh}}_{/Y})_{\mathrm{v}}$ is the v-site of coherent Y-schemes, $\mathbf{I}_{Y\to X^Y}$ is the v-site of Y-integrally closed coherent X^Y-schemes (1.4). Moreover, ε is actually a morphism of ringed sites

(1.7.3)
$$\varepsilon : (\mathbf{I}_{Y\to X^Y}, \mathscr{O}) \longrightarrow (\mathbf{E}^{\text{ét}}_{Y\to X}, \overline{\mathscr{B}}).$$

Recall the following facts:

(1) (Acyclicity of Ψ, see 3.8) The morphism Ψ is acyclic for any torsion abelian sheaf $\mathcal{F}$ on $Y_{\text{ét}}$, i.e., $\Psi_*(a^{-1}\mathcal{F}) = \mathrm{R}\Psi_*(a^{-1}\mathcal{F})$, which allows more general coefficients and whose proof [He23, 3.27] is much easier than that for ψ, as we could reduce to the case of valuation rings.
(2) (Cohomological descent for étale cohomology, see 3.4) For any torsion abelian sheaf $\mathcal{F}$ on $Y_{\text{ét}}$, the canonical morphism $\mathcal{F} \to \mathrm{R}a_*a^{-1}\mathcal{F}$ is an isomorphism.
(3) (Cohomological descent for Faltings ringed topos, see 3.9) For any finite locally constant abelian sheaf $\mathbb{L}$ over $\mathbf{E}^{\text{ét}}_{Y\to X}$, the canonical morphism $\mathbb{L}\otimes_{\mathbb{Z}}\overline{\mathscr{B}} \to \mathrm{R}\varepsilon_*(\varepsilon^{-1}\mathbb{L}\otimes_{\mathbb{Z}}\mathscr{O})$ is an almost isomorphism.

From these facts, we obtain a new construction of Faltings' comparison morphism, which does not give a "real morphism" (1.5.2) but a canonical morphism in the derived category of almost modules (see 6.6).

1.8 As we have seen above, the v-variant of Faltings site $\mathbf{I}_{Y\to X^Y}$ computes the étale cohomology of Y for torsion abelian sheaves (by the facts (1) and (2) in Section 1.7), and it describes well the cohomological descent for Faltings ringed topos ((3) in Section 1.7). These facts enable us to reformulate Faltings' main comparison theorem using $\mathbf{I}_{Y\to X^Y}$ instead of $\mathbf{E}^{\text{ét}}_{Y\to X}$, which then provides a relative statement for more general coefficients.

Theorem 1.9 (see 6.12) *With the same notation in 1.5, let $\mathcal{F}$ be a torsion abelian sheaf on $Y_{\text{ét}}$ and we set $\mathscr{F} = \Psi_*a^{-1}\mathcal{F}$. Then, there is a canonical almost isomorphism*

(1.9.1)
$$\mathrm{R}\Gamma(Y_{\text{ét}},\mathcal{F})\otimes^{\mathrm{L}}_{\mathbb{Z}} A \longrightarrow \mathrm{R}\Gamma(\mathbf{I}_{Y\to X^Y},\mathscr{F}\otimes_{\mathbb{Z}}\mathscr{O}).$$

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Indeed, the canonical morphism (1.9.1) is obtained from the canonical morphisms

$$\mathrm{R}\Gamma(Y_{\text{ét}}, \mathcal{F}) \otimes^{\mathrm{L}}_{\mathbb{Z}} A \xleftarrow{\sim} \mathrm{R}\Gamma(\mathbf{I}_{Y\to X^Y}, \mathscr{F}) \otimes^{\mathrm{L}}_{\mathbb{Z}} A \longrightarrow \mathrm{R}\Gamma(\mathbf{I}_{Y\to X^Y}, \mathscr{F} \otimes_{\mathbb{Z}} \mathscr{O}). \tag{1.9.2}$$

Thus, 1.9 is a direct corollary of the following relative statement.

Theorem 1.10 (see 6.11) *Let $\overline{\mathbb{Z}_p}$ be the integral closure of $\mathbb{Z}_p$ in an algebraic closure $\overline{\mathbb{Q}_p}$ of $\mathbb{Q}_p$. Consider a Cartesian square of coherent schemes:*

$$\begin{tikzcd} Y' \arrow[r] \arrow[d] & X' \arrow[d] \\ Y \arrow[r] & X \end{tikzcd} \tag{1.10.1}$$

where $Y \to X$ is Cartesian over $\mathrm{Spec}(\overline{\mathbb{Q}_p}) \to \mathrm{Spec}(\overline{\mathbb{Z}_p})$. Assume that $X' \to X$ is proper of finite presentation. Let $\mathcal{F}'$ be a torsion abelian sheaf on $Y'_{\text{ét}}$ and $\mathscr{F}' = \Psi'_* a'^{-1}\mathcal{F}'$ (see (1.7.2)). Then, the canonical morphism

$$(\mathrm{R}f_{\mathbf{I}*}\mathscr{F}') \otimes^{\mathrm{L}}_{\mathbb{Z}} \mathscr{O} \longrightarrow \mathrm{R}f_{\mathbf{I}*}(\mathscr{F}' \otimes_{\mathbb{Z}} \mathscr{O}') \tag{1.10.2}$$

is an almost isomorphism, where $f_{\mathbf{I}} : (\mathbf{I}_{Y'\to X'^{Y'}}, \mathscr{O}') \longrightarrow (\mathbf{I}_{Y\to X^Y}, \mathscr{O})$ is the natural morphism of ringed sites defined by the functoriality of (1.7.2).

1.11 We remark that if $\mathcal{F}' = \mathbb{Z}/p^n\mathbb{Z}$, then $\mathscr{F}' = \mathbb{Z}/p^n\mathbb{Z}$ (see 6.2), and that $\mathrm{R}^q f_{\mathbf{I}*}\mathscr{F}'$ is the sheafification in v-topology of the qth étale cohomologies of Y' over Y with coefficients in $\mathcal{F}'$ (see 6.13). Roughly speaking, objects of $\mathbf{I}_{Y\to X^Y}$ are "locally" the spectrums of valuation rings, and the "stalks" of (1.10.2) are Faltings' comparison morphisms (1.5.2) when $\mathcal{F}'$ is finite locally constant (see 6.5). This enables us to prove 1.10 for such $\mathcal{F}'$ by reducing to the absolute case 1.5. Then, standard techniques in [SGA 4_{III}, IX.2] allow us to extend the conclusion to general $\mathcal{F}'$. Theorem 1.10 can be regarded as a generalization of Scholze's comparison theorem for proper morphisms of algebraic varieties [Sch13b, 3.13]. Finally, we generalize Faltings' main comparison theorem in the relative case for the original Faltings site using 1.10 and the cohomological descent for Faltings ringed topos (3.9).

Theorem 1.12 (see 6.14 and 6.15) *With the same notation in 1.10, assume that $Y' \to Y$ is smooth and that $X' \to X$ is proper of finite presentation. Then, for any finite locally constant abelian sheaf $\mathbb{F}'$ on $Y'_{\text{ét}}$, there exists a canonical morphism*

$$(\mathrm{R}\psi_* \mathrm{R}f_{\text{ét}*}\mathbb{F}') \otimes^{\mathrm{L}}_{\mathbb{Z}} \overline{\mathscr{B}} \longrightarrow \mathrm{R}f_{\mathbf{E}*}(\psi'_*\mathbb{F}' \otimes_{\mathbb{Z}} \overline{\mathscr{B}}'), \tag{1.12.1}$$

which is an almost isomorphism, and where $f_{\text{ét}} : Y'_{\text{ét}} \to Y_{\text{ét}}$ and $f_{\mathbf{E}} : \mathbf{E}^{\text{ét}}_{Y'\to X'} \to \mathbf{E}^{\text{ét}}_{Y\to X}$ are the natural morphisms of sites. In particular, there exists a canonical morphism

$$(\psi_* \mathrm{R}^q f_{\text{ét}*}\mathbb{F}') \otimes_{\mathbb{Z}} \overline{\mathscr{B}} \longrightarrow \mathrm{R}^q f_{\mathbf{E}*}(\psi'_*\mathbb{F}' \otimes_{\mathbb{Z}} \overline{\mathscr{B}}'), \tag{1.12.2}$$

which is an almost isomorphism, for any integer q.

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1.13 The article is structured as follows. In Section 3, we summarize the main results of [He23] including the cohomological descent for Faltings ringed topos. In Section 4, we review de Jong–Gabber–Illusie–Temkin's alteration theorem and apply it to schemes over a valuation ring of height 1. Section 5 is devoted to proving our generalization of Faltings' main comparison theorem in the absolute case. Finally, we give a new construction of Faltings' comparison morphism and our generalization of Faltings' main comparison theorem in the relative case in Section 6.

2 Notation and conventions

2.1 We fix a prime number p throughout this paper. For a ring R, we denote by $R^\times$ the group of units of R. A ring R is called *absolutely integrally closed* if any monic polynomial $f \in R[T]$ has a root in R [Sta23, 0DCK]. We remark that quotients, localizations, and products of absolutely integrally closed rings are still absolutely integrally closed.

Recall that a valuation ring is a domain V such that for any element x in its fraction field, if $x \notin V$, then $x^{-1} \in V$. The family of ideals of V is totally ordered by the inclusion relation [Bou06, VI.§1.2, Theorem 1]. In particular, a radical ideal of V is a prime ideal. Moreover, any quotient of V by a prime ideal and any localization of V are still valuations rings [Sta23, 088Y]. We remark that V is normal, and that V is absolutely integrally closed if and only if its fraction field is algebraically closed. An *extension of valuation rings* is an injective and local homomorphism of valuation rings.

2.2 Following [SGA 4_{II}, VI.1.22], a *coherent* scheme (resp. morphism of schemes) stands for a quasi-compact and quasi-separated scheme (resp. morphism of schemes). For a coherent morphism $Y \to X$ of schemes, we denote by X^Y the integral closure of X in Y [Sta23, 0BAK]. For an X-scheme Z, we say that Z is *Y-integrally closed* if $Z = Z^{Y \times_X Z}$.

2.3 Throughout this paper, we fix two universes $\mathbb{U}$ and $\mathbb{V}$ such that the set of natural numbers $\mathbb{N}$ is an element of $\mathbb{U}$ and that $\mathbb{U}$ is an element of $\mathbb{V}$ [SGA 4_I, I.0]. In most cases, we won't emphasize this set theoretical issue. Unless stated otherwise, we only consider $\mathbb{U}$-small schemes and we denote by **Sch** the category of $\mathbb{U}$-small schemes, which is a $\mathbb{V}$-small category.

2.4 Let C be a category. We denote by $\widehat{C}$ the category of presheaves of $\mathbb{V}$-small sets on C. If C is a $\mathbb{V}$-site [SGA 4_I, II.3.0.2], we denote by $\widetilde{C}$ the topos of sheaves of $\mathbb{V}$-small sets on C. We denote by $h^C : C \to \widehat{C}$, $x \mapsto h^C_x$ the Yoneda embedding [SGA 4_I, I.1.3], and by $\widehat{C} \to \widetilde{C}$, $\mathcal{F} \mapsto \mathcal{F}^{\mathrm{a}}$ the sheafification functor [SGA 4_I, II.3.4].

2.5 Let $u^+ : C \to D$ be a functor of categories. We denote by $u^{\mathrm{p}} : \widehat{D} \to \widehat{C}$ the functor that associates to a presheaf $\mathcal{G}$ of $\mathbb{V}$-small sets on D the presheaf $u^{\mathrm{p}}\mathcal{G} = \mathcal{G} \circ u^+$. If C is $\mathbb{V}$-small and D is a $\mathbb{V}$-category, then u^{p} admits a left adjoint u_{p} [Sta23, 00VC] and a right adjoint ${}_{\mathrm{p}}u$ [Sta23, 00XF] (cf. [SGA 4_I, I.5]). So we have a sequence of adjoint functors

$$u_{\mathrm{p}},\ u^{\mathrm{p}},\ {}_{\mathrm{p}}u. \tag{2.5.1}$$

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If, moreover, C and D are $\mathbb{V}$-sites, then we denote by $u_{\mathrm{s}}, u^{\mathrm{s}}, {}_{\mathrm{s}}u$ the functors of the topoi $\widetilde{C}$ and $\widetilde{D}$ of sheaves of $\mathbb{V}$-small sets induced by composing the sheafification functor with the functors $u_{\mathrm{p}}, u^{\mathrm{p}}, {}_{\mathrm{p}}u$, respectively. If finite limits are representable in C and D and if u^+ is left exact and continuous, then u^+ gives a morphism of sites $u : D \to C$ [SGA 4_I, IV.4.9.2] and we also denote by

$$u = (u^{-1}, u_*) : \widetilde{D} \to \widetilde{C} \tag{2.5.2}$$

the associated morphism of topoi, where $u^{-1} = u_{\mathrm{s}}$ and $u_* = u^{\mathrm{s}} = u^{\mathrm{p}}|_{\widetilde{D}}$. If, moreover, u is a morphism of ringed sites $u : (D, \mathcal{O}_D) \to (C, \mathcal{O}_C)$, then we denote by $u^* = \mathcal{O}_D \otimes_{u^{-1}\mathcal{O}_C} u^{-1}$ the pullback functor of modules. We remark that the notation here, adopted by [Sta23], is slightly different from that in [SGA 4_I] (see [Sta23, 0CMZ]).

3 Brief review on cohomological descent for faltings ringed topos

For the convenience of readers and preparation of new notation used later, we briefly summarize some main notions and results in [He23].

3.1 Firstly, we recall the definition of the Faltings site associated with a morphism of coherent schemes $Y \to X$ (see [He23, 7.7]). Let $\mathbf{E}^{\text{ét}}_{Y\to X}$ be the category of morphisms of coherent schemes $V \to U$ over $Y \to X$, i.e., commutative diagrams

$$\begin{tikzcd} V \arrow[r] \arrow[d] & U \arrow[d] \\ Y \arrow[r] & X \end{tikzcd} \tag{3.1.1}$$

such that U is étale over X and that V is finite étale over $Y \times_X U$. We endow $\mathbf{E}^{\text{ét}}_{Y\to X}$ with the topology generated by the following types of families of morphisms:

(v) $\{(V_m \to U) \to (V \to U)\}_{m\in M}$, where M is a finite set and $\coprod_{m\in M} V_m \to V$ is surjective;
(c) $\{(V \times_U U_n \to U_n) \to (V \to U)\}_{n\in N}$, where N is a finite set and $\coprod_{n\in N} U_n \to U$ is surjective.

Consider the presheaf $\overline{\mathscr{B}}$ on $\mathbf{E}^{\text{ét}}_{Y\to X}$ defined by

$$\overline{\mathscr{B}}(V \to U) = \Gamma(U^V, \mathcal{O}_{U^V}), \tag{3.1.2}$$

where U^V is the integral closure of U in V. It is indeed a sheaf of rings, called the structural sheaf of $\mathbf{E}^{\text{ét}}_{Y\to X}$ (see [He23, 7.6]).

3.2 Let $Y \to X$ be a morphism of coherent schemes. The natural left exact and continuous functors

$$\psi^+ : \mathbf{E}^{\text{ét}}_{Y\to X} \longrightarrow Y_{\text{ét}},\ (V \to U) \longmapsto V, \tag{3.2.1}$$

$$\beta^+ : Y_{\text{fét}} \longrightarrow \mathbf{E}^{\text{ét}}_{Y\to X},\ V \longmapsto (V \to X), \tag{3.2.2}$$

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induce a natural commutative diagram of sites (2.5),

$$\begin{tikzcd} Y_{\text{ét}} \arrow[r, "\psi"'] \arrow[rr, bend left, "\rho"] & \mathbf{E}^{\text{ét}}_{Y\to X} \arrow[r, "\beta"'] & Y_{\text{fét}}, \end{tikzcd} \tag{3.2.3}$$

where $\rho : Y_{\text{ét}} \to Y_{\text{fét}}$ is defined by the inclusion functor (see [He23, 7.8]).

3.3 Recall that a morphism of coherent schemes $T \to S$ is called a v-covering if for any morphism $\mathrm{Spec}(A) \to S$ with A a valuation ring, there exists an extension of valuation rings $A \to B$ and a lifting $\mathrm{Spec}(B) \to T$ [He23, 3.1(1)]. Let $\mathbf{Sch}^{\text{coh}}$ be the category of coherent schemes. We endow it with the topology generated by the pretopology formed by families of morphisms $\{S_i \to S\}_{i\in I}$ with I finite such that $\coprod_{i\in I} S_i \to S$ is a v-covering, and we denote the corresponding site by $\mathbf{Sch}^{\text{coh}}_{\text{v}}$ [He23, 3.3]. For any object S of $\mathbf{Sch}^{\text{coh}}_{\text{v}}$, we denote by $(\mathbf{Sch}^{\text{coh}}_{/S})_{\text{v}}$ the localization of the v-site $\mathbf{Sch}^{\text{coh}}_{\text{v}}$ at S. The cohomological descent for étale cohomology can be stated as follows:

Theorem 3.4 ([He23, 3.9]) *Let S be a coherent scheme, let $\mathcal{F}$ be a torsion abelian sheaf on the site $S_{\text{ét}}$ formed by coherent étale S-schemes endowed with the étale topology, and let $a : (\mathbf{Sch}^{\text{coh}}_{/S})_{\text{v}} \to S_{\text{ét}}$ be the morphism of sites defined by the inclusion functor. Then, the canonical morphism $\mathcal{F} \to \mathrm{R}a_* a^{-1}\mathcal{F}$ is an isomorphism.*

Definition 3.5 ([He23, 3.23]) Let $S^\circ \to S$ be an open immersion of coherent schemes such that S is integrally closed in S°. We define a site $\mathbf{I}_{S^\circ\to S}$ as follows:

(1) The underlying category is formed by coherent S-schemes T which are integrally closed in $S^\circ \times_S T$.
(2) The topology is generated by covering families $\{T_i \to T\}_{i\in I}$ in the v-topology.

We call $\mathbf{I}_{S^\circ\to S}$ the *v-site of S°-integrally closed coherent S-schemes*, and we call the sheaf $\mathscr{O}$ on $\mathbf{I}_{S^\circ\to S}$ associated with the presheaf $T \mapsto \Gamma(T, \mathcal{O}_T)$ the *structural sheaf of* $\mathbf{I}_{S^\circ\to S}$.

