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# ON COMPLEX HOMOGENEOUS SINGULARITIES

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**Abstract**

We study the singularity at the origin of $\mathbb{C}^{n+1}$ of an arbitrary homogeneous polynomial in $n+1$ variables with complex coefficients, by investigating the monodromy characteristic polynomials $\Delta_l(t)$ as well as the relation between the monodromy zeta function and the Hodge spectrum of the singularity. In the case $n = 2$, we give a description of $\Delta_C(t) = \Delta_1(t)$ in terms of the multiplier ideal.

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## 1. Introduction

In this paper we extend the work of the first author in [16].

Let $f$ be a homogeneous polynomial (not necessarily reduced) of degree $d$ in $n+1$ variables with coefficients in $\mathbb{C}$, which defines a holomorphic function germ at the origin $O$ of $\mathbb{C}^{n+1}$. In general, according to [20] and [15], the Milnor fibre of the hypersurface germ $(f = 0, O)$ is up to diffeomorphism a manifold $M = f^{-1}(\delta) \cap B_\varepsilon$, for $B_\varepsilon \subseteq \mathbb{C}^{n+1}$ the ball of radius $\varepsilon$ around $O$ and $0 < \delta \ll \varepsilon \ll 1$. Since $f$ is a homogeneous polynomial, $f^{-1}(\delta) \cap B_\varepsilon$ is a deformation retract of $f^{-1}(\delta) \cong f^{-1}(1)$, and we may consider $M$ as $f^{-1}(1)$. The monodromy

$$T^* : H^*(M, \mathbb{C}) \to H^*(M, \mathbb{C})$$

of the singularity is given explicitly by the $\mathbb{C}$-linear endomorphism induced by the map

$$T : M \to M, \quad (x_0, \ldots, x_n) \mapsto (e^{2\pi i/d} x_0, \ldots, e^{2\pi i/d} x_n).$$

When $f$ is an isolated homogeneous singularity, several invariants such as the Milnor number, the characteristic polynomials of $T^*$, the signature and Hodge numbers of $M$ can be computed