3.6 Let $Y \to X$ be a morphism of coherent schemes such that $Y \to X^Y$ is an open immersion, where X^Y denotes the integral closure of X in Y (2.2). The natural left exact and continuous functors

$$\Psi^+ : \mathbf{I}_{Y\to X^Y} \longrightarrow (\mathbf{Sch}^{\text{coh}}_{/Y})_{\text{v}},\ Z \longmapsto Y \times_{X^Y} Z, \tag{3.6.1}$$

$$\varepsilon^+ : \mathbf{E}^{\text{ét}}_{Y\to X} \longrightarrow \mathbf{I}_{Y\to X^Y},\ (V \to U) \longmapsto U^V, \tag{3.6.2}$$

induce natural morphisms of sites (see [He23, 3.26, 8.6])

$$(\mathbf{Sch}^{\text{coh}}_{/Y})_{\text{v}} \xrightarrow{\Psi} \mathbf{I}_{Y\to X^Y} \xrightarrow{\varepsilon} \mathbf{E}^{\text{ét}}_{Y\to X}. \tag{3.6.3}$$

Combining with the morphisms of sites defined in 3.2 and 3.4, we obtain the following natural commutative diagram of sites:

$$\begin{tikzcd} (\mathbf{Sch}^{\text{coh}}_{/Y})_{\text{v}} \arrow[r, "a"] \arrow[d, "\Psi"'] & Y_{\text{ét}} \arrow[d, "\psi"] \arrow[dr, "\rho"] & \\ \mathbf{I}_{Y\to X^Y} \arrow[r, "\varepsilon"] & \mathbf{E}^{\text{ét}}_{Y\to X} \arrow[r, "\beta"] & Y_{\text{fét}} \end{tikzcd} \tag{3.6.4}$$

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Moreover, ε^+ actually defines a morphism of ringed sites (see [He23, 8.6])

$$\varepsilon : (\mathbf{I}_{Y\to X^Y}, \mathscr{O}) \longrightarrow (\mathbf{E}^{\text{ét}}_{Y\to X}, \overline{\mathscr{B}}). \tag{3.6.5}$$

Lemma 3.7 (cf. [He23, 7.9]) *Let X be the spectrum of an absolutely integrally closed valuation ring, and let Y be a quasi-compact open subscheme of X. Then, for any presheaf $\mathcal{F}$ on $\mathbf{E}^{\text{ét}}_{Y\to X}$ (resp. $\mathbf{I}_{Y\to X^Y}$), we have $\mathcal{F}^{\mathrm{a}}(Y \to X) = \mathcal{F}(Y \to X)$ (resp. $\mathcal{F}^{\mathrm{a}}(X^Y) = \mathcal{F}(X^Y)$). In particular, the associated topos of $\mathbf{E}^{\text{ét}}_{Y\to X}$ (resp. $\mathbf{I}_{Y\to X^Y}$) is local [SGA 4_{II}, VI.8.4.6].*

Proof The statement for $\mathbf{E}^{\text{ét}}_{Y\to X^Y}$ is proved in [He23, 7.9]. The same arguments work for $\mathbf{I}_{Y\to X^Y}$. ■

Proposition 3.8 ([He23, 3.27]) *We keep the notation in 3.6.*

(1) *For any torsion abelian sheaf $\mathcal{F}$ on $Y_{\text{ét}}$, the canonical morphism $\Psi_*(a^{-1}\mathcal{F}) \to \mathrm{R}\Psi_*(a^{-1}\mathcal{F})$ is an isomorphism.*
(2) *For any locally constant torsion abelian sheaf $\mathbb{L}$ on $\mathbf{I}_{Y\to X^Y}$, the canonical morphism $\mathbb{L} \to \mathrm{R}\Psi_*\Psi^{-1}\mathbb{L}$ is an isomorphism.*

Combining 3.8(1) with 3.4, we see that the cohomology of $\mathbf{I}_{Y\to X^Y}$ computes the étale cohomology of Y. Moreover, it also computes the cohomology of Faltings ringed topos in the following sense.

Theorem 3.9 ([He23, 8.14]) *Let K be a pre-perfectoid field of mixed characteristic $(0, p)$ (i.e., a valuation field whose valuation ring $\mathcal{O}_K$ is non-discrete, extension of $\mathbb{Z}_p$ and of height 1 such that the Frobenius map on $\mathcal{O}_K/p\mathcal{O}_K$ is surjective, see [He23, 5.1]), $\eta = \mathrm{Spec}(K)$, $S = \mathrm{Spec}(\mathcal{O}_K)$, $Y \to X$ a morphism of coherent schemes such that X^Y is an S-scheme with generic fiber $(X^Y)_\eta = Y$ (in particular, X^Y is an object of $\mathbf{I}_{\eta\to S}$). Then, for any finite locally constant abelian sheaf $\mathbb{L}$ on $\mathbf{E}^{\text{ét}}_{Y\to X}$, the canonical morphism (3.6)*

$$\mathbb{L} \otimes_{\mathbb{Z}} \overline{\mathscr{B}} \longrightarrow \mathrm{R}\varepsilon_*(\varepsilon^{-1}\mathbb{L} \otimes_{\mathbb{Z}} \mathscr{O}) \tag{3.9.1}$$

is an almost isomorphism [He23, 5.7].

The cohomological descent for Faltings ringed topos along a proper hypercovering is stated as follows:

Corollary 3.10 ([He23, 8.18]) *Under the assumptions in 3.9 and with the same notation, let $X_\bullet \to X$ be an augmentation of simplicial coherent scheme, and let $Y_\bullet = Y \times_X X_\bullet$, $b : \mathbf{E}^{\text{ét}}_{Y_\bullet\to X_\bullet} \to \mathbf{E}^{\text{ét}}_{Y\to X}$ be the augmentation of simplicial site [He23, 8.17]. If $X_\bullet^{Y_\bullet} \to X^Y$ is a hypercovering in $\mathbf{I}_{\eta\to S}$, then the canonical morphism*

$$\mathbb{L} \otimes_{\mathbb{Z}} \overline{\mathscr{B}} \to \mathrm{R}b_*(b^{-1}\mathbb{L} \otimes_{\mathbb{Z}} \overline{\mathscr{B}}_\bullet) \tag{3.10.1}$$

is an almost isomorphism [He23, 5.7].

4 Complements on logarithmic geometry

We briefly recall some notions and facts of logarithmic geometry which will be used in the rest of the paper. We refer to [GR04, Kat89, Kat94, Ogu18] for a systematic development of logarithmic geometry, and to [AGT16, II.5] for a brief summary of the theory.

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4.1 We only consider logarithmic structures in étale topology. More precisely, let X be a scheme, let $X_{ét}$ be the étale site of X, let $\mathcal{O}_{X_{ét}}$ be the structure sheaf on $X_{ét}$, and let $\mathcal{O}_{X_{ét}}^\times$ be the subsheaf of units of $\mathcal{O}_{X_{ét}}$. A logarithmic structure on X is a homomorphism of sheaves of monoids $\alpha : \mathscr{M} \to \mathcal{O}_{X_{ét}}$ on $X_{ét}$ which induces an isomorphism $\alpha^{-1}(\mathcal{O}_{X_{ét}}^\times) \xrightarrow{\sim} \mathcal{O}_{X_{ét}}^\times$. We denote by $(X, \mathscr{M})$ the associated logarithmic scheme (cf. [AGT16, II.5.11]).

4.2 Let $(X, \mathscr{M})$ be a coherent log scheme (cf. [AGT16, II.5.15]). Then, there is a maximal open subscheme X^{tr} of X on which $\mathscr{M}$ is trivial, and moreover it is functorial in $(X, \mathscr{M})$ [Ogu18, III.1.2.8]. Let $(X, \mathscr{M}) \to (S, \mathscr{L}) \leftarrow (Y, \mathscr{N})$ be a diagram of fine and saturated log schemes (cf. [AGT16, II.5.15]). Then, the fibered product is representable in the category of fine and saturated log schemes by $(Z, \mathscr{P}) = (X, \mathscr{M}) \times^{fs}_{(S,\mathscr{L})} (Y, \mathscr{N})$. We remark that $Z^{tr} = X^{tr} \times_{S^{tr}} Y^{tr}$, that $Z \to X \times_S Y$ is finite, and that $Z^{tr} \to Z$ is Cartesian over $X^{tr} \times_{S^{tr}} Y^{tr} \to X \times_S Y$ [Ogu18, III 2.1.2, 2.1.6]. Moreover, if $X^{tr} = X$, then $Z = X \times_S Y$ [Ogu18, III.2.1.3].

4.3 For an open immersion $j : Y \to X$, we denote by $j_{ét} : Y_{ét} \to X_{ét}$ the morphism of their étale sites defined by the base change by j. Let $\mathscr{M}_{Y\to X}$ be the preimage of $j_{ét*}\mathcal{O}_{Y_{ét}}^\times$ under the natural map $\mathcal{O}_{X_{ét}} \to j_{ét*}\mathcal{O}_{Y_{ét}}$, and we endow X with the logarithmic structure $\mathscr{M}_{Y\to X} \to \mathcal{O}_{X_{ét}}$, which is called the compactifying log structure associated with the open immersion j [Ogu18, III.1.6.1]. Sometimes we write $\mathscr{M}_{Y\to X}$ as $\mathscr{M}_X$ if Y is clear in the context.

4.4 Let $(X, \mathscr{M})$ be a fine and saturated log scheme which is regular ([Kat94, 2.1], [Niz06, 2.3]). Then, X is locally Noetherian and normal, and X^{tr} is regular and dense in X [Kat94, 4.1]. Moreover, there is a natural isomorphism $\mathscr{M} \xrightarrow{\sim} \mathscr{M}_{X^{tr}\to X}$ ([Kat94, 11.6], [Niz06, 2.6]). We remark that if X is a regular scheme with a strict normal crossings divisor D, then $(X, \mathscr{M}_{X\setminus D\to X})$ is fine, saturated, and regular [Ogu18, III.1.11.9].

Let $f : (X, \mathscr{M}) \to (S, \mathscr{L})$ be a smooth (resp. saturated) morphism of fine and saturated log schemes (cf. [AGT16, II 5.25, 5.18]). Then, f remains smooth (resp. saturated) under the base change in the category of fine and saturated log schemes ([Ogu18, IV.3.1.2, IV.3.1.11], resp. [Ogu18, III.2.5.3]). We remark that if f is smooth, then $f^{tr} : X^{tr} \to S^{tr}$ is a smooth morphism of schemes. If, moreover, $(S, \mathscr{L})$ is regular, then $(X, \mathscr{M})$ is also regular [Ogu18, IV.3.5.3]. We also remark that if f is saturated, then for any fibered product in the category of fine and saturated log schemes $(Z, \mathscr{P}) = (X, \mathscr{M}) \times^{fs}_{(S,\mathscr{L})} (Y, \mathscr{N})$, we have $Z = X \times_S Y$ [Tsu19, II.2.13].

4.5 Let K be a complete discrete valuation field with valuation ring $\mathcal{O}_K$, let k be the residue field of $\mathcal{O}_K$, and let π be a uniformizer of $\mathcal{O}_K$. We set $\eta = \mathrm{Spec}(K)$, $S = \mathrm{Spec}(\mathcal{O}_K)$ and $s = \mathrm{Spec}(k)$. Then, $(S, \mathscr{M}_{\eta\to S})$ is fine, saturated, and regular, since $\mathbb{N} \to \Gamma(S, \mathscr{M}_{\eta\to S})$ sending 1 to π forms a chart of $(S, \mathscr{M}_{\eta\to S})$ (cf. [AGT16, II.5.13, II.6.1]). Recall that an open immersion $Y \to X$ of quasi-compact and separated schemes over $\eta \to S$ is *strictly semi-stable* [dJ96, 6.3] if and only if the following conditions are satisfied ([dJ96, 6.4], [EGA IV_4, 17.5.3]):

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(i) For each point x of the generic fiber X_η, there is an open neighborhood $U \subseteq X_\eta$ of x and a smooth K-morphism

$$f : U \longrightarrow \mathrm{Spec}(K[s_1, \dots, s_m]) \tag{4.5.1}$$

such that f maps x to the point associated with the maximal ideal $(s_1, \dots, s_m)$ and that $U \backslash Y$ is the inverse image of the closed subset defined by $s_1 \dots s_m = 0$.

(ii) For each point x of the special fiber X_s, there is an open neighborhood $U \subseteq X$ of x and a smooth $\mathcal{O}_K$-morphism

$$f : U \longrightarrow \mathrm{Spec}(\mathcal{O}_K[t_1, \dots, t_n, s_1, \dots, s_m]/(\pi - t_1 \dots t_n)) \tag{4.5.2}$$

such that f maps x to the point associated with the maximal ideal $(t_1, \dots, t_n, s_1, \dots, s_m)$ and that $U \backslash Y$ is the inverse image of the closed subset defined by $t_1 \dots t_n \cdot s_1 \dots s_m = 0$.

We say that an open immersion $Y \to X$ of quasi-compact and separated schemes over $\eta \to S$ is *semi-stable* if for any point x of X there is an étale neighborhood U of x such that $Y \times_X U \to U$ is strictly semi-stable. In this case, $(X, \mathscr{M}_{Y \to X})$ is a fine, saturated, and regular log scheme smooth and saturated over $(S, \mathscr{M}_{\eta \to S})$, since for any point x of X there is an étale neighborhood U of x such that there exists a chart for the morphism $(U, \mathscr{M}_{Y \times_X U \to U}) \to (S, \mathscr{M}_{\eta \to S})$ subordinate to the morphism $\mathbb{N} \to \mathbb{N}^n \oplus \mathbb{N}^m$ sending 1 to $(1, \dots, 1, 0, \dots, 0)$ such that the induced morphism $U \to S \times_{\mathbb{A}_\mathbb{N}} \mathbb{A}_{\mathbb{N}^n \oplus \mathbb{N}^m}$ is smooth (cf. [Ogu18, II.2.4.1, IV.3.1.18]).

4.6 Recall that a morphism of schemes $f : X \to S$ is called *generically finite* if there exists a dense open subscheme U of S such that $f^{-1}(U) \to U$ is finite. We remark that for a morphism $f : X \to S$ of finite type between Noetherian schemes which maps generic points to generic points, f is generically finite if and only if the residue field of any generic point η of X is a finite field extension of the residue field of $f(\eta)$ [ILO14, II.1.1.7].

4.7 Let K be a complete discrete valuation field with valuation ring $\mathcal{O}_K$, let L be an algebraically closed valuation field of height 1 extension of K with valuation ring $\mathcal{O}_L$, and let $\overline{K}$ be the algebraic closure of K in L.

Consider the category $\mathscr{C}$ of open immersions between integral affine schemes $U \to T$ over $\mathrm{Spec}(K) \to \mathrm{Spec}(\mathcal{O}_K)$ under $\mathrm{Spec}(L) \to \mathrm{Spec}(\mathcal{O}_L)$ such that T is of finite type over $\mathcal{O}_K$ and that $\mathrm{Spec}(L) \to U$ is dominant. Let $\mathscr{C}_{\mathrm{car}}$ be the full subcategory of $\mathscr{C}$ formed by those objects $U \to T$ Cartesian over $\mathrm{Spec}(K) \to \mathrm{Spec}(\mathcal{O}_K)$.

$$\begin{tikzcd} \mathrm{Spec}(L) \arrow[r] \arrow[d] & \mathrm{Spec}(\mathcal{O}_L) \arrow[d] \\ U = \mathrm{Spec}(B) \arrow[r] \arrow[d] & T = \mathrm{Spec}(A) \arrow[d] \\ \mathrm{Spec}(K) \arrow[r] & \mathrm{Spec}(\mathcal{O}_K) \end{tikzcd} \tag{4.7.1}$$

We note that the objects of $\mathscr{C}$ are of the form $(U = \mathrm{Spec}(B) \to T = \mathrm{Spec}(A))$, where A (resp. B) is a finitely generated $\mathcal{O}_K$-subalgebra of $\mathcal{O}_L$ (resp. K-subalgebra of L) with $A \subseteq B$ such that $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is an open immersion.

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Lemma 4.8 *With the notation in 4.7, we have:*

(1) *The category $\mathscr{C}$ is cofiltered, and the subcategory $\mathscr{C}_{\mathrm{car}}$ is initial in $\mathscr{C}$.*
(2) *The morphism $\mathrm{Spec}(L) \to \mathrm{Spec}(\mathcal{O}_L)$ represents the cofiltered limit of morphisms $U \to T$ indexed by $\mathscr{C}$ in the category of morphisms of schemes (see [He23, 7.1]).*
(3) *There exists a directed inverse system $(U_\lambda \to T_\lambda)_{\lambda\in\Lambda}$ of objects of $\mathscr{C}_{\mathrm{car}}$ over a directed inverse system $(\mathrm{Spec}(K_\lambda) \to \mathrm{Spec}(\mathcal{O}_{K_\lambda}))_{\lambda\in\Lambda}$ of objects of $\mathscr{C}_{\mathrm{car}}$ such that K_λ is a finite field extension of K in L, that $\overline{K} = \bigcup_{\lambda\in\Lambda} K_\lambda$, that $U_\lambda \to T_\lambda$ is strictly semi-stable over $\mathrm{Spec}(K_\lambda) \to \mathrm{Spec}(\mathcal{O}_{K_\lambda})$ (4.5), and that $(U_\lambda \to T_\lambda)_{\lambda\in\Lambda}$ forms an initial full subcategory of $\mathscr{C}_{\mathrm{car}}$.*

Proof (1) For a diagram $(U_1 \to T_1) \to (U_0 \to T_0) \leftarrow (U_2 \to T_2)$ in $\mathscr{C}$, let T be the scheme theoretic image of $\mathrm{Spec}(L) \to T_1 \times_{T_0} T_2$, and let U be the intersection of $U_1 \times_{U_0} U_2$ with T. It is clear that T is of finite type over $\mathcal{O}_K$ as $\mathcal{O}_K$ is Noetherian, that U and T are integral and affine, that $\mathrm{Spec}(L) \to U$ is dominant, and that $\mathrm{Spec}(L) \to T$ factors through $\mathrm{Spec}(\mathcal{O}_L)$. Thus, $U \to T$ is an object of $\mathscr{C}$, which shows that $\mathscr{C}$ is cofiltered. For an object $(U = \mathrm{Spec}(B) \to T = \mathrm{Spec}(A))$ of $\mathscr{C}$, we write $\mathcal{O}_L$ as a filtered union of finitely generated A-subalgebras A_i. Let π be a uniformizer of K. Notice that $L = \mathcal{O}_L[1/\pi] = \mathrm{colim}\, A_i[1/\pi]$ and that $\mathrm{Hom}_{K\text{-}\mathbf{Alg}}(B, L) = \mathrm{colim}\, \mathrm{Hom}_{K\text{-}\mathbf{Alg}}(B, A_i[1/\pi])$ by [EGA IV$_3$, 8.14.2.2]. Thus, there exists an index i such that $\mathrm{Spec}(A_i[1/\pi]) \to \mathrm{Spec}(A_i)$ is an object of $\mathscr{C}_{\mathrm{car}}$ over $U \to T$.

(2) It follows immediately from the arguments above.

(3) Consider the category $\mathscr{D}$ of morphisms of $\mathscr{C}_{\mathrm{car}}$,

$$\begin{tikzcd} U' \arrow[r] \arrow[d] & T' \arrow[d] \\ \mathrm{Spec}(K') \arrow[r] & \mathrm{Spec}(\mathcal{O}_{K'}) \end{tikzcd} \tag{4.8.1}$$

such that K' is a finite field extension of K. Similarly, this category is also cofiltered with limit of diagrams of schemes $(\mathrm{Spec}(L) \to \mathrm{Spec}(\mathcal{O}_L)) \to (\mathrm{Spec}(\overline{K}) \to \mathrm{Spec}(\mathcal{O}_{\overline{K}}))$. It suffices to show that the full subcategory of $\mathscr{D}$ formed by strictly semi-stable objects is initial. For any object $U \to T$ of $\mathscr{C}_{\mathrm{car}}$, by de Jong's alteration theorem [dJ96, 6.5], there exists a proper surjective and generically finite morphism $T' \to T$ of integral schemes such that $U' = U \times_T T' \to T'$ is strictly semi-stable over $\mathrm{Spec}(K') \to \mathrm{Spec}(\mathcal{O}_{K'})$ for a finite field extension $K \to K'$. Since L is algebraically closed, the dominant morphism $\mathrm{Spec}(L) \to U$ lifts to a dominant morphism $\mathrm{Spec}(L) \to U'$ (4.6), which further extends to a lifting $\mathrm{Spec}(\mathcal{O}_L) \to T'$ of $\mathrm{Spec}(\mathcal{O}_L) \to T$ by the valuative criterion. After replacing T' by an affine open neighborhood of the image of the closed point of $\mathrm{Spec}(\mathcal{O}_L)$, we obtain a strictly semi-stable object of $\mathscr{D}$ over $(U \to T) \to (\mathrm{Spec}(K) \to \mathrm{Spec}(\mathcal{O}_K))$, which completes the proof. ∎

Theorem 4.9 ([ILO14, X 3.5, 3.7]) *Let K be a complete discrete valuation field with valuation ring $\mathcal{O}_K$, and let $(Y \to X) \to (U \to T)$ be a morphism of dominant open immersions over $\mathrm{Spec}(K) \to \mathrm{Spec}(\mathcal{O}_K)$ between irreducible $\mathcal{O}_K$-schemes of finite type such that $X \to T$ is proper surjective. Then, there exists a commutative diagram of dominant open immersions between irreducible $\mathcal{O}_K$-schemes of finite type*

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$$\begin{tikzcd}
(Y' \to X') \arrow[r, "(\beta^\circ{,}\beta)"] \arrow[d, "(f'^\circ{,}f')"'] & (Y \to X) \arrow[d, "(f^\circ{,}f)"] \\
(U' \to T') \arrow[r, "(\alpha^\circ{,}\alpha)"'] & (U \to T)
\end{tikzcd} \tag{4.9.1}$$

satisfying the following conditions:

(i) *We have* $Y' = \beta^{-1}(Y) \cap f'^{-1}(U')$*, i.e.,* $Y' \to X'$ *is Cartesian over* $U' \times_U Y \to T' \times_T X$ *(see [He23, 7.1]).*
(ii) *The morphism* $(X', \mathscr{M}_{Y' \to X'}) \to (T', \mathscr{M}_{U' \to T'})$ *induced by* (f'°, f') *is a smooth and saturated morphism of fine, saturated, and regular log schemes.*
(iii) *The morphisms* α *and* β *are proper surjective and generically finite, and* f' *is projective surjective.*

Proof We may assume that T is nonempty. Recall that $\mathrm{Spec}(\mathcal{O}_K)$ is universally $\mathbb{Q}$-resolvable [ILO14, X.3.3] by de Jong's alteration theorem [dJ96, 6.5]. Thus, T is also universally $\mathbb{Q}$-resolvable by [ILO14, X 3.5, 3.5.2] so that we can apply [ILO14, X.3.5] to the proper surjective morphism f and the nowhere dense closed subset $X \backslash Y$. Then, we obtain a commutative diagram of schemes

$$\begin{tikzcd}
X' \arrow[r, "\beta"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\
T' \arrow[r, "\alpha"'] & T
\end{tikzcd} \tag{4.9.2}$$

and dense open subsets $U' \subseteq T'$, $Y' = \beta^{-1}(Y) \cap f'^{-1}(U') \subseteq X'$ such that $(X', \mathscr{M}_{Y' \to X'})$ and $(T', \mathscr{M}_{U' \to T'})$ are fine, saturated, and regular, that $(X', \mathscr{M}_{Y' \to X'}) \to (T', \mathscr{M}_{U' \to T'})$ is smooth, that α, β are proper surjective and generically finite morphisms which map generic points to generic points, and that f' is projective (since f is proper, cf. [ILO14, X 3.1.6, 3.1.7]). Since X (resp. T) is irreducible and X' (resp. T') is a disjoint union of normal integral schemes (4.4), after firstly replacing X' by an irreducible component and then replacing T' by the irreducible component under X', we may assume that X' and T' are irreducible. Then, $Y' \to U'$ is dominant (so that f' is projective surjective), since it is smooth and Y' is nonempty [EGA IV_2, 2.3.4]. We claim that α maps U' into U. Indeed, if there exists a point $u \in U'$ with $\alpha(u) \notin U$, then $f'^{-1}(u) \cap Y' = \emptyset$. However, endowing u with the trivial log structure, the log scheme $(u, \mathcal{O}^\times_{u_{\text{ét}}})$ is fine, saturated, and regular, and the fibered product in the category of fine and saturated log schemes

$$(u, \mathcal{O}^\times_{u_{\text{ét}}}) \times^{\mathrm{fs}}_{(T', \mathscr{M}_{U' \to T'})} (X', \mathscr{M}_{Y' \to X'}) \tag{4.9.3}$$

is regular with underlying scheme $f'^{-1}(u)$ (4.4, 4.2). Thus, $f'^{-1}(u) \cap Y'$ is dense in $f'^{-1}(u)$, which contradicts the assumption that $f'^{-1}(u) \cap Y' = \emptyset$ since f' is surjective. Thus, we obtain a diagram (4.9.1) satisfying all the conditions except the saturatedness of $(X', \mathscr{M}_{Y' \to X'}) \to (T', \mathscr{M}_{U' \to T'})$.

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To make $(X', \mathscr{M}_{Y' \to X'}) \to (T', \mathscr{M}_{U' \to T'})$ saturated, we apply [ILO14, X.3.7] to the morphism (f'°, f'). We obtain a Cartesian morphism $(\gamma^{\circ}, \gamma) : (U'' \to T'') \to (U' \to T')$ of dominant open immersions such that $(T'', \mathscr{M}_{U'' \to T''})$ is a fine, saturated, and regular log scheme, that γ is a proper surjective and generically finite morphism which maps generic points of T'' to the generic point of T', and that the fibered product in the category of fine and saturated log schemes

$$(T'', \mathscr{M}_{U'' \to T''}) \times^{\mathrm{fs}}_{(T', \mathscr{M}_{U' \to T'})} (X', \mathscr{M}_{Y' \to X'}) \tag{4.9.4}$$

is saturated over $(T'', \mathscr{M}_{U'' \to T''})$. The fibered product (4.9.4) is still smooth over $(T'', \mathscr{M}_{U'' \to T''})$, and thus it is regular (4.4). Let X'' be the underlying scheme of it, and let $Y'' = (X'')^{\mathrm{tr}}$. Then, the fibered product (4.9.4) is isomorphic to $(X'', \mathscr{M}_{Y'' \to X''})$ (4.4). Thus, we obtain a commutative diagram of dominant open immersions of schemes:

$$\begin{tikzcd} (Y'' \to X'') \arrow[r, "(\delta^{\circ}, \delta)"] \arrow[d, "(f''^{\circ}, f'')"'] & (Y' \to X') \arrow[d, "(f'^{\circ}, f')"] \\ (U'' \to T'') \arrow[r, "(\gamma^{\circ}, \gamma)"'] & (U' \to T') \end{tikzcd} \tag{4.9.5}$$

Notice that $Y'' = U'' \times_{U'} Y'$ and $X'' \to T'' \times_{T'} X'$ is finite, and that $Y'' \to X''$ is Cartesian over $U'' \times_{U'} Y' \to T'' \times_{T'} X'$ (4.2). Thus, we see that $Y'' \to X''$ is Cartesian over $U'' \times_U Y \to T'' \times_T X$ and that f'' is projective. Since T' (resp. X') is irreducible and T'' (resp. X'') is a disjoint union of normal integral schemes (4.4), after firstly replacing T'' by an irreducible component and then replacing X'' by an irreducible component on which the restriction of δ° is dominant, we may assume that T'' and X'' are irreducible. In particular, δ is generically finite and so is $\beta \circ \delta$ (4.6), and again $Y'' \to U''$ is dominant so that f'' is projective surjective. ∎

Lemma 4.10 *Let X be a scheme of finite type over a valuation ring A of height* 1. *Then, the underlying topological space of X is Noetherian.*

Proof Let η and s be the generic point and closed point of $\mathrm{Spec}(A)$, respectively. Then, the generic fiber X_η and the special fiber X_s are both Noetherian. As a union of X_η and X_s, the underlying topological space of X is also Noetherian [Sta23, 0053]. ∎

Proposition 4.11 *With the notation in 4.7 and 4.8, let $Y \to X$ be a quasi-compact dominant open immersion over $\mathrm{Spec}(L) \to \mathrm{Spec}(\mathcal{O}_L)$ such that $X \to \mathrm{Spec}(\mathcal{O}_L)$ is proper of finite presentation. Then, there exists a proper surjective $\mathcal{O}_L$-morphism of finite presentation $X' \to X$, an index $\lambda_1 \in \Lambda$, and a directed inverse system of open immersions $(Y'_\lambda \to X'_\lambda)_{\lambda \geq \lambda_1}$ over $(U_\lambda \to T_\lambda)_{\lambda \geq \lambda_1}$ satisfying the following conditions for each $\lambda \geq \lambda_1$:*

(i) *We have $Y' = Y \times_X X' = \lim_{\lambda \geq \lambda_1} Y'_\lambda$ and $X' = \lim_{\lambda \geq \lambda_1} X'_\lambda$.*
(ii) *The log scheme $(X'_\lambda, \mathscr{M}_{Y'_\lambda \to X'_\lambda})$ is fine, saturated, and regular.*
(iii) *The morphism $(X'_\lambda, \mathscr{M}_{Y'_\lambda \to X'_\lambda}) \to (T_\lambda, \mathscr{M}_{U_\lambda \to T_\lambda})$ is smooth and saturated, and $X'_\lambda \to T_\lambda$ is projective.*
(iv) *If, moreover, $Y = \mathrm{Spec}(L) \times_{\mathrm{Spec}(\mathcal{O}_L)} X$, then we can require that $Y'_\lambda = U_\lambda \times_{T_\lambda} X'_\lambda$.*

Proof We follow closely the proof of [ALPT19, 5.2.19]. Since the underlying topological space of X is Noetherian by 4.10, each irreducible component Z of X admits a closed subscheme structure such that $Z \to X$ is of finite presentation [Sta23, 01PH]. After replacing X by the disjoint union of its irreducible components, we may assume that X is irreducible. Then, the generic fiber of $X \to \mathrm{Spec}(\mathcal{O}_L)$ is also irreducible as an open subset of X. Using [EGA IV$_3$, 8.8.2, 8.10.5], there exists an index $\lambda_0 \in \Lambda$, a proper T_{λ_0}-scheme X_{λ_0}, and an open subscheme Y_{λ_0} of $U_{\lambda_0} \times_{T_{\lambda_0}} X_{\lambda_0}$, such that $X = \mathrm{Spec}(\mathcal{O}_L) \times_{T_{\lambda_0}} X_{\lambda_0}$ and that $Y = \mathrm{Spec}(L) \times_{U_{\lambda_0}} Y_{\lambda_0}$. Let η denote the generic point of X, η_{λ_0} the image of η under the morphism $X \to X_{\lambda_0}$, Z_{λ_0} the scheme theoretic closure of η_{λ_0} in X_{λ_0}. Notice that $\mathrm{Spec}(\mathcal{O}_L) \times_{T_{\lambda_0}} Z_{\lambda_0} \to X$ is a surjective finitely presented closed immersion. After replacing X by $\mathrm{Spec}(\mathcal{O}_L) \times_{T_{\lambda_0}} Z_{\lambda_0}$ and replacing X_{λ_0} by Z_{λ_0}, we may assume that $X \to X_{\lambda_0}$ is a dominant morphism of irreducible schemes. Since T_{λ_0} is irreducible and L is algebraically closed, the generic fiber of $f : X_{\lambda_0} \to T_{\lambda_0}$ is geometrically irreducible. In particular, if ξ_{λ_0} (resp. η_{λ_0}) denotes the generic point of T_{λ_0} (resp. X_{λ_0}), then $\eta = \mathrm{Spec}(L) \times_{\xi_{\lambda_0}} \eta_{\lambda_0}$ [EGA IV$_2$, 4.5.9]. In the situation of (iv), we can moreover assume that $Y_{\lambda_0} = U_{\lambda_0} \times_{T_{\lambda_0}} X_{\lambda_0}$.

By 4.9, there exists a commutative diagram of dominant open immersions of irreducible schemes,

(4.11.1)
$$\begin{tikzcd}
(Y'_{\lambda_0} \to X'_{\lambda_0}) \arrow[r, "(\beta^\circ,\beta)"] \arrow[d, "(f'^\circ,f')"'] & (Y_{\lambda_0} \to X_{\lambda_0}) \arrow[d, "(f^\circ,f)"] \\
(U'_{\lambda_0} \to T'_{\lambda_0}) \arrow[r, "(\alpha^\circ,\alpha)"'] & (U_{\lambda_0} \to T_{\lambda_0})
\end{tikzcd}$$

where $Y'_{\lambda_0} \to X'_{\lambda_0}$ is Cartesian over $U'_{\lambda_0} \times_{U_{\lambda_0}} Y_{\lambda_0} \to T'_{\lambda_0} \times_{T_{\lambda_0}} X_{\lambda_0}$, and where $(X'_{\lambda_0}, \mathscr{M}_{Y'_{\lambda_0} \to X'_{\lambda_0}}) \to (T'_{\lambda_0}, \mathscr{M}_{U'_{\lambda_0} \to T'_{\lambda_0}})$ is a smooth and saturated morphism of fine, saturated, and regular log schemes, and where α and β are proper surjective and generically finite, and where f' is projective surjective. We take a dominant morphism $\gamma^\circ : \mathrm{Spec}(L) \to U'_{\lambda_0}$ which lifts $\mathrm{Spec}(L) \to U_{\lambda_0}$ since L is algebraically closed and α is generically finite, the morphism $\mathrm{Spec}(\mathcal{O}_L) \to T_{\lambda_0}$ lifts to $\gamma : \mathrm{Spec}(\mathcal{O}_L) \to T'_{\lambda_0}$ by the valuative criterion. We set $Y' = \mathrm{Spec}(L) \times_{U'_{\lambda_0}} Y'_{\lambda_0}$ and $X' = \mathrm{Spec}(\mathcal{O}_L) \times_{T'_{\lambda_0}} X'_{\lambda_0}$. It is clear that $Y' \to X'$ is Cartesian over $Y \to X$ by base change. Let ξ'_{λ_0} (resp. η'_{λ_0}) be the generic point of T'_{λ_0} (resp. X'_{λ_0}). Since the generic fiber of f is geometrically irreducible, $\xi'_{\lambda_0} \times_{\xi_{\lambda_0}} \eta_{\lambda_0}$ is a single point and η'_{λ_0} maps to it [EGA IV$_2$, 4.5.9]. Since $\mathrm{Spec}(L) \times_{\xi_{\lambda_0}} \eta_{\lambda_0}$ is the generic point of X, we see that $X' \to X$ is proper surjective and of finite presentation. It remains to construct $(Y'_\lambda \to X'_\lambda)_{\lambda \geq \lambda_1}$.

After replacing T'_{λ_0} by an affine open neighborhood of the image of the closed point of $\mathrm{Spec}(\mathcal{O}_L)$, Lemma 4.8 implies that there exists an index $\lambda_1 \geq \lambda_0$ such that the transition morphism $(U_{\lambda_1} \to T_{\lambda_1}) \to (U_{\lambda_0} \to T_{\lambda_0})$ factors through $(U'_{\lambda_0} \to T'_{\lambda_0})$. For each index $\lambda \geq \lambda_1$, consider the fibered product in the category of fine and saturated log schemes

(4.11.2)
$$(X'_\lambda, \mathscr{M}_{Y'_\lambda \to X'_\lambda}) = (T_\lambda, \mathscr{M}_{U_\lambda \to T_\lambda}) \times^{\mathrm{fs}}_{(T'_{\lambda_0}, \mathscr{M}_{U'_{\lambda_0} \to T'_{\lambda_0}})} (X'_{\lambda_0}, \mathscr{M}_{Y'_{\lambda_0} \to X'_{\lambda_0}}),$$

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which is a fine, saturated, and regular log scheme smooth and saturated over $(T_\lambda, \mathscr{M}_{U_\lambda \to T_\lambda})$ (4.2, 4.4). Moreover, we have $Y'_\lambda = U_\lambda \times_{U'_{\lambda_0}} Y'_{\lambda_0}$, $X'_\lambda = T_\lambda \times_{T'_{\lambda_0}} X'_{\lambda_0}$, and in the situation of (iv), $Y'_\lambda = U_\lambda \times_{T_\lambda} X'_\lambda$ by base change. Therefore, $(Y'_\lambda \to X'_\lambda)_{\lambda \geq \lambda_1}$ meets our requirements. ∎

5 Faltings' main *p*-adic comparison theorem: the absolute case

Lemma 5.1 *Let Y be a coherent scheme, and let V be a finite étale Y-scheme. Then, there exists a finite étale surjective morphism $Y' \to Y$ such that $Y' \times_Y V$ is isomorphic to a finite disjoint union of Y'.*

Proof If Y is connected, let $\overline{y}$ be a geometric point of Y, $\pi_1(Y, \overline{y})$ the fundamental group of Y with base point $\overline{y}$. Then, $Y_{\text{fét}}$ is equivalent to the category of finite $\pi_1(Y, \overline{y})$-sets so that the lemma holds [Sta23, 0BND].

In general, for any connected component Z of Y, let $(Y_\lambda)_{\lambda \in \Lambda_Z}$ be the directed inverse system of all open and closed subschemes of Y which contain Z and whose transition morphisms are inclusions. Notice that $\lim_{\lambda \in \Lambda_Z} Y_\lambda$ is a closed subscheme of Y with underlying topological space Z by [Sta23, 04PL] and [EGA IV$_3$, 8.2.9]. We endow Z with the closed subscheme structure of $\lim_{\lambda \in \Lambda_Z} Y_\lambda$. The first paragraph shows that there exists a finite étale surjective morphism $Z' \to Z$ such that $Z' \times_Y V = \coprod_{i=1}^r Z'$. Using [EGA IV$_3$, 8.8.2, 8.10.5] and [EGA IV$_4$, 17.7.8], there exists an index $\lambda_0 \in \Lambda_Z$, a finite étale surjective morphism $Y'_{\lambda_0} \to Y_{\lambda_0}$ and an isomorphism $Y'_{\lambda_0} \times_Y V = \coprod_{i=1}^r Y'_{\lambda_0}$. Notice that Y'_{λ_0} is also finite étale over Y. Since Z is an arbitrary connected component of Y, the conclusion follows from the quasi-compactness of Y. ∎

Lemma 5.2 *Let Y be a coherent scheme, and let $\rho : Y_{\text{ét}} \to Y_{\text{fét}}$ be the morphism of sites defined by the inclusion functor. Then, the functor $\rho^{-1} : \widetilde{Y_{\text{fét}}} \to \widetilde{Y_{\text{ét}}}$ of the associated topoi induces an equivalence $\rho^{-1} : \mathbf{LocSys}(Y_{\text{fét}}) \to \mathbf{LocSys}(Y_{\text{ét}})$ between the categories of finite locally constant abelian sheaves with quasi-inverse ρ_*.*

Proof Since any finite locally constant sheaf on $Y_{\text{ét}}$ (resp. $Y_{\text{fét}}$) is representable by a finite étale Y-scheme by faithfully flat descent (cf. [Sta23, 03RV]), the Yoneda embeddings induce a commutative diagram:

$$\begin{tikzcd} \mathbf{LocSys}(Y_{\text{fét}}) \arrow[r] \arrow[d, "\rho^{-1}"'] & Y_{\text{fét}} \arrow[r, "h^{\text{fét}}"] \arrow[d] & \widetilde{Y_{\text{fét}}} \arrow[d] \\ \mathbf{LocSys}(Y_{\text{ét}}) \arrow[r] & Y_{\text{ét}} \arrow[r, "h^{\text{ét}}"] & \widetilde{Y_{\text{ét}}} \end{tikzcd} \tag{5.2.1}$$

where the horizontal arrows are fully faithful. In particular, ρ^{-1} is fully faithful. For a finite locally constant abelian sheaf $\mathbb{F}$ on $Y_{\text{ét}}$, let V be a finite étale Y-scheme representing $\mathbb{F}$, and let $h_V^{\text{ét}}$ (resp. $h_V^{\text{fét}}$) be the representable sheaf associated with V on $Y_{\text{ét}}$ (resp. $Y_{\text{fét}}$) (see 2.4). We have $\mathbb{F} = h_V^{\text{ét}} = \rho^{-1} h_V^{\text{fét}}$ [Sta23, 04D3]. By 5.1, $h_V^{\text{fét}}$ is finite locally constant. It is clear that the adjunction morphism $h_V^{\text{fét}} \to \rho_* h_V^{\text{ét}}$ is an isomorphism, which shows that $h_V^{\text{fét}}$ is an abelian sheaf. Thus, ρ^{-1} is essentially surjective. Moreover, the argument also shows that ρ_* induces a functor $\rho_* : \mathbf{LocSys}(Y_{\text{ét}}) \to \mathbf{LocSys}(Y_{\text{fét}})$ which is a quasi-inverse of ρ^{-1}. ∎

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Proposition 5.3 *Let $Y \to X$ be a morphism of coherent schemes. With the notation in 3.2, the functors between the categories of finite locally constant abelian sheaves*

(5.3.1) $$\mathbf{LocSys}(Y_{\text{fét}}) \xrightarrow{\beta^{-1}} \mathbf{LocSys}(\mathbf{E}^{\text{ét}}_{Y\to X}) \xrightarrow{\psi^{-1}} \mathbf{LocSys}(Y_{\text{ét}})$$

are equivalences with quasi-inverses β_* and ψ_*, respectively.

Proof Notice that for any finite locally constant abelian sheaf $\mathbb{G}$ on $Y_{\text{fét}}$, the canonical morphism $\beta^{-1}\mathbb{G} \to \psi_*\rho^{-1}\mathbb{G}$, which is induced by the adjunction $\mathrm{id} \to \psi_*\psi^{-1}$ and by the identity $\psi^{-1}\beta^{-1} = \rho^{-1}$, is an isomorphism by 5.2 and the proof of [AGT16, VI.6.3(iii)]. For a finite locally constant abelian sheaf $\mathbb{F}$ over $Y_{\text{ét}}$, we write $\mathbb{F} = \rho^{-1}\mathbb{G}$ by 5.2. Then, $\mathbb{F} = \psi^{-1}\beta^{-1}\mathbb{G} \xrightarrow{\sim} \psi^{-1}\psi_*\rho^{-1}\mathbb{G} = \psi^{-1}\psi_*\mathbb{F}$, whose inverse is the adjunction map $\psi^{-1}\psi_*\mathbb{F} \to \mathbb{F}$ since the composition of $\psi^{-1}(\beta^{-1}\mathbb{G}) \to \psi^{-1}(\psi_*\psi^{-1})(\beta^{-1}\mathbb{G}) = (\psi^{-1}\psi_*)\psi^{-1}(\beta^{-1}\mathbb{G}) \to \psi^{-1}(\beta^{-1}\mathbb{G})$ is the identity. On the other hand, for a finite locally constant abelian sheaf $\mathbb{L}$ over $\mathbf{E}^{\text{ét}}_{Y\to X}$, we claim that $\mathbb{L} \to \psi_*\psi^{-1}\mathbb{L}$ is an isomorphism. The problem is local on $\mathbf{E}^{\text{ét}}_{Y\to X}$. Thus, we may assume that $\mathbb{L}$ is the constant sheaf with value L, where L is a finite abelian group. Let $\underline{L}$ be the constant sheaf with value L on $Y_{\text{fét}}$. Then, $\mathbb{L} = \beta^{-1}\underline{L}$, and the isomorphism $\mathbb{L} = \beta^{-1}\underline{L} \xrightarrow{\sim} \psi_*\rho^{-1}\underline{L} = \psi_*\psi^{-1}\mathbb{L}$ coincides with the adjunction map $\mathbb{L} \to \psi_*\psi^{-1}\mathbb{L}$. Therefore, $\psi^{-1} : \mathbf{LocSys}(\mathbf{E}^{\text{ét}}_{Y\to X}) \to \mathbf{LocSys}(Y_{\text{ét}})$ is an equivalence with quasi-inverse ψ_*. The conclusion follows from 5.2. ■

5.4 Let $f : (Y' \to X') \to (Y \to X)$ be a morphism of morphisms between coherent schemes over $\mathrm{Spec}(\mathbb{Q}_p) \to \mathrm{Spec}(\mathbb{Z}_p)$. The base change by f induces a commutative diagram of sites (see 3.2):

(5.4.1) $$\begin{array}{ccc} Y'_{\text{ét}} & \xrightarrow{\psi'} & \mathbf{E}^{\text{ét}}_{Y'\to X'} \\ \downarrow{\scriptstyle f_{\text{ét}}} & & \downarrow{\scriptstyle f_{\mathbf{E}}} \\ Y_{\text{ét}} & \xrightarrow{\psi} & \mathbf{E}^{\text{ét}}_{Y\to X} \end{array}$$

Let $\mathbb{F}'$ be a finite locally constant abelian sheaf on $Y'_{\text{ét}}$. Remark that the sheaf $\overline{\mathscr{B}}$ on $\mathbf{E}^{\text{ét}}_{Y\to X}$ is flat over $\mathbb{Z}$. Consider the natural morphisms in the derived category $\mathbf{D}(\overline{\mathscr{B}}\text{-}\mathbf{Mod}_{\mathbf{E}^{\text{ét}}_{Y\to X}})$,

(5.4.2) $$(\mathrm{R}\psi_*\mathrm{R}f_{\text{ét}*}\mathbb{F}') \otimes^{\mathrm{L}}_{\mathbb{Z}} \overline{\mathscr{B}} \xleftarrow{\alpha_1} (\mathrm{R}f_{\mathbf{E}*}\psi'_*\mathbb{F}') \otimes^{\mathrm{L}}_{\mathbb{Z}} \overline{\mathscr{B}} \xrightarrow{\alpha_2} \mathrm{R}f_{\mathbf{E}*}(\psi'_*\mathbb{F}' \otimes_{\mathbb{Z}} \overline{\mathscr{B}}'),$$

where α_1 is induced by the canonical morphism $\psi'_*\mathbb{F}' \to \mathrm{R}\psi'_*\mathbb{F}'$, and α_2 is the canonical morphism.

5.5 We keep the notation in 5.4 and assume that X is the spectrum of an absolutely integrally closed valuation ring A and that Y is a quasi-compact open subscheme of X. Then, the associated topos of $\mathbf{E}^{\text{ét}}_{Y\to X}$ is local (3.7). By applying the functor $\mathrm{R}\Gamma(Y \to X, -)$ on (5.4.2), we obtain the natural morphisms in the derived category $\mathbf{D}(A\text{-}\mathbf{Mod})$,

(5.5.1)
$$\mathrm{R}\Gamma(Y'_{\text{ét}}, \mathbb{F}') \otimes^{\mathrm{L}}_{\mathbb{Z}} A \xleftarrow{\alpha_1} \mathrm{R}\Gamma(\mathbf{E}^{\text{ét}}_{Y'\to X'}, \psi'_*\mathbb{F}') \otimes^{\mathrm{L}}_{\mathbb{Z}} A \xrightarrow{\alpha_2} \mathrm{R}\Gamma(\mathbf{E}^{\text{ét}}_{Y'\to X'}, \psi'_*\mathbb{F}' \otimes_{\mathbb{Z}} \overline{\mathscr{B}}').$$

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Definition 5.6 ([AG20, 4.8.13, 5.7.4]) Under the assumptions in 5.4 (resp. 5.5) and with the same notation, if α_1 is an isomorphism (for instance, if the canonical morphism $\psi'_*\mathbb{F}' \to \mathrm{R}\psi'_*\mathbb{F}'$ is an isomorphism), then we call the canonical morphism

$$\alpha_2 \circ \alpha_1^{-1} : (\mathrm{R}\psi_*\mathrm{R}f_{\text{ét}*}\mathbb{F}') \otimes^{\mathrm{L}}_{\mathbb{Z}} \overline{\mathscr{B}} \longrightarrow \mathrm{R}f_{\mathbf{E}*}(\psi'_*\mathbb{F}' \otimes_{\mathbb{Z}} \overline{\mathscr{B}}') \tag{5.6.1}$$

$$(\text{resp. } \alpha_2 \circ \alpha_1^{-1} : \mathrm{R}\Gamma(Y'_{\text{ét}}, \mathbb{F}') \otimes^{\mathrm{L}}_{\mathbb{Z}} A \longrightarrow \mathrm{R}\Gamma(\mathbf{E}^{\text{ét}}_{Y'\to X'}, \psi'_*\mathbb{F}' \otimes_{\mathbb{Z}} \overline{\mathscr{B}}')) \tag{5.6.2}$$

the *relative* (resp. *absolute*) *Faltings' comparison morphism* associated with $f : (Y' \to X') \to (Y \to X)$ and $\mathbb{F}'$. In this case, we say that the *relative* (resp. *absolute*) *Faltings' comparison morphism exists.*

Theorem 5.7 [Ach17, Corollary 6.9], cf. [AG20, 4.4.2] *Let $\mathcal{O}_K$ be a strictly Henselian discrete valuation ring with fraction field K of characteristic* 0 *and residue field of characteristic p. We fix an algebraic closure $\overline{K}$ of K. Let X be an $\mathcal{O}_K$-scheme of finite type, let $\mathbb{F}$ be a finite locally constant abelian sheaf on $X_{\overline{K},\text{ét}}$, and let $\psi : X_{\overline{K},\text{ét}} \to \mathbf{E}^{\text{ét}}_{X_{\overline{K}}\to X}$ be the morphism of sites defined in* 3.2. *Then, the canonical morphism $\psi_*\mathbb{F} \to \mathrm{R}\psi_*\mathbb{F}$ is an isomorphism.*

Corollary 5.8 *Let $\mathcal{O}_K$ be a strictly Henselian discrete valuation ring with fraction field K of characteristic* 0 *and residue field of characteristic p. We fix an algebraic closure $\overline{K}$ of K. Let X be a coherent $\mathcal{O}_{\overline{K}}$-scheme, $Y = \mathrm{Spec}(\overline{K}) \times_{\mathrm{Spec}(\mathcal{O}_{\overline{K}})} X$, let $\mathbb{F}$ be a finite locally constant abelian sheaf on $Y_{\text{ét}}$, and let $\psi : Y_{\text{ét}} \to \mathbf{E}^{\text{ét}}_{Y\to X}$ be the morphism of sites defined in* 3.2. *Then, the canonical morphism $\psi_*\mathbb{F} \to \mathrm{R}\psi_*\mathbb{F}$ is an isomorphism.*

We emphasize that we don't need any finiteness condition of X over $\mathcal{O}_{\overline{K}}$ in 5.8. In fact, one can replace $\mathcal{O}_{\overline{K}}$ by $\overline{\mathbb{Z}_p}$ without loss of generality, where $\overline{\mathbb{Z}_p}$ is the integral closure of $\mathbb{Z}_p$ in an algebraic closure of $\mathbb{Q}_p$. We keep working over $\mathcal{O}_{\overline{K}}$ only for the continuation of our usage of notation.

Proof of 5.8 We take a directed inverse system $(X_\lambda \to \mathrm{Spec}(\mathcal{O}_{K_\lambda}))_{\lambda\in\Lambda}$ of morphisms of finite type of schemes by Noetherian approximation, such that K_λ is a finite field extension of K and $\overline{K} = \bigcup_{\lambda\in\Lambda} K_\lambda$, and that the transition morphisms $X_{\lambda'} \to X_\lambda$ are affine and $X = \lim_{\lambda\in\Lambda} X_\lambda$ (cf. [Sta23, 09MV]). For each $\lambda \in \Lambda$, we set $Y_\lambda = \mathrm{Spec}(\overline{K}) \times_{\mathrm{Spec}(\mathcal{O}_{K_\lambda})} X_\lambda$. Notice that $Y = \lim Y_\lambda$. Then, there exists an index $\lambda_0 \in \Lambda$ and a finite locally constant abelian sheaf $\mathbb{F}_{\lambda_0}$ on $Y_{\lambda_0,\text{ét}}$ such that $\mathbb{F}$ is the pullback of $\mathbb{F}_{\lambda_0}$ by $Y_{\text{ét}} \to Y_{\lambda_0,\text{ét}}$ (cf. [Sta23, 09YU]). Let $\mathbb{F}_\lambda$ be the pullback of $\mathbb{F}_{\lambda_0}$ by $Y_{\lambda,\text{ét}} \to Y_{\lambda_0,\text{ét}}$ for each $\lambda \geq \lambda_0$. Notice that $\mathcal{O}_{K_\lambda}$ also satisfies the conditions in 5.7. Let $\psi_\lambda : Y_{\lambda,\text{ét}} \to \mathbf{E}^{\text{ét}}_{Y_\lambda\to X_\lambda}$ be the morphism of sites defined in 3.2, and let $\varphi_\lambda : \mathbf{E}^{\text{ét}}_{Y\to X} \to \mathbf{E}^{\text{ét}}_{Y_\lambda\to X_\lambda}$ be the morphism of sites defined by the transition morphism. Then, we have $\mathrm{R}^q\psi_{\lambda*}\mathbb{F}_\lambda = 0$ for each integer $q > 0$ by 5.7, and moreover,

$$\mathrm{R}^q\psi_*\mathbb{F} = \operatorname*{colim}_{\lambda\geq\lambda_0} \varphi_\lambda^{-1}\mathrm{R}^q\psi_{\lambda*}\mathbb{F}_\lambda = 0 \tag{5.8.1}$$

by [He23, 7.12], [SGA 4_{II}, VII.5.6], and [SGA 4_{II}, VI.8.7.3] whose conditions are satisfied because each object in each concerned site is quasi-compact. ∎

Lemma 5.9 *With the notation in* 5.4, *let $\mathbb{F}$ be a finite locally constant abelian sheaf on $Y_{\text{ét}}$. Then, the canonical morphism $f_{\mathbf{E}}^{-1}\psi_*\mathbb{F} \to \psi'_* f_{\text{ét}}^{-1}\mathbb{F}$ is an isomorphism.*

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Proof The base change morphism $f_{\mathbf{E}}^{-1}\psi_*\mathbb{F} \to \psi'_* f_{\text{ét}}^{-1}\mathbb{F}$ is the composition of the adjunction morphisms [SGA 4_{III}, XVII.2.1.3]

$$f_{\mathbf{E}}^{-1}\psi_*\mathbb{F} \to \psi'_*\psi'^{-1}(f_{\mathbf{E}}^{-1}\psi_*\mathbb{F}) = \psi'_* f_{\text{ét}}^{-1}(\psi^{-1}\psi_*\mathbb{F}) \to \psi'_* f_{\text{ét}}^{-1}\mathbb{F} \tag{5.9.1}$$

which are both isomorphisms by 5.3. ∎

5.10 Let K be a complete discrete valuation field of characteristic 0 with valuation ring $\mathcal{O}_K$ whose residue field k is algebraically closed (a condition required by [AG20, 4.1.3, 5.1.3]) of characteristic $p > 0$, and let $\overline{K}$ be an algebraic closure of K, $\mathcal{O}_{\overline{K}}$ the integral closure of $\mathcal{O}_K$ in $\overline{K}$, $\eta = \mathrm{Spec}(K)$, $\overline{\eta} = \mathrm{Spec}(\overline{K})$, $S = \mathrm{Spec}(\mathcal{O}_K)$, $\overline{S} = \mathrm{Spec}(\mathcal{O}_{\overline{K}})$, $s = \mathrm{Spec}(k)$. Remark that $\overline{K}$ is a pre-perfectoid field with valuation ring $\mathcal{O}_{\overline{K}}$ so we are also in the situation of 3.9.

5.11 With the notation in 5.10, let X be an S-scheme, and let Y be an open subscheme of the generic fiber X_η. We simply denote by $\mathscr{M}_X$ the compactifying log structure $\mathscr{M}_{X_\eta \to X}$ (4.3). Following [AGT16, III.4.7], we say that $Y \to X$ is *adequate* over $\eta \to S$ if the following conditions are satisfied:

(i) X is of finite type over S.
(ii) Any point of the special fiber X_s admits an étale neighborhood U such that $U_\eta \to \eta$ is smooth and that $U_\eta \backslash Y$ is the support of a strict normal crossings divisor on U_η.
(iii) $(X, \mathscr{M}_{Y\to X})$ is a fine log scheme and the structure morphism $(X, \mathscr{M}_{Y\to X}) \to (S, \mathscr{M}_S)$ is smooth and saturated.

In this case, $(X, \mathscr{M}_{Y\to X}) \to (S, \mathscr{M}_S)$ is adequate in the sense of [AGT16, III.4.7]. We remark that for any adequate $(S, \mathscr{M}_S)$-log scheme $(X, \mathscr{M})$, $X^{\text{tr}} \to X$ is adequate over $\eta \to S$ and $(X, \mathscr{M}) = (X, \mathscr{M}_{X^{\text{tr}}\to X})$ (see 4.4 and 4.5). Note that if $Y \to X$ is semi-stable over $\eta \to S$ then it is adequate (see 4.5).

5.12 We recall the statement of Faltings' main p-adic comparison theorem following Abbes and Gros [AG20]. We take the notation and assumptions in 5.10. Firstly, recall that for any adequate open immersion of schemes $X^\circ \to X$ over $\eta \to S$ and any finite locally constant abelian sheaf $\mathbb{F}$ on $X^\circ_{\overline{\eta},\text{ét}}$, the canonical morphism $\psi_*\mathbb{F} \to \mathrm{R}\psi_*\mathbb{F}$ is an isomorphism, where $\psi : X^\circ_{\overline{\eta},\text{ét}} \to \mathbf{E}^{\text{ét}}_{X^\circ_{\overline{\eta}}\to X}$ is the morphism of sites defined in 3.2 [AG20, 4.4.2].

Let $(X'^{\triangleright} \to X') \to (X^\circ \to X)$ be a morphism of adequate open immersions of schemes over $\eta \to S$ such that $X' \to X$ is projective and that the induced morphism $(X', \mathscr{M}_{X'^{\triangleright}\to X'}) \to (X, \mathscr{M}_{X^\circ\to X})$ is smooth and saturated. Let $Y' = \overline{\eta} \times_\eta X'^{\triangleright}$, $Y = \overline{\eta} \times_\eta X^\circ$, $f : (Y' \to X') \to (Y \to X)$ be the natural morphism, and let $\mathbb{F}'$ be a finite locally constant abelian sheaf on $Y'_{\text{ét}}$. By the first paragraph, we have the relative Faltings' comparison morphism associated with f and $\mathbb{F}'$ (5.6.1),

$$(\mathrm{R}\psi_*\mathrm{R}f_{\text{ét}*}\mathbb{F}') \otimes^{\mathrm{L}}_{\mathbb{Z}} \overline{\mathscr{B}} \longrightarrow \mathrm{R}f_{\mathbf{E}*}(\psi'_*\mathbb{F}' \otimes_{\mathbb{Z}} \overline{\mathscr{B}}'). \tag{5.12.1}$$

Remark that under our assumption, the sheaf $\mathrm{R}^q f_{\text{ét}*}\mathbb{F}'$ on $Y_{\text{ét}}$ is finite locally constant for each integer q [AG20, 2.2.25].

Theorem 5.13 [Fal02, Theorem 6, page 266], [AG20, 5.7.4] *Under the assumptions in 5.12 and with the same notation, the relative Faltings' comparison morphism associated*

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with f and $\mathbb{F}'$ is an almost isomorphism in the derived category $\mathbf{D}(\mathcal{O}_{\overline{K}}\text{-}\mathbf{Mod}_{\mathbf{E}^{\text{ét}}_{Y\to X}})$ [He23, 5.7], and it induces an almost isomorphism

$$(\psi_* \mathrm{R}^q f_{\text{ét}*}\mathbb{F}') \otimes_{\mathbb{Z}} \overline{\mathscr{B}} \longrightarrow \mathrm{R}^q f_{\mathbf{E}*}(\psi'_*\mathbb{F}' \otimes_{\mathbb{Z}} \overline{\mathscr{B}}') \tag{5.13.1}$$

of $\mathcal{O}_{\overline{K}}$-modules for each integer q.

Proposition 5.14 *With the notation in 5.10, let A be an absolutely integrally closed valuation ring of height 1 extension of $\mathcal{O}_{\overline{K}}$, let X be a proper A-scheme of finite presentation, $Y = \mathrm{Spec}(A[1/p]) \times_{\mathrm{Spec}(A)} X$, and let $\mathbb{F}$ be a finite locally constant abelian sheaf on $Y_{\text{ét}}$. Then, there exists a proper surjective morphism $X' \to X$ of finite presentation such that the relative and absolute Faltings' comparison morphisms associated with $f' : (Y' \to X') \to (\mathrm{Spec}(A[1/p]) \to \mathrm{Spec}(A))$ and $\mathbb{F}'$ (which exist by 5.8) are almost isomorphisms, where $Y' = Y \times_X X'$ and $\mathbb{F}'$ is the pullback of $\mathbb{F}$ on $Y'_{\text{ét}}$.*

Proof Since the underlying topological space of X is Noetherian by 4.10, each irreducible component Z of X admits a closed subscheme structure such that $Z \to X$ is of finite presentation [Sta23, 01PH]. After replacing X by the disjoint union of its irreducible components, we may assume that X is irreducible. If Y is empty, then we take $X' = X$ and thus the relative (resp. absolute) Faltings' comparison morphism associated with f' and $\mathbb{F}'$ is an isomorphism between zero objects. If Y is not empty, then we are in the situation of 4.11(iv) by taking $\mathcal{O}_L = A$. With the notation in 4.11, we check that the morphism $X' \to X$ meets our requirements. We set $\eta_\lambda = \mathrm{Spec}(K_\lambda)$, $S_\lambda = \mathrm{Spec}(\mathcal{O}_{K_\lambda})$, $T_{\lambda,\overline{\eta_\lambda}} = \overline{\eta} \times_{\eta_\lambda} U_\lambda$, $X'_{\lambda,\overline{\eta_\lambda}} = \overline{\eta} \times_{\eta_\lambda} Y'_\lambda$, and denote by $f'_\lambda : (X'_{\lambda,\overline{\eta_\lambda}} \to X'_\lambda) \to (T_{\lambda,\overline{\eta_\lambda}} \to T_\lambda)$ the natural morphism. We obtain a commutative diagram:

(5.14.1)

$$\begin{tikzcd}
\mathbf{E}^{\text{ét}}_{Y'\to X'} \arrow[rrr, "g_{\lambda,\mathbf{E}}"] \arrow[ddd, "f'_{\mathbf{E}}"'] & & & \mathbf{E}^{\text{ét}}_{X'_{\lambda,\overline{\eta_\lambda}}\to X'_\lambda} \arrow[ddd, "f'_{\lambda,\mathbf{E}}"] \\
& Y'_{\text{ét}} \arrow[ul, "\psi'"'] \arrow[r, "g_{\lambda,\text{ét}}"] \arrow[d, "f'_{\text{ét}}"'] & X'_{\lambda,\overline{\eta_\lambda},\text{ét}} \arrow[ur, "\psi'_\lambda"] \arrow[d, "f'_{\lambda,\text{ét}}"] & \\
& \mathrm{Spec}(A[1/p])_{\text{ét}} \arrow[r, "h_{\lambda,\text{ét}}"] \arrow[dl, "\psi"'] & T_{\lambda,\overline{\eta_\lambda},\text{ét}} \arrow[dr, "\psi_\lambda"] & \\
\mathbf{E}^{\text{ét}}_{\mathrm{Spec}(A[1/p])\to\mathrm{Spec}(A)} \arrow[rrr, "h_{\lambda,\mathbf{E}}"] & & & \mathbf{E}^{\text{ét}}_{T_{\lambda,\overline{\eta_\lambda}}\to T_\lambda}
\end{tikzcd}$$

Firstly, notice that the site $Y'_{\text{ét}}$ (resp. $\mathrm{Spec}(A[1/p])_{\text{ét}}$) is the limit of the sites $X'_{\lambda,\overline{\eta_\lambda},\text{ét}}$ (resp. $T_{\lambda,\overline{\eta_\lambda},\text{ét}}$) and the site $\mathbf{E}^{\text{ét}}_{Y'\to X'}$ (resp. $\mathbf{E}^{\text{ét}}_{\mathrm{Spec}(A[1/p])\to\mathrm{Spec}(A)}$) is the limit of the sites $\mathbf{E}^{\text{ét}}_{X'_{\lambda,\overline{\eta_\lambda}}\to X'_\lambda}$ (resp. $\mathbf{E}^{\text{ét}}_{T_{\lambda,\overline{\eta_\lambda}}\to T_\lambda}$) ([SGA 4_{II}, VII.5.6] and [He23, 7.12]). There exists an index $\lambda_0 \in \Lambda$ and a finite locally constant abelian sheaf $\mathbb{F}'_{\lambda_0}$ on $X'_{\lambda_0,\overline{\eta_{\lambda_0}},\text{ét}}$ such that $\mathbb{F}'$ is

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the pullback of $\mathbb{F}'_{\lambda_0}$ by $Y'_{\text{ét}} \to X'_{\lambda_0,\overline{\eta_{\lambda_0}},\text{ét}}$ (cf. [Sta23, 09YU]). Let $\mathbb{F}'_\lambda$ be the pullback of $\mathbb{F}'_{\lambda_0}$ by $X'_{\lambda,\overline{\eta_\lambda},\text{ét}} \to X'_{\lambda_0,\overline{\eta_{\lambda_0}},\text{ét}}$ for each $\lambda \geq \lambda_0$. We also have $\overline{\mathscr{B}}' = \operatorname{colim} g^{-1}_{\lambda,\mathbf{E}}\overline{\mathscr{B}}'$ (resp. $\overline{\mathscr{B}} = \operatorname{colim} h^{-1}_{\lambda,\mathbf{E}}\overline{\mathscr{B}}$) by [He23, 7.12]. According to [SGA 4_{II}, VI.8.7.3], whose conditions are satisfied because each object in each concerned site is quasi-compact, there are canonical isomorphisms for each integer q,

$$(\mathrm{R}^q(\psi \circ f'_{\text{ét}})_*\mathbb{F}') \otimes_{\mathbb{Z}} \overline{\mathscr{B}} \xrightarrow{\sim} \operatorname{colim} h^{-1}_{\lambda,\mathbf{E}}((\mathrm{R}^q(\psi_\lambda \circ f'_{\lambda,\text{ét}})_*\mathbb{F}'_\lambda) \otimes_{\mathbb{Z}} \overline{\mathscr{B}}), \tag{5.14.2}$$

$$\mathrm{R}^q f'_{\mathbf{E}*}(\psi'_*\mathbb{F}' \otimes_{\mathbb{Z}} \overline{\mathscr{B}}') \xrightarrow{\sim} \operatorname{colim} h^{-1}_{\lambda,\mathbf{E}}\mathrm{R}^q f'_{\lambda,\mathbf{E}*}(\psi'_{\lambda*}\mathbb{F}'_\lambda \otimes_{\mathbb{Z}} \overline{\mathscr{B}}'). \tag{5.14.3}$$

On the other hand, $(X'_\lambda, \mathscr{M}_{X'_\lambda}) \to (T_\lambda, \mathscr{M}_{T_\lambda})$ is a smooth and saturated morphism of adequate $(S_\lambda, \mathscr{M}_{S_\lambda})$-log schemes with $X'_\lambda \to T_\lambda$ projective for each $\lambda \in \Lambda$ by construction. Thus, we are in the situation of 5.13, which implies that the relative Faltings' comparison morphism associated with f'_λ and $\mathbb{F}'_\lambda$,

$$(\mathrm{R}^q(\psi_\lambda \circ f'_{\lambda,\text{ét}})_*\mathbb{F}'_\lambda) \otimes_{\mathbb{Z}} \overline{\mathscr{B}} \longrightarrow \mathrm{R}^q f'_{\lambda,\mathbf{E}*}(\psi'_{\lambda*}\mathbb{F}'_\lambda \otimes_{\mathbb{Z}} \overline{\mathscr{B}}') \tag{5.14.4}$$

is an almost isomorphism for each $\lambda \geq \lambda_0$. Combining with (5.14.2) and (5.14.3), we see that the relative Faltings' comparison morphism associated with f' and $\mathbb{F}'$,

$$\mathrm{R}\psi_*(\mathrm{R}f'_{\text{ét}*}\mathbb{F}') \otimes^{\mathrm{L}}_{\mathbb{Z}} \overline{\mathscr{B}} \longrightarrow \mathrm{R}f'_{\mathbf{E}*}(\psi'_*\mathbb{F}' \otimes_{\mathbb{Z}} \overline{\mathscr{B}}'), \tag{5.14.5}$$

is an almost isomorphism (and thus so is the absolute one). ∎

Corollary 5.15 *Under the assumptions in 5.14 and with the same notation, there exists a proper hypercovering $X_\bullet \to X$ of coherent schemes [Sta23, 0DHI] such that for each degree n, the relative and absolute Faltings' comparison morphisms associated with $f_n : (Y_n \to X_n) \to (\operatorname{Spec}(A[1/p]) \to \operatorname{Spec}(A))$ and $\mathbb{F}_n$ (which exist by 5.8) are almost isomorphisms, where $Y_n = Y \times_X X_n$ and $\mathbb{F}_n$ is the pullback of $\mathbb{F}$ by the natural morphism $Y_{n,\text{ét}} \to Y_{\text{ét}}$. In particular, $Y_\bullet \to Y$ is a proper hypercovering and $X_\bullet^{Y_\bullet} \to X^Y$ is a hypercovering in $\mathbf{I}_{\overline{\eta}\to\overline{S}}$.*

Proof Let $\mathscr{C}$ be the category of proper A-schemes of finite presentation endowed with the pretopology formed by families of morphisms $\{f_i : X_i \to X\}_{i\in I}$ with I finite and $X = \bigcup_{i\in I} f_i(X_i)$. Consider the functor $u^+ : \mathscr{C} \to \mathbf{I}_{\operatorname{Spec}(A[1/p])\to\operatorname{Spec}(A)}$ sending X to X^Y, where $Y = \operatorname{Spec}(A[1/p]) \times_{\operatorname{Spec}(A)} X$. It is well-defined by [He23, 3.19(4)] and commutes with fibered products by [He23, 3.21] and is continuous by [He23, 3.15]. Proposition 5.14 allows us to take a hypercovering $X_\bullet \to X$ in $\mathscr{C}$ meeting our requirement by [Sta23, 094K and 0DB1]. We see that $Y_\bullet \to Y$ is a proper hypercovering and that $X_\bullet^{Y_\bullet} \to X^Y$ is a hypercovering in $\mathbf{I}_{\overline{\eta}\to\overline{S}}$ by the properties of u^+ [Sta23, 0DAY]. ∎

Lemma 5.16 *Let $\overline{\mathbb{Z}_p}$ be the integral closure of $\mathbb{Z}_p$ in an algebraic closure of $\mathbb{Q}_p$, let A be a $\overline{\mathbb{Z}_p}$-algebra which is an absolutely integrally closed valuation ring, let X be a proper A-scheme of finite presentation, $Y = \operatorname{Spec}(A[1/p]) \times_{\operatorname{Spec}(A)} X$, and let $\mathbb{F}$ be a finite locally constant abelian sheaf on $Y_{\text{ét}}$. Let $A' = ((A/\cap_{n>0} p^n A)_{\sqrt{pA}})^\wedge$ (p-adic completion), $X' = X_{A'}$, $Y' = Y_{A'}$, $\mathbb{F}'$ the pullback of $\mathbb{F}$ on $Y'_{\text{ét}}$. Then, the following statements are equivalent:*

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(1) *The absolute Faltings' comparison morphism associated with* $f : (Y \to X) \to (\mathrm{Spec}(A[1/p]) \to \mathrm{Spec}(A))$ *and* $\mathbb{F}$ *(which exists by 5.8) is an almost isomorphism.*
(2) *The absolute Faltings' comparison morphism associated with* $f' : (Y' \to X') \to (\mathrm{Spec}(A'[1/p]) \to \mathrm{Spec}(A'))$ *and* $\mathbb{F}'$ *(which exists by 5.8) is an almost isomorphism.*

Proof If p is zero (resp. invertible) in A, then the absolute Faltings' comparison morphisms are both isomorphisms between zero objects, since Y and Y' are empty (resp. the abelian sheaves $\mathbb{F}$ and $\mathbb{F}'$ are zero after inverting p). Thus, we may assume that p is a nonzero element of the maximal ideal of A. Notice that $\cap_{n>0} p^n A$ is the maximal prime ideal of A not containing p and that $\sqrt{pA}$ is the minimal prime ideal of A containing p (2.1). Thus, $(A/\cap_{n>0} p^n A)_{\sqrt{pA}}$ is an absolutely integrally closed valuation ring of height 1 extension of $\overline{\mathbb{Z}_p}$ (2.1) and thus so is its p-adic completion A'.

We denote by $u : (Y' \to X') \to (Y \to X)$ the natural morphism. We have $\mathbb{F}' = u_{\text{ét}}^{-1}\mathbb{F}$. The natural morphisms in (5.5.1) induce a commutative diagram:

(5.16.1)
$$\begin{tikzcd}
\mathrm{R}\Gamma(Y_{\text{ét}}, \mathbb{F}) \otimes^{\mathrm{L}}_{\mathbb{Z}} A \arrow[d, "\gamma_1"] & \mathrm{R}\Gamma(\mathbf{E}^{\text{ét}}_{Y\to X}, \psi_*\mathbb{F}) \otimes^{\mathrm{L}}_{\mathbb{Z}} A \arrow[l, "\alpha_1"'] \arrow[r, "\alpha_2"] \arrow[d, "\gamma_2"] & \mathrm{R}\Gamma(\mathbf{E}^{\text{ét}}_{Y\to X}, \psi_*\mathbb{F} \otimes_{\mathbb{Z}} \overline{\mathscr{B}}) \arrow[d, "\gamma_3"] \\
\mathrm{R}\Gamma(Y'_{\text{ét}}, \mathbb{F}') \otimes^{\mathrm{L}}_{\mathbb{Z}} A' & \mathrm{R}\Gamma(\mathbf{E}^{\text{ét}}_{Y'\to X'}, \psi'_*\mathbb{F}') \otimes^{\mathrm{L}}_{\mathbb{Z}} A' \arrow[l, "\alpha'_1"'] \arrow[r, "\alpha'_2"] & \mathrm{R}\Gamma(\mathbf{E}^{\text{ét}}_{Y'\to X'}, \psi'_*\mathbb{F}' \otimes_{\mathbb{Z}} \overline{\mathscr{B}}')
\end{tikzcd}$$

where γ_1 is induced by the canonical morphism $\mathbb{F} \to \mathrm{R}u_{\text{ét}*}u_{\text{ét}}^{-1}\mathbb{F}$, and γ_2 (resp. γ_3) is induced by the composition of $\psi_*\mathbb{F} \to \mathrm{R}u_{\mathbf{E}*}u_{\mathbf{E}}^{-1}\psi_*\mathbb{F} \to \mathrm{R}u_{\mathbf{E}*}\psi'_*u_{\text{ét}}^{-1}\mathbb{F}$ (resp. and by the canonical morphism $\overline{\mathscr{B}} \to \mathrm{R}u_{\mathbf{E}*}\overline{\mathscr{B}}'$). Since α_1 and α'_1 are isomorphisms by 5.8, it suffices to show that γ_1 and γ_3 are almost isomorphisms.

Since $A/\cap_{n>0} p^n A \to (A/\cap_{n>0} p^n A)_{\sqrt{pA}}$ is injective whose cokernel is killed by $\sqrt{pA}$ [He23, 4.7], the morphism $A \to A'$ induces an almost isomorphism $A/p^n A \to A'/p^n A'$ for each n. Then, for any torsion abelian group M, the natural morphism $M \otimes_{\mathbb{Z}} A \to M \otimes_{\mathbb{Z}} A'$ is an almost isomorphism. Therefore, γ_1 is an almost isomorphism by the proper base change theorem over the strictly Henselian local ring $A[1/p]$ [SGA 4_{III}, XII 5.5, 5.4]. For γ_3, it suffices to show that the canonical morphism $\psi_*\mathbb{F} \otimes \overline{\mathscr{B}} \to \mathrm{R}u_{\mathbf{E}*}(\psi'_*\mathbb{F}' \otimes \overline{\mathscr{B}}')$ is an almost isomorphism. The problem is local on $\mathbf{E}^{\text{ét}}_{Y\to X}$, thus we may assume that $\psi_*\mathbb{F}$ is the constant sheaf with value $\mathbb{Z}/p^n\mathbb{Z}$ by 5.3. Then, $\psi'_*\mathbb{F}'$ is also the constant sheaf with value $\mathbb{Z}/p^n\mathbb{Z}$ by 5.9. Thus, it remains to show that $\overline{\mathscr{B}}/p^n\overline{\mathscr{B}} \to \mathrm{R}u_{\mathbf{E}*}(\overline{\mathscr{B}}'/p^n\overline{\mathscr{B}}')$ is an almost isomorphism. Consider the following commutative diagram of ringed sites (see 3.6):

(5.16.2)
$$\begin{tikzcd}
(\mathbf{I}_{Y'\to X'^{Y'}}, \mathscr{O}') \arrow[r, "\varepsilon"] \arrow[d, "u_{\mathbf{I}}"] & (\mathbf{E}^{\text{ét}}_{Y'\to X'}, \overline{\mathscr{B}}') \arrow[d, "u_{\mathbf{E}}"] \\
(\mathbf{I}_{Y\to X^{Y}}, \mathscr{O}) \arrow[r, "\varepsilon"] & (\mathbf{E}^{\text{ét}}_{Y\to X}, \overline{\mathscr{B}})
\end{tikzcd}$$

By the cohomological descent for Faltings ringed topos (3.9), it remains to show that $\mathscr{O}/p^n\mathscr{O} \to \mathrm{R}u_{\mathbf{E}*}(\mathscr{O}'/p^n\mathscr{O}')$ is an almost isomorphism. Let W be an object of

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$\mathbf{I}_{Y\to X^Y}$ such that W is the spectrum of a $\overline{\mathbb{Z}_p}$-algebra R which is almost pre-perfectoid [He23, 5.19]. Since the almost isomorphisms $R/p^n \to (R\otimes_A A')/p^n$ $(n\geq 1)$ induces an almost isomorphism of the p-adic completions $\widehat{R}\to R\widehat{\otimes}_A A'$, the $\overline{\mathbb{Z}_p}$-algebra $R\otimes_A A'$ is still almost pre-perfectoid [He23, 5.19]. The pullback of W in $\mathbf{I}_{Y'\to X'^{Y'}}$ is the spectrum of the integral closure R' of $R\otimes_A A'$ in $R\otimes_A A'[1/p]$ [He23, (3.21.1)]. Since $R\otimes_A A'$ is almost pre-perfectoid, R' is also almost pre-perfectoid and the morphism $(R\otimes_A A')/p^n\to R'/p^n$ is an almost isomorphism by [He23, 5.30]. Recall that such W forms a topological generating family of $\mathbf{I}_{Y\to X^Y}$ [He23, 8.10]. Therefore, the morphism $\mathscr{O}/p^n\mathscr{O}\to \mathrm{R}u_{\mathbf{E}*}(\mathscr{O}'/p^n\mathscr{O}')$ is an almost isomorphism by [He23, 8.11]. ■

Theorem 5.17 *Let $\overline{\mathbb{Z}_p}$ be the integral closure of $\mathbb{Z}_p$ in an algebraic closure of $\mathbb{Q}_p$, let A be a $\overline{\mathbb{Z}_p}$-algebra which is an absolutely integrally closed valuation ring, let X be a proper A-scheme of finite presentation, $Y=\mathrm{Spec}(A[1/p])\times_{\mathrm{Spec}(A)} X$, and let $\mathbb{F}$ be a finite locally constant abelian sheaf on $Y_{\text{ét}}$. Then, the absolute Faltings' comparison morphism associated with $f:(Y\to X)\to(\mathrm{Spec}(A[1/p])\to\mathrm{Spec}(A))$ and $\mathbb{F}$ (5.6.2) (which exists by 5.8),*

(5.17.1)
$$\mathrm{R}\Gamma(Y_{\text{ét}},\mathbb{F})\otimes^{\mathrm{L}}_{\mathbb{Z}} A\longrightarrow \mathrm{R}\Gamma(\mathbf{E}^{\text{ét}}_{Y\to X},\psi_*\mathbb{F}\otimes_{\mathbb{Z}}\overline{\mathscr{B}}),$$

is an almost isomorphism in $\mathbf{D}(\overline{\mathbb{Z}_p}\text{-}\mathbf{Mod})$ [He23, 5.7].

Proof Let K be the p-adic completion of the maximal unramified extension of $\mathbb{Q}_p$. By 5.16, we may assume that A is a valuation ring of height 1 extension of $\mathcal{O}_{\overline{K}}$. Let $X_\bullet\to X$ be the proper hypercovering of coherent schemes constructed in 5.15. For each degree n, the canonical morphisms (5.6.2),

(5.17.2)
$$\mathrm{R}\Gamma(Y_{n,\text{ét}},\mathbb{F}_n)\otimes^{\mathrm{L}}_{\mathbb{Z}} A\longleftarrow \mathrm{R}\Gamma(\mathbf{E}^{\text{ét}}_{Y_n\to X_n},\psi_{n*}\mathbb{F}_n)\otimes^{\mathrm{L}}_{\mathbb{Z}} A\longrightarrow \mathrm{R}\Gamma(\mathbf{E}^{\text{ét}}_{Y_n\to X_n},\psi_{n*}\mathbb{F}_n\otimes_{\mathbb{Z}}\overline{\mathscr{B}})$$

are, respectively, an isomorphism and an almost isomorphism, where $\mathbb{F}_n$ is the pullback of $\mathbb{F}$ by the natural morphism $Y_{n,\text{ét}}\to Y_{\text{ét}}$. Consider the commutative diagram:

(5.17.3)
$$\begin{array}{ccccc}
\mathrm{R}\Gamma(Y_{\text{ét}},\mathbb{F})\otimes^{\mathrm{L}}_{\mathbb{Z}} A & \xleftarrow{\alpha_1} & \mathrm{R}\Gamma(\mathbf{E}^{\text{ét}}_{Y\to X},\psi_*\mathbb{F})\otimes^{\mathrm{L}}_{\mathbb{Z}} A & \xrightarrow{\alpha_2} & \mathrm{R}\Gamma(\mathbf{E}^{\text{ét}}_{Y\to X},\psi_*\mathbb{F}\otimes_{\mathbb{Z}}\overline{\mathscr{B}})\\
\downarrow & & \downarrow & & \downarrow\\
\mathrm{R}\Gamma(Y_{\bullet,\text{ét}},\mathbb{F}_\bullet)\otimes^{\mathrm{L}}_{\mathbb{Z}} A & \xleftarrow{\alpha_{1\bullet}} & \mathrm{R}\Gamma(\mathbf{E}^{\text{ét}}_{Y_\bullet\to X_\bullet},\psi_{\bullet*}\mathbb{F}_\bullet)\otimes^{\mathrm{L}}_{\mathbb{Z}} A & \xrightarrow{\alpha_{2\bullet}} & \mathrm{R}\Gamma(\mathbf{E}^{\text{ét}}_{Y_\bullet\to X_\bullet},\psi_{\bullet*}\mathbb{F}_\bullet\otimes_{\mathbb{Z}}\overline{\mathscr{B}}_\bullet)
\end{array}$$

where $\mathbb{F}_\bullet=(\mathbb{F}_n)_{[n]\in\mathrm{Ob}(\Delta)}$ (see [He23, 6.5] for the notation). By the functorial spectral sequence of simplicial sites [Sta23, 09WJ], we deduce from (5.17.2) that $\alpha_{1\bullet}$ is an isomorphism and $\alpha_{2\bullet}$ is an almost isomorphism. Since α_1 is an isomorphism by 5.8, it remains to show that the left vertical arrow is an isomorphism and the right vertical arrow is an almost isomorphism.

We denote by $b:\mathbf{E}^{\text{ét}}_{Y_\bullet\to X_\bullet}\to\mathbf{E}^{\text{ét}}_{Y\to X}$ the augmentation of simplicial site and by $b_n:\mathbf{E}^{\text{ét}}_{Y_n\to X_n}\to\mathbf{E}^{\text{ét}}_{Y\to X}$ the natural morphism of sites. Notice that $b^{-1}\psi_*\mathbb{F}=(b_n^{-1}\psi_*\mathbb{F})_{[n]\in\mathrm{Ob}(\Delta)}=(\psi_{n*}\mathbb{F}_n)_{[n]\in\mathrm{Ob}(\Delta)}=\psi_{\bullet*}\mathbb{F}_\bullet$ by 5.9 [Sta23, 0D70]. Since $X_\bullet^{Y_\bullet}\to X^Y$ forms a hypercovering in $\mathbf{I}_{\overline{\eta}\to\overline{S}}$, the right vertical arrow is an almost isomorphism

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by 5.3 and 3.10. Finally, the left vertical arrow is an isomorphism by the cohomological descent for étale cohomology [Sta23, 0DHL]. ■

6 Faltings' main p-adic comparison theorem: the relative case

6.1 Let $Y \to X$ be a morphism of coherent schemes such that $Y \to X^Y$ is an open immersion. Recall that there is a natural commutative diagram of sites (see 3.6):

(6.1.1)
$$\begin{tikzcd}
(\mathbf{Sch}^{\mathrm{coh}}_{/Y})_{\mathrm{v}} \arrow[r, "a"] \arrow[d, "\Psi"'] & Y_{\mathrm{\acute{e}t}} \arrow[d, "\psi"] \arrow[dr, "\rho"] & \\
\mathbf{I}_{Y\to X^Y} \arrow[r, "\varepsilon"] & \mathbf{E}^{\mathrm{\acute{e}t}}_{Y\to X} \arrow[r, "\beta"] & Y_{\mathrm{f\acute{e}t}}
\end{tikzcd}$$

Lemma 6.2 *With the notation in 6.1, for any finite locally constant abelian sheaf $\mathbb{F}$ on $Y_{\mathrm{\acute{e}t}}$, the canonical morphism $\varepsilon^{-1}\psi_*\mathbb{F} \to \Psi_* a^{-1}\mathbb{F}$ is an isomorphism.*

Proof The base change morphism $\varepsilon^{-1}\psi_*\mathbb{F} \to \Psi_* a^{-1}\mathbb{F}$ is the composition of the adjunction morphisms [SGA 4_{III}, XVII.2.1.3]

(6.2.1)
$$\varepsilon^{-1}\psi_*\mathbb{F} \to \Psi_*\Psi^{-1}(\varepsilon^{-1}\psi_*\mathbb{F}) = \Psi_* a^{-1}(\psi^{-1}\psi_*\mathbb{F}) \to \Psi_* a^{-1}\mathbb{F}$$

which are both isomorphisms by 3.8(2) and 5.3. ■

6.3 We fix an algebraic closure $\overline{\mathbb{Q}_p}$ of $\mathbb{Q}_p$ and we denote by $\overline{\mathbb{Z}_p}$ the integral closure of $\mathbb{Z}_p$ in $\overline{\mathbb{Q}_p}$. We set $\eta = \mathrm{Spec}(\mathbb{Q}_p)$, $\overline{\eta} = \mathrm{Spec}(\overline{\mathbb{Q}_p})$, $S = \mathrm{Spec}(\mathbb{Z}_p)$, $\overline{S} = \mathrm{Spec}(\overline{\mathbb{Z}_p})$. Remark that $\overline{\mathbb{Q}_p}$ is a pre-perfectoid field with valuation ring $\overline{\mathbb{Z}_p}$ so we are also in the situation of 3.9. Let $f : (Y' \to X') \to (Y \to X)$ be a Cartesian morphism of morphisms of coherent schemes with a Cartesian morphism $(Y \to X^Y) \to (\overline{\eta} \to \overline{S})$ (then, $Y' \to X'^{Y'}$ is Cartesian over $\overline{\eta} \to \overline{S}$ by [He23, 3.19(4)]). Thus, X^Y and $X'^{Y'}$ are objects of $\mathbf{I}_{\overline{\eta}\to\overline{S}}$. Consider the following commutative diagram of sites associated with f (see 3.6):

(6.3.1)
$$\begin{tikzcd}
Y'_{\mathrm{\acute{e}t}} \arrow[d, "f_{\mathrm{\acute{e}t}}"'] \arrow[rrr, bend left, "\psi'"] & (\mathbf{Sch}^{\mathrm{coh}}_{/Y'})_{\mathrm{v}} \arrow[l, "a'"'] \arrow[r, "\Psi'"] \arrow[d, "f_{\mathrm{v}}"'] & \mathbf{I}_{Y'\to X'^{Y'}} \arrow[r, "\varepsilon'"] \arrow[d, "f_{\mathrm{I}}"] & \mathbf{E}^{\mathrm{\acute{e}t}}_{Y'\to X'} \arrow[d, "f_{\mathrm{E}}"] \\
Y_{\mathrm{\acute{e}t}} \arrow[rrr, bend right, "\psi"'] & (\mathbf{Sch}^{\mathrm{coh}}_{/Y})_{\mathrm{v}} \arrow[l, "a"'] \arrow[r, "\Psi"] & \mathbf{I}_{Y\to X^Y} \arrow[r, "\varepsilon"] & \mathbf{E}^{\mathrm{\acute{e}t}}_{Y\to X}
\end{tikzcd}$$

6.4 Following 6.3, let $g : (\widetilde{Y} \to \widetilde{X}) \to (Y \to X)$ be a morphism of coherent schemes such that $\widetilde{Y} \to \widetilde{X}^{\widetilde{Y}}$ is also Cartesian over $\overline{\eta} \to \overline{S}$. We denote by $g' : (\widetilde{Y}' \to \widetilde{X}') \to (Y' \to X')$ the base change of g by f, and denote by $\widetilde{f} : (\widetilde{Y}' \to \widetilde{X}') \to (\widetilde{Y} \to \widetilde{X})$ the natural morphism which is Cartesian by base change. Thus, $\widetilde{X}^{\widetilde{Y}}$ and $\widetilde{X}'^{\widetilde{Y}'}$ are also objects of $\mathbf{I}_{\overline{\eta}\to\overline{S}}$. We write the diagram (6.3.1) associated with $\widetilde{f}$ by equipping all labels with tildes.

Lemma 6.5 *With the notation in 6.3 and 6.4, let $\mathbb{F}'$ be a finite locally constant abelian sheaf on $Y'_{\text{ét}}$ and we set $\mathscr{F}' = \Psi'_* a'^{-1}\mathbb{F}'$. Let $\widetilde{X}$ be an object of $\mathbf{I}_{Y\to X^Y}$, $\widetilde{Y} = \overline{\eta} \times_{\overline{S}} \widetilde{X}$, $\widetilde{\mathbb{F}'} = g'^{-1}_{\text{ét}}\mathbb{F}'$, q an integer.*

(1) *The sheaf $\mathrm{R}^q f_{\mathbf{I}*}\mathscr{F}'$ on $\mathbf{I}_{Y\to X^Y}$ is canonically isomorphic to the sheaf associated with the presheaf $\widetilde{X} \mapsto H^q_{\text{ét}}(\widetilde{Y}', \widetilde{\mathbb{F}'})$.*
(2) *The sheaf $\mathrm{R}^q f_{\mathbf{I}*}(\mathscr{F}' \otimes_{\mathbb{Z}} \mathscr{O}')$ on $\mathbf{I}_{Y\to X^Y}$ is canonically almost isomorphic to the sheaf associated with the presheaf $\widetilde{X} \mapsto H^q(\mathbf{E}^{\text{ét}}_{\widetilde{Y}'\to\widetilde{X}'}, \widetilde{\psi}'_*\widetilde{\mathbb{F}'} \otimes_{\mathbb{Z}} \overline{\mathscr{B}}')$.*
(3) *The canonical morphism $(\mathrm{R}^q f_{\mathbf{I}*}\mathscr{F}') \otimes_{\mathbb{Z}} \mathscr{O} \to (\mathrm{R}^q f_{\mathbf{I}*}\mathscr{F}' \otimes_{\mathbb{Z}} \mathscr{O}')$ is compatible with the canonical morphisms $H^q_{\text{ét}}(\widetilde{Y}', \widetilde{\mathbb{F}'}) \otimes_{\mathbb{Z}} R \xleftarrow{\alpha_1} H^q(\mathbf{E}^{\text{ét}}_{\widetilde{Y}'\to\widetilde{X}'}, \widetilde{\psi}'_*\widetilde{\mathbb{F}'}) \otimes_{\mathbb{Z}} R \xrightarrow{\alpha_2} H^q(\mathbf{E}^{\text{ét}}_{\widetilde{Y}'\to\widetilde{X}'}, \widetilde{\psi}'_*\widetilde{\mathbb{F}'} \otimes_{\mathbb{Z}} \overline{\mathscr{B}}')$, where $R = \overline{\mathscr{B}}(\widetilde{Y} \to \widetilde{X})$ (cf. (5.5.1)).*

Proof Let $\widetilde{\mathscr{F}'}$ be the restriction of $\mathscr{F}'$ on $\mathbf{I}_{\widetilde{Y}'\to\widetilde{X}'^{\widetilde{Y}'}}$. We have $\widetilde{\mathscr{F}'} = \widetilde{\Psi}'_*\widetilde{a'}^{-1}\widetilde{\mathbb{F}'}$. We set $\widetilde{\mathbb{L}'} = \widetilde{\psi}'_*\widetilde{\mathbb{F}'}$ which is a finite locally constant abelian sheaf on $\mathbf{E}^{\text{ét}}_{\widetilde{Y}'\to\widetilde{X}'}$ by 5.3. Notice that the canonical morphisms $\widetilde{\psi'}^{-1}\widetilde{\mathbb{L}'} \to \widetilde{\mathbb{F}'}$ and $\widetilde{\varepsilon'}^{-1}\widetilde{\mathbb{L}'} \to \widetilde{\mathscr{F}'}$ are isomorphisms by 5.3 and 6.2, respectively.

(1) It follows from the canonical isomorphisms

(6.5.1)
$$H^q(\mathbf{I}_{\widetilde{Y}'\to\widetilde{X}'^{\widetilde{Y}'}}, \widetilde{\varepsilon'}^{-1}\widetilde{\mathbb{L}'}) \xrightarrow{\gamma_1} H^q_{\text{v}}(\widetilde{Y}', \widetilde{\Psi'}^{-1}\widetilde{\varepsilon'}^{-1}\widetilde{\mathbb{L}'}) = H^q_{\text{v}}(\widetilde{Y}', \widetilde{a'}^{-1}\widetilde{\psi'}^{-1}\widetilde{\mathbb{L}'}) \xleftarrow{\gamma_2} H^q_{\text{ét}}(\widetilde{Y}', \widetilde{\psi'}^{-1}\widetilde{\mathbb{L}'}),$$

where γ_1 is induced by the canonical isomorphism $\widetilde{\varepsilon'}^{-1}\widetilde{\mathbb{L}'} \xrightarrow{\sim} \mathrm{R}\widetilde{\Psi}'_*\widetilde{\Psi'}^{-1}\widetilde{\varepsilon'}^{-1}\widetilde{\mathbb{L}'}$ (3.8(2)), and γ_2 is induced by the canonical isomorphism $\widetilde{\psi'}^{-1}\widetilde{\mathbb{L}'} \to \mathrm{R}\widetilde{a'}_*\widetilde{a'}^{-1}\widetilde{\psi'}^{-1}\widetilde{\mathbb{L}'}$ (3.4).

(2) It follows from the canonical almost isomorphism

(6.5.2)
$$\gamma_3 : H^q(\mathbf{E}^{\text{ét}}_{\widetilde{Y}'\to\widetilde{X}'}, \widetilde{\mathbb{L}'} \otimes_{\mathbb{Z}} \overline{\mathscr{B}}') \longrightarrow H^q(\mathbf{I}_{\widetilde{Y}'\to\widetilde{X}'^{\widetilde{Y}'}}, \widetilde{\varepsilon'}^{-1}\widetilde{\mathbb{L}'} \otimes \mathscr{O}'),$$

which is induced by the canonical almost isomorphism $\widetilde{\mathbb{L}'} \otimes_{\mathbb{Z}} \overline{\mathscr{B}}' \to \mathrm{R}\widetilde{\varepsilon}'_*(\widetilde{\varepsilon'}^{-1}\widetilde{\mathbb{L}'} \otimes \mathscr{O}')$ (3.9).

(3) Consider the following diagram:

(6.5.3)
$$\begin{tikzcd}
H^q_{\text{ét}}(\widetilde{Y}', \widetilde{\psi'}^{-1}\widetilde{\mathbb{L}'}) \otimes R \arrow[d, "\gamma_2\otimes\mathrm{id}_R"', "\wr"] & H^q(\mathbf{E}^{\text{ét}}_{\widetilde{Y}'\to\widetilde{X}'}, \widetilde{\mathbb{L}'}) \otimes R \arrow[l, "\alpha_1"'] \arrow[r, "\alpha_2"] \arrow[d] & H^q(\mathbf{E}^{\text{ét}}_{\widetilde{Y}'\to\widetilde{X}'}, \widetilde{\mathbb{L}'} \otimes_{\mathbb{Z}} \overline{\mathscr{B}}') \arrow[d, "\gamma_3"] \\
H^q_{\text{v}}(\widetilde{Y}', \widetilde{\Psi'}^{-1}\widetilde{\varepsilon'}^{-1}\widetilde{\mathbb{L}'}) \otimes R & H^q(\mathbf{I}_{\widetilde{Y}'\to\widetilde{X}'^{\widetilde{Y}'}}, \widetilde{\varepsilon'}^{-1}\widetilde{\mathbb{L}'}) \otimes R \arrow[l, "\sim", "\gamma_1\otimes\mathrm{id}_R"'] \arrow[r] & H^q(\mathbf{I}_{\widetilde{Y}'\to\widetilde{X}'^{\widetilde{Y}'}}, \widetilde{\varepsilon'}^{-1}\widetilde{\mathbb{L}'} \otimes \mathscr{O}')
\end{tikzcd}$$

where the unlabeled vertical arrow is induced by the canonical morphism $\widetilde{\mathbb{L}'} \to \mathrm{R}\widetilde{\varepsilon}'_*\widetilde{\varepsilon'}^{-1}\widetilde{\mathbb{L}'}$, and the unlabeled horizontal arrow is the canonical morphism which induces $(\mathrm{R}^q f_{\mathbf{I}*}\mathscr{F}') \otimes_{\mathbb{Z}} \mathscr{O} \to \mathrm{R}^q f_{\mathbf{I}*}(\mathscr{F}' \otimes_{\mathbb{Z}} \mathscr{O}')$ on $\mathbf{I}_{Y\to X^Y}$ by sheafification. It is clear that the diagram (6.5.3) is commutative, which completes the proof. ■

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6.6 We remark that 6.5 gives a new definition of the relative (resp. absolute) Faltings' comparison morphism without using 5.8. Following 6.3, let $\mathbb{F}'$ be a finite locally constant abelian sheaf on $Y'_{\text{ét}}$ and we set $\mathscr{F}' = \Psi'_* a'^{-1}\mathbb{F}'$. We set $\mathbb{L}' = \psi'_*\mathbb{F}'$, which is a finite locally constant abelian sheaf on $\mathbf{E}^{\text{ét}}_{Y'\to X'}$ by 5.3. Remark that the canonical morphisms $\psi'^{-1}\mathbb{L}' \to \mathbb{F}'$ and $\varepsilon'^{-1}\mathbb{L}' \to \mathscr{F}'$ are isomorphisms by 5.3 and 6.2, respectively. We also remark that $\overline{\mathscr{B}}$, $\mathscr{O}$ are flat over $\mathbb{Z}$. The canonical morphisms in the derived category $\mathbf{D}(\overline{\mathscr{B}}\text{-}\mathbf{Mod}_{\mathbf{E}^{\text{ét}}_{Y\to X}})$ (cf. (5.4.2)),

$$(6.6.1)\quad (\mathrm{R}\psi_*\mathrm{R}f_{\text{ét}*}\psi'^{-1}\mathbb{L}')\otimes^{\mathrm{L}}_{\mathbb{Z}}\overline{\mathscr{B}} \xleftarrow{\alpha_1} (\mathrm{R}f_{\mathbf{E}*}\mathbb{L}')\otimes^{\mathrm{L}}_{\mathbb{Z}}\overline{\mathscr{B}} \xrightarrow{\alpha_2} \mathrm{R}f_{\mathbf{E}*}(\mathbb{L}'\otimes_{\mathbb{Z}}\overline{\mathscr{B}}'),$$

fit into the following commutative diagram:

(6.6.2)

$$\begin{tikzcd}
\mathrm{R}\psi_*(\mathrm{R}f_{\text{ét}*}\psi'^{-1}\mathbb{L}')\otimes^{\mathrm{L}}_{\mathbb{Z}}\overline{\mathscr{B}} \arrow[d, "\alpha_3"', "\wr"] & (\mathrm{R}f_{\mathbf{E}*}\mathbb{L}')\otimes^{\mathrm{L}}_{\mathbb{Z}}\overline{\mathscr{B}} \arrow[l, "\alpha_1"'] \arrow[r, "\alpha_2"] \arrow[d] & \mathrm{R}f_{\mathbf{E}*}(\mathbb{L}'\otimes_{\mathbb{Z}}\overline{\mathscr{B}}') \arrow[d, "\alpha_4"] \\
\mathrm{R}\psi_*(\mathrm{R}a_*\mathrm{R}f_{\mathbf{v}*}\Psi'^{-1}\varepsilon'^{-1}\mathbb{L}')\otimes^{\mathrm{L}}_{\mathbb{Z}}\overline{\mathscr{B}} & \mathrm{R}\varepsilon_*(\mathrm{R}f_{\mathbf{I}*}\varepsilon'^{-1}\mathbb{L}')\otimes^{\mathrm{L}}_{\mathbb{Z}}\overline{\mathscr{B}} \arrow[l, "\sim", "\alpha_5"'] \arrow[r, "\alpha_6"'] & \mathrm{R}\varepsilon_*\mathrm{R}f_{\mathbf{I}*}(\varepsilon'^{-1}\mathbb{L}'\otimes_{\mathbb{Z}}\mathscr{O}')
\end{tikzcd}$$

(1) The morphism α_3 is induced by the canonical isomorphism $\psi'^{-1}\mathbb{L}' \to \mathrm{R}a'_*a'^{-1}(\psi'^{-1}\mathbb{L}')$ by 3.4, and thus α_3 is an isomorphism.
(2) The morphism α_5 is induced by the canonical isomorphism $\varepsilon'^{-1}\mathbb{L}' \to \mathrm{R}\Psi'_*\Psi'^{-1}\varepsilon'^{-1}\mathbb{L}'$ by 3.8(2), and thus α_5 is an isomorphism.
(3) The unlabeled arrow is induced by the canonical morphism $\mathbb{L} \to \mathrm{R}\varepsilon'_*\varepsilon'^{-1}\mathbb{L}$.
(4) The morphism α_4 is induced by the canonical almost isomorphism $\mathbb{L}'\otimes_{\mathbb{Z}}\overline{\mathscr{B}}' \to \mathrm{R}\varepsilon'_*(\varepsilon'^{-1}\mathbb{L}'\otimes_{\mathbb{Z}}\mathscr{O}')$ by 3.9, and thus α_4 is an almost isomorphism.
(5) The morphism α_6 is the composition of

$$(6.6.3)\qquad \mathrm{R}\varepsilon_*(\mathrm{R}f_{\mathbf{I}*}\varepsilon'^{-1}\mathbb{L}')\otimes^{\mathrm{L}}_{\mathbb{Z}}\overline{\mathscr{B}} \longrightarrow \mathrm{R}\varepsilon_*((\mathrm{R}f_{\mathbf{I}*}\varepsilon'^{-1}\mathbb{L}')\otimes^{\mathrm{L}}_{\mathbb{Z}}\mathscr{O})$$

$$(6.6.4)\qquad \text{with } \mathrm{R}\varepsilon_*((\mathrm{R}f_{\mathbf{I}*}\varepsilon'^{-1}\mathbb{L}')\otimes^{\mathrm{L}}_{\mathbb{Z}}\mathscr{O}) \longrightarrow \mathrm{R}\varepsilon_*\mathrm{R}f_{\mathbf{I}*}(\varepsilon'^{-1}\mathbb{L}'\otimes_{\mathbb{Z}}\mathscr{O}').$$

In conclusion, the arrows α_3, α_5, α_6 and α_4 induce an arrow

$$(6.6.5)\qquad \alpha_4^{-1}\circ\alpha_6\circ\alpha_5^{-1}\circ\alpha_3 : \mathrm{R}\psi_*(\mathrm{R}f_{\text{ét}*}\mathbb{F}')\otimes^{\mathrm{L}}_{\mathbb{Z}}\overline{\mathscr{B}} \longrightarrow \mathrm{R}f_{\mathbf{E}*}(\psi'_*\mathbb{F}'\otimes_{\mathbb{Z}}\overline{\mathscr{B}}')$$

in the derived category of almost $\overline{\mathbb{Z}_p}$-modules on $\mathbf{E}^{\text{ét}}_{Y\to X}$ [He23, 5.7]. Remark that we don't assume that α_1 is an isomorphism here. We also call (6.6.5) the *relative Faltings' comparison morphism*. Indeed, if α_1 is an isomorphism, then the relative Faltings' comparison morphism (5.6.1) induces (6.6.5) in $\mathbf{D}(\overline{\mathbb{Z}_p}^{\mathrm{al}}\text{-}\mathbf{Mod})$ (the derived category of almost $\overline{\mathbb{Z}_p}$-modules, see [He23, 5.7]) due to the commutativity of the diagram (6.6.2).

If X is the spectrum of an absolutely integrally closed valuation ring A and if $Y = \overline{\eta}\times_{\overline{S}} X$, then applying the functor $\mathrm{R}\Gamma(Y\to X, -)$ on (6.6.2) we obtain the natural morphisms in the derived category $\mathbf{D}(A\text{-}\mathbf{Mod})$ by 3.7 making the following diagram commutative.

https://doi.org/10.4153/S0008414X24000051 Published online by Cambridge University Press

(6.6.6)

$$\begin{tikzcd} \mathrm{R}\Gamma(Y'_{\text{ét}},\psi'^{-1}\mathbb{L}')\otimes^{\mathrm{L}}_{\mathbb{Z}}A \arrow[d, "\alpha_3"', "\wr"] & \mathrm{R}\Gamma(\mathbf{E}^{\text{ét}}_{Y'\to X'},\mathbb{L}')\otimes^{\mathrm{L}}_{\mathbb{Z}}A \arrow[l, "\alpha_1"'] \arrow[r, "\alpha_2"] \arrow[d] & \mathrm{R}\Gamma(\mathbf{E}^{\text{ét}}_{Y'\to X'},\mathbb{L}'\otimes_{\mathbb{Z}}\overline{\mathscr{B}}') \arrow[d, "\alpha_4"] \\ \mathrm{R}\Gamma((\mathbf{Sch}^{\mathrm{coh}}_{/Y'})_{\mathrm{v}},\Psi'^{-1}\varepsilon'^{-1}\mathbb{L}')\otimes^{\mathrm{L}}_{\mathbb{Z}}A & \mathrm{R}\Gamma(\mathbf{I}_{Y'\to X'^{Y'}},\varepsilon'^{-1}\mathbb{L}')\otimes^{\mathrm{L}}_{\mathbb{Z}}A \arrow[l, "\sim", "\alpha_5"'] \arrow[r, "\alpha_6"'] & \mathrm{R}\Gamma(\mathbf{I}_{Y'\to X'^{Y'}},\varepsilon'^{-1}\mathbb{L}'\otimes_{\mathbb{Z}}\mathscr{O}') \end{tikzcd}$$

The arrows α_3, α_5, α_6, and α_4 induce an arrow

(6.6.7) $$\alpha_4^{-1}\circ\alpha_6\circ\alpha_5^{-1}\circ\alpha_3 : \mathrm{R}\Gamma(Y'_{\text{ét}},\mathbb{F}')\otimes^{\mathrm{L}}_{\mathbb{Z}}A \longrightarrow \mathrm{R}\Gamma(\mathbf{E}^{\text{ét}}_{Y'\to X'},\psi'_*\mathbb{F}'\otimes_{\mathbb{Z}}\overline{\mathscr{B}}')$$

in the derived category $\mathbf{D}(\overline{\mathbb{Z}_p}^{\mathrm{al}}\text{-}\mathbf{Mod})$ of almost $\overline{\mathbb{Z}_p}$-modules [He23, 5.7]. We also call (6.6.7) the *absolute Faltings' comparison morphism.*

Lemma 6.7 *With the notation in 6.3, let $\mathbb{F}'$ be a finite locally constant abelian sheaf on $Y'_{\text{ét}}$ and we set $\mathscr{F}'=\Psi'_*a'^{-1}\mathbb{F}'$. Assume that $X'\to X$ is proper of finite presentation. Then, the canonical morphism*

(6.7.1) $$(\mathrm{R}f_{\mathbf{I}*}\mathscr{F}')\otimes^{\mathrm{L}}_{\mathbb{Z}}\mathscr{O}\longrightarrow \mathrm{R}f_{\mathbf{I}*}(\mathscr{F}'\otimes_{\mathbb{Z}}\mathscr{O}')$$

is an almost isomorphism.

Proof Following 6.5, consider the following presheaves on $\mathbf{I}_{Y\to X^Y}$ for each integer q:

(6.7.2) $$\mathcal{H}_1^q : \widetilde{X}\longmapsto H^q_{\text{ét}}(\widetilde{Y}',\widetilde{\mathbb{F}}')\otimes_{\mathbb{Z}}\overline{\mathscr{B}}(\widetilde{Y}\to\widetilde{X}),$$

(6.7.3) $$\mathcal{H}_2^q : \widetilde{X}\longmapsto H^q(\mathbf{E}^{\text{ét}}_{\widetilde{Y}'\to\widetilde{X}'},\widetilde{\psi}'_*\widetilde{\mathbb{F}}')\otimes_{\mathbb{Z}}\overline{\mathscr{B}}(\widetilde{Y}\to\widetilde{X}),$$

(6.7.4) $$\mathcal{H}_3^q : \widetilde{X}\longmapsto H^q(\mathbf{E}^{\text{ét}}_{\widetilde{Y}'\to\widetilde{X}'},\widetilde{\psi}'_*\widetilde{\mathbb{F}}'\otimes_{\mathbb{Z}}\overline{\mathscr{B}}').$$

They satisfy the limit-preserving condition [He23, 3.25(ii)] by [He23, 7.12], [SGA 4_{II}, VII.5.6], and [SGA 4_{II}, VI 8.5.9, 8.7.3]. Moreover, if $\widetilde{X}=\mathrm{Spec}(A)$, where A is an absolutely integrally closed valuation ring with p nonzero in A, then the canonical morphisms

(6.7.5) $$\mathcal{H}_1^q(\mathrm{Spec}(A))\leftarrow\mathcal{H}_2^q(\mathrm{Spec}(A))\rightarrow\mathcal{H}_3^q(\mathrm{Spec}(A))$$

are, respectively, an isomorphism and an almost isomorphism by 5.17. Thus, the canonical morphisms $\mathcal{H}_1^q\leftarrow\mathcal{H}_2^q\rightarrow\mathcal{H}_3^q$ induce an isomorphism and an almost isomorphism of their sheafifications by [He23, 3.25]. The conclusion follows from 6.5. ■

Lemma 6.8 *Let $Y\to X$ be an open immersion of coherent schemes, $Y'\to Y$ a finite morphism of finite presentation. Then, there exists a finite morphism $X'\to X$ of finite presentation whose base change by $Y\to X$ is $Y'\to Y$.*

Proof Firstly, assume that X is Noetherian. We have $Y'=Y\times_X X^Y$ by [He23, 3.19(4)]. We write $X^Y=\mathrm{Spec}_X(\mathcal{A})$ where $\mathcal{A}$ is an integral quasi-coherent $\mathcal{O}_X$-algebra on X, and we write $\mathcal{A}$ as a filtered colimit of its finite quasi-coherent $\mathcal{O}_X$-subalgebras $\mathcal{A}=\operatorname{colim}\mathcal{A}_\alpha$ [Sta23, 0817]. Let $\mathcal{B}_\alpha$ be the restriction of $\mathcal{A}_\alpha$ to Y. Then, $\mathcal{B}=\operatorname{colim}\mathcal{B}_\alpha$ is a filtered colimit of finite quasi-coherent $\mathcal{O}_Y$-algebras with injective transition morphisms. Since $Y'=\mathrm{Spec}_Y(\mathcal{B})$ is finite over Y, there exists an index α_0 such that $Y'=\mathrm{Spec}_Y(\mathcal{B}_{\alpha_0})$. Therefore, $X'=\mathrm{Spec}_X(\mathcal{A}_{\alpha_0})$ meets our requirements.

https://doi.org/10.4153/S0008414X24000051 Published online by Cambridge University Press

In general, we write X as a cofiltered limit of coherent schemes of finite type over $\mathbb{Z}$ with affine transition morphisms $X = \lim_{\lambda\in\Lambda} X_\lambda$ [Sta23, 01ZA]. Since $Y \to X$ is an open immersion of finite presentation, using [EGA IV$_3$, 8.8.2, 8.10.5] there exists an index $\lambda_0 \in \Lambda$, an open immersion $Y_{\lambda_0} \to X_{\lambda_0}$ and a finite morphism $Y'_{\lambda_0} \to Y_{\lambda_0}$ such that the base change of the morphisms $Y'_{\lambda_0} \to Y_{\lambda_0} \to X_{\lambda_0}$ by $X \to X_{\lambda_0}$ are the morphisms $Y' \to Y \to X$. By the first paragraph, there exists a finite morphism $X'_{\lambda_0} \to X_{\lambda_0}$ of finite presentation such that $Y'_{\lambda_0} = Y_{\lambda_0} \times_{X_{\lambda_0}} X'_{\lambda_0}$. We see that the base change $X' \to X$ of $X'_{\lambda_0} \to X_{\lambda_0}$ by $X \to X_{\lambda_0}$ meets our requirements. ■

Lemma 6.9 *With the notation in 6.3, let $g : Y'' \to Y'$ be a finite morphism of finite presentation, let $\mathbb{F}''$ be a finite locally constant abelian sheaf on $Y''_{\text{ét}}$ and we set $\mathscr{F}' = \Psi'_* a'^{-1}(g_{\text{ét}*}\mathbb{F}'')$. Assume that $X' \to X$ is proper of finite presentation. Then, the canonical morphism*

$$(6.9.1)\qquad (\mathrm{R}f_{\mathbf{I}*}\mathscr{F}') \otimes^{\mathrm{L}}_{\mathbb{Z}} \mathscr{O} \longrightarrow \mathrm{R}f_{\mathbf{I}*}(\mathscr{F}' \otimes_{\mathbb{Z}} \mathscr{O}')$$

is an almost isomorphism.

Proof There exists a Cartesian morphism $g : (Y'' \to X'') \to (Y' \to X^Y \times_X X')$ of open immersions of coherent schemes such that $X'' \to X^Y \times_X X'$ is finite and of finite presentation by 6.8. Consider the diagram (6.3.1) associated with g:

$$(6.9.2)\qquad \begin{tikzcd} Y''_{\text{ét}} \arrow[d, "g_{\text{ét}}"'] & (\mathbf{Sch}^{\text{coh}}_{/Y''})_{\text{v}} \arrow[l, "a''"'] \arrow[r, "\Psi''"] \arrow[d, "g_{\text{v}}"] & \mathbf{I}_{Y''\to X''^{Y''}} \arrow[d, "g_{\mathbf{I}}"] \\ Y'_{\text{ét}} & (\mathbf{Sch}^{\text{coh}}_{/Y'})_{\text{v}} \arrow[l, "a'"'] \arrow[r, "\Psi'"] & \mathbf{I}_{Y'\to X'^{Y'}} \end{tikzcd}$$

We set $\mathscr{G}'' = \Psi''_* a''^{-1}\mathbb{F}''$. As $g : Y'' \to Y'$ is finite, the base change morphism $a'^{-1}g_{\text{ét}*} \to g_{\text{v}*}a''^{-1}$ induces a canonical isomorphism $\mathscr{F}' \xrightarrow{\sim} g_{\mathbf{I}*}\mathscr{G}''$ by [He23, 3.10]. Moreover, the canonical morphism $g_{\mathbf{I}*}\mathscr{G}'' \to \mathrm{R}g_{\mathbf{I}*}\mathscr{G}''$ is an isomorphism by 6.5(1) and [He23, 3.25], since $g : Y'' \to Y'$ is finite [SGA 4$_{II}$, VIII.5.6]. By applying 6.7 to g and $\mathbb{F}''$, the canonical morphism

$$(6.9.3)\qquad (\mathrm{R}g_{\mathbf{I}*}\mathscr{G}'') \otimes^{\mathrm{L}}_{\mathbb{Z}} \mathscr{O}' \longrightarrow \mathrm{R}g_{\mathbf{I}*}(\mathscr{G}'' \otimes_{\mathbb{Z}} \mathscr{O}'')$$

is an almost isomorphism. Let h be the composition of $(Y'' \to X'') \to (Y' \to X^Y \times_X X') \to (Y \to X^Y)$. Note that $X'' \to X^Y$ is also proper of finite presentation. By applying 6.7 to h and $\mathbb{F}''$, the canonical morphism

$$(6.9.4)\qquad (\mathrm{R}h_{\mathbf{I}*}\mathscr{G}'') \otimes^{\mathrm{L}}_{\mathbb{Z}} \mathscr{O} \longrightarrow \mathrm{R}h_{\mathbf{I}*}(\mathscr{G}'' \otimes_{\mathbb{Z}} \mathscr{O}'')$$

is an almost isomorphism. It is clear that $h_{\mathbf{I}} = f_{\mathbf{I}} \circ g_{\mathbf{I}}$. The conclusion follows from the canonical isomorphism $\mathscr{F}' \to \mathrm{R}g_{\mathbf{I}*}\mathscr{G}''$ and the canonical almost isomorphisms (6.9.3) and (6.9.4). ■

Lemma 6.10 *With the notation in 6.3, let $\mathcal{F}'$ be a constructible abelian sheaf on $Y'_{\text{ét}}$ and we set $\mathscr{F}' = \Psi'_* a'^{-1}\mathcal{F}'$. Assume that $X' \to X$ is proper of finite presentation. Then, the canonical morphism*

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$$(6.10.1)\qquad (\mathrm{R}f_{\mathbf{I}*}\mathscr{F}')\otimes^{\mathrm{L}}_{\mathbb{Z}}\mathscr{O}\longrightarrow \mathrm{R}f_{\mathbf{I}*}(\mathscr{F}'\otimes_{\mathbb{Z}}\mathscr{O}')$$

is an almost isomorphism.

Proof We prove by induction on an integer q that the canonical morphism $(\mathrm{R}^q f_{\mathbf{I}*}\mathscr{F}')\otimes_{\mathbb{Z}}\mathscr{O}\to \mathrm{R}^q f_{\mathbf{I}*}(\mathscr{F}'\otimes_{\mathbb{Z}}\mathscr{O}')$ is an almost isomorphism. It holds trivially for each $q\leq -1$. Notice that there exists a finite morphism $g: Y''\to Y'$ of finite presentation, a finite locally constant abelian sheaf $\mathbb{F}''$ on $Y''_{\text{ét}}$ and an injective morphism $\mathcal{F}'\to g_{\text{ét}*}\mathbb{F}''$ by [Sta23, 09Z7] (cf. [SGA 4_{III}, IX.2.14]). Let $\mathcal{G}'$ be the quotient of $\mathcal{F}'\to g_{\text{ét}*}\mathbb{F}''$, which is also a constructible abelian sheaf on $Y'_{\text{ét}}$ since $g_{\text{ét}*}\mathbb{F}''$ is so [Sta23, 095R, 03RZ]. The exact sequence $0\to\mathcal{F}'\to g_{\text{ét}*}\mathbb{F}''\to\mathcal{G}'\to 0$ induces an exact sequence by 3.8(1),

$$(6.10.2)\qquad 0\longrightarrow \Psi'_*a'^{-1}\mathcal{F}'\longrightarrow \Psi'_*a'^{-1}(g_{\text{ét}*}\mathbb{F}'')\longrightarrow \Psi'_*a'^{-1}\mathcal{G}'\longrightarrow 0.$$

We set $\mathscr{H}'=\Psi'_*a'^{-1}(g_{\text{ét}*}\mathbb{F}'')$ and $\mathscr{G}'=\Psi'_*a'^{-1}\mathcal{G}'$. Then, we obtain a morphism of long exact sequences:

(6.10.3)

$$\begin{array}{ccccccccc}
(\mathrm{R}^{q-1}f_{\mathbf{I}*}\mathscr{H}')\otimes\mathscr{O} & \to & (\mathrm{R}^{q-1}f_{\mathbf{I}*}\mathscr{G}')\otimes\mathscr{O} & \to & (\mathrm{R}^{q}f_{\mathbf{I}*}\mathscr{F}')\otimes\mathscr{O} & \to & (\mathrm{R}^{q}f_{\mathbf{I}*}\mathscr{H}')\otimes\mathscr{O} & \to & (\mathrm{R}^{q}f_{\mathbf{I}*}\mathscr{G}')\otimes\mathscr{O}\\
\downarrow{\scriptstyle \gamma_1} & & \downarrow{\scriptstyle \gamma_2} & & \downarrow{\scriptstyle \gamma_3} & & \downarrow{\scriptstyle \gamma_4} & & \downarrow{\scriptstyle \gamma_5}\\
\mathrm{R}^{q-1}f_{\mathbf{I}*}(\mathscr{H}'\otimes\mathscr{O}') & \to & \mathrm{R}^{q-1}f_{\mathbf{I}*}(\mathscr{G}'\otimes\mathscr{O}') & \to & \mathrm{R}^{q}f_{\mathbf{I}*}(\mathscr{F}'\otimes\mathscr{O}') & \to & \mathrm{R}^{q}f_{\mathbf{I}*}(\mathscr{H}'\otimes\mathscr{O}') & \to & \mathrm{R}^{q}f_{\mathbf{I}*}(\mathscr{G}'\otimes\mathscr{O}')
\end{array}$$

Notice that γ_1 and γ_2 are almost isomorphisms by induction, and that γ_4 is an almost isomorphism by 6.9. Thus, applying the 5-lemma [Sta23, 05QA] in the abelian category of almost $\overline{\mathbb{Z}_p}$-modules over $\mathbf{I}_{Y\to X^Y}$, we see that γ_3 is almost injective. Since $\mathcal{F}'$ is an arbitrary constructible abelian sheaf, the morphism γ_5 is also almost injective. Thus, γ_3 is an almost isomorphism. ∎

Theorem 6.11 *With the notation in 6.3, let $\mathcal{F}'$ be a torsion abelian sheaf on $Y'_{\text{ét}}$ and we set $\mathscr{F}'=\Psi'_*a'^{-1}\mathcal{F}'$. Assume that $X'\to X$ is proper of finite presentation. Then, the canonical morphism*

$$(6.11.1)\qquad (\mathrm{R}f_{\mathbf{I}*}\mathscr{F}')\otimes^{\mathrm{L}}_{\mathbb{Z}}\mathscr{O}\longrightarrow \mathrm{R}f_{\mathbf{I}*}(\mathscr{F}'\otimes_{\mathbb{Z}}\mathscr{O}')$$

is an almost isomorphism in the derived category $\mathbf{D}(\overline{\mathbb{Z}_p}\text{-}\mathbf{Mod}_{\mathbf{I}_{Y\to X^Y}})$ [He23, 5.7].

Proof We write $\mathcal{F}'$ as a filtered colimit of constructible abelian sheaves $\mathcal{F}'=\mathrm{colim}_{\lambda\in\Lambda}\mathcal{F}'_\lambda$ ([Sta23, 03SA], cf. [SGA 4_{III}, IX.2.7.2]). We set $\mathscr{F}'_\lambda=\Psi'_*a'^{-1}\mathcal{F}_\lambda$. We have $\mathscr{F}'=\mathrm{colim}_{\lambda\in\Lambda}\mathscr{F}'_\lambda$ by [SGA 4_{II}, VI.5.1] whose conditions are satisfied since each object in each concerned site is quasi-compact. Moreover, for each integer q, we have

$$(6.11.2)\qquad (\mathrm{R}^q f_{\mathbf{I}*}\mathscr{F}')\otimes_{\mathbb{Z}}\mathscr{O}=\operatorname*{colim}_{\lambda\in\Lambda}(\mathrm{R}^q f_{\mathbf{I}*}\mathscr{F}'_\lambda)\otimes_{\mathbb{Z}}\mathscr{O},$$

$$(6.11.3)\qquad \mathrm{R}^q f_{\mathbf{I}*}(\mathscr{F}'\otimes_{\mathbb{Z}}\mathscr{O}')=\operatorname*{colim}_{\lambda\in\Lambda}\mathrm{R}^q f_{\mathbf{I}*}(\mathscr{F}'_\lambda\otimes_{\mathbb{Z}}\mathscr{O}').$$

The conclusion follows from 6.10. ∎

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Corollary 6.12 *Let $\overline{\mathbb{Z}_p}$ be the integral closure of $\mathbb{Z}_p$ in an algebraic closure of $\mathbb{Q}_p$, let A be a $\overline{\mathbb{Z}_p}$-algebra which is an absolutely integrally closed valuation ring, let X be a proper A-scheme of finite presentation, $Y = \mathrm{Spec}(A[1/p]) \times_{\mathrm{Spec}(A)} X$, and let $\mathcal{F}$ be a torsion abelian sheaf on $Y_{\text{ét}}$ and we set $\mathscr{F} = \Psi_* a^{-1}\mathcal{F}$ with the notation in 3.6. Then, there is a canonical isomorphism*

$$\mathrm{R}\Gamma(Y_{\text{ét}}, \mathcal{F}) \xrightarrow{\sim} \mathrm{R}\Gamma(\mathbf{I}_{Y\to X^Y}, \mathscr{F}), \tag{6.12.1}$$

and the canonical morphism

$$\mathrm{R}\Gamma(\mathbf{I}_{Y\to X^Y}, \mathscr{F}) \otimes^{\mathrm{L}}_{\mathbb{Z}} A \longrightarrow \mathrm{R}\Gamma(\mathbf{I}_{Y\to X^Y}, \mathscr{F} \otimes_{\mathbb{Z}} \mathscr{O}) \tag{6.12.2}$$

is an almost isomorphism in the derived category $\mathbf{D}(\overline{\mathbb{Z}_p}\text{-}\mathbf{Mod}_{\mathbf{I}_{Y\to X^Y}})$ [He23, 5.7].

Proof The first assertion follows from the canonical isomorphisms $\mathrm{R}\Gamma(Y_{\text{ét}}, \mathcal{F}) \xrightarrow{\sim} \mathrm{R}\Gamma((\mathbf{Sch}^{\mathrm{coh}}_{/Y})_{\mathrm{v}}, a^{-1}\mathcal{F}) = \mathrm{R}\Gamma(\mathbf{I}_{Y\to X^Y}, \mathrm{R}\Psi_* a^{-1}\mathcal{F}) \xleftarrow{\sim} \mathrm{R}\Gamma(\mathbf{I}_{Y\to X^Y}, \Psi_* a^{-1}\mathcal{F})$ by 3.4 and 3.8(1) (cf. 6.13(1)). The second assertion follows from applying 6.11 to the morphism $X \to \mathrm{Spec}(A)$ and from the fact that the associated topos of $\mathbf{I}_{\mathrm{Spec}(A[1/p])\to\mathrm{Spec}(A)}$ is local (3.7). ∎

Lemma 6.13 (cf. 6.5(1)) *With the notation in 6.3 and 6.4, let $\mathcal{F}'$ be a torsion abelian sheaf on $Y'_{\text{ét}}$, $\mathcal{H} = \mathrm{R}f_{\text{ét}*}\mathcal{F}'$, and we set $\mathscr{F}' = \Psi'_* a'^{-1}\mathcal{F}'$, $\mathscr{H} = \mathrm{R}\Psi_* a^{-1}\mathcal{H}$. Let $\widetilde{X}$ be an object of $\mathbf{I}_{Y\to X^Y}$, $\widetilde{Y} = \overline{\eta} \times_{\overline{S}} \widetilde{X}$, $\widetilde{\mathcal{F}'} = g'^{-1}_{\text{ét}}\mathcal{F}'$.*

1. *The sheaf $\mathrm{R}^q f_{\mathbf{I}*}\mathscr{F}'$ is canonically isomorphic to the presheaf $\widetilde{X} \mapsto H^q_{\text{ét}}(\widetilde{Y}', \widetilde{\mathcal{F}'})$ for each integer q.*
2. *If $Y' \to Y$ is proper, then there exists a canonical isomorphism $\mathscr{H} \xrightarrow{\sim} \mathrm{R}f_{\mathbf{I}*}\mathscr{F}'$.*

Proof Note that the canonical morphism $\mathscr{F}' \to \mathrm{R}\Psi'_* a'^{-1}\mathcal{F}'$ is an isomorphism by 3.8(1). Thus, $\mathrm{R}f_{\mathbf{I}*}\mathscr{F}' = \mathrm{R}(\Psi \circ f_{\mathrm{v}})_* a'^{-1}\mathcal{F}'$, whose qth cohomology is the sheaf associated with the presheaf $\widetilde{X} \mapsto H^q_{\mathrm{v}}(\widetilde{Y}', \widetilde{a'}^{-1}\widetilde{\mathcal{F}'}) = H^q_{\text{ét}}(\widetilde{Y}', \widetilde{\mathcal{F}'})$ by 3.4, and thus (1) follows. If $Y' \to Y$ is proper, then the base change morphism $a^{-1}\mathrm{R}f_{\text{ét}*} \to \mathrm{R}f_{\mathrm{v}*}a'^{-1}$ induces an isomorphism $a^{-1}\mathcal{H} \xrightarrow{\sim} \mathrm{R}f_{\mathrm{v}*}a'^{-1}\mathcal{F}'$ by [He23, 3.10], and thus (2) follows. ∎

Theorem 6.14 *With the notation in 6.3, let $\mathbb{F}'$ be a finite locally constant abelian sheaf on $Y'_{\text{ét}}$. Assume that:*

- (i) *the morphism $X' \to X$ is proper of finite presentation, and that*
- (ii) *the sheaf $\mathrm{R}^q f_{\text{ét}*}\mathbb{F}'$ is finite locally constant for each integer q and nonzero for finitely many q, and that*
- (iii) *we have $\mathrm{R}^q \psi_* \mathbb{H} = 0$ (resp. $\mathrm{R}^q \psi'_* \mathbb{H} = 0$) for any finite locally constant abelian sheaf $\mathbb{H}$ on $Y_{\text{ét}}$ (resp. $Y'_{\text{ét}}$) and any integer $q > 0$.*

Then, the relative Faltings' comparison morphism associated with f and $\mathbb{F}'$ (5.6.1) (which exists by (iii)) is an almost isomorphism in the derived category $\mathbf{D}(\overline{\mathbb{Z}_p}\text{-}\mathbf{Mod}_{\mathbf{E}^{\text{ét}}_{Y\to X}})$ [He23, 5.7], and it induces an almost isomorphism

$$(\psi_* \mathrm{R}^q f_{\text{ét}*}\mathbb{F}') \otimes_{\mathbb{Z}} \overline{\mathscr{B}} \longrightarrow \mathrm{R}^q f_{\mathbf{E}*}(\psi'_* \mathbb{F}' \otimes_{\mathbb{Z}} \overline{\mathscr{B}}') \tag{6.14.1}$$

of $\overline{\mathbb{Z}_p}$-modules for each integer q.

https://doi.org/10.4153/S0008414X24000051 Published online by Cambridge University Press

Proof We follow the discussion of 6.6 and set $\mathscr{F}' = \Psi'_* a'^{-1}\mathbb{F}'$. The canonical morphism (6.6.4)

$$\mathrm{R}\varepsilon_*((\mathrm{R}f_{\mathbf{I}*}\mathscr{F}') \otimes^{\mathrm{L}}_{\mathbb{Z}} \mathscr{O}) \longrightarrow \mathrm{R}\varepsilon_*\mathrm{R}f_{\mathbf{I}*}(\mathscr{F}' \otimes_{\mathbb{Z}} \mathscr{O}') \tag{6.14.2}$$

is an almost isomorphism by 6.7. It remains to show that the canonical morphism (6.6.3)

$$\mathrm{R}\varepsilon_*(\mathrm{R}f_{\mathbf{I}*}\mathscr{F}') \otimes^{\mathrm{L}}_{\mathbb{Z}} \overline{\mathscr{B}} \longrightarrow \mathrm{R}\varepsilon_*((\mathrm{R}f_{\mathbf{I}*}\mathscr{F}') \otimes^{\mathrm{L}}_{\mathbb{Z}} \mathscr{O}) \tag{6.14.3}$$

is also an almost isomorphism. With the notation in 6.13 by taking $\mathcal{F}' = \mathbb{F}'$, the complex $\mathcal{H}$ is a bounded complex whose cohomologies $H^q(\mathcal{H})$ are finite locally constant abelian sheaves by condition (ii). Consider the commutative diagram (3.6.4):

$$\begin{tikzcd} (\mathbf{Sch}^{\mathrm{coh}}_{/Y})_{\mathrm{v}} \arrow[r, "a"] \arrow[d, "\Psi"'] & Y_{\text{ét}} \arrow[d, "\psi"] \\ \mathbf{I}_{Y\to X^Y} \arrow[r, "\varepsilon"] & \mathbf{E}^{\text{ét}}_{Y\to X} \end{tikzcd} \tag{6.14.4}$$

We set $\mathcal{L} = \mathrm{R}\psi_*\mathcal{H}$. Then, $H^q(\mathcal{L}) = \psi_* H^q(\mathcal{H})$ by Cartan–Leray spectral sequence and condition (iii). Hence, $\mathcal{L}$ is a bounded complex of abelian sheaves whose cohomologies are finite locally constant by 5.3 so that the canonical morphism

$$\mathcal{L} \otimes^{\mathrm{L}}_{\mathbb{Z}} \overline{\mathscr{B}} \longrightarrow \mathrm{R}\varepsilon_*(\varepsilon^{-1}\mathcal{L} \otimes^{\mathrm{L}}_{\mathbb{Z}} \mathscr{O}) \tag{6.14.5}$$

is an almost isomorphism by 3.9 (cf. [He23, 8.15]).

On the other hand, $H^q(\mathscr{H}) = \Psi_* a^{-1} H^q(\mathcal{H})$ by Cartan–Leray spectral sequence and 3.8(1). Thus, the base change morphism $\varepsilon^{-1}\mathrm{R}\psi_* \to \mathrm{R}\Psi_* a^{-1}$ induces an isomorphism $\varepsilon^{-1}\mathcal{L} \xrightarrow{\sim} \mathscr{H}$ by 6.2. Moreover, the canonical morphism $\mathcal{L} \to \mathrm{R}\varepsilon_*\varepsilon^{-1}\mathcal{L} = \mathrm{R}\varepsilon_*\mathscr{H} = \mathrm{R}\psi_*\mathrm{R}a_* a^{-1}\mathcal{H}$ is an isomorphism by 3.4. Thus, the canonical morphism

$$(\mathrm{R}\varepsilon_*\varepsilon^{-1}\mathcal{L}) \otimes^{\mathrm{L}}_{\mathbb{Z}} \overline{\mathscr{B}} \longrightarrow \mathrm{R}\varepsilon_*(\varepsilon^{-1}\mathcal{L} \otimes_{\mathbb{Z}} \mathscr{O}) \tag{6.14.6}$$

is an almost isomorphism by (6.14.5). In conclusion, (6.14.3) is an almost isomorphism by (6.14.6) and by the canonical isomorphisms $\varepsilon^{-1}\mathcal{L} \xrightarrow{\sim} \mathscr{H} \xrightarrow{\sim} \mathrm{R}f_{\mathbf{I}*}\mathscr{F}'$. ■

Remark 6.15 We give two concrete situations where the conditions in 6.14 are satisfied:

(1) Let $\overline{\mathbb{Z}_p}$ be the integral closure of $\mathbb{Z}_p$ in an algebraic closure $\overline{\mathbb{Q}_p}$ of $\mathbb{Q}_p$, let $X' \to X$ be a proper and finitely presented morphism of coherent $\overline{\mathbb{Z}_p}$-schemes, and let $Y' \to Y$ be the base change of $X' \to X$ by $\mathrm{Spec}(\overline{\mathbb{Q}_p}) \to \mathrm{Spec}(\overline{\mathbb{Z}_p})$. Assume that $Y' \to Y$ is smooth. Then, the condition (ii) is guaranteed by [SGA 4_{III}, XVI.2.2 and XVII.5.2.8.1], and the condition (iii) is guaranteed by 5.8.

(2) Let $\mathcal{O}_K$ be a strictly Henselian discrete valuation ring with fraction field K of characteristic 0 and residue field of characteristic p, let $\overline{K}$ be an algebraic closure of K, let $X' \to X$ be a proper morphism of $\mathcal{O}_K$-schemes of finite type, and let $Y' \to Y$ be the base change of $X' \to X$ by $\mathrm{Spec}(\overline{K}) \to \mathrm{Spec}(\mathcal{O}_K)$. Assume that $Y' \to Y$ is smooth. Then, the condition (ii) is guaranteed by [SGA 4_{III}, XVI.2.2 and XVII.5.2.8.1], and the condition (iii) is guaranteed by 5.7.

https://doi.org/10.4153/S0008414X24000051 Published online by Cambridge University Press

Acknowledgments This work is part of my thesis prepared at Université Paris-Saclay and Institut des Hautes Études Scientifiques. I would like to express my sincere gratitude to my doctoral supervisor, Ahmed Abbes, for his guidance to this project, his thorough review of this work, and his plenty of helpful suggestions on both research and writing.

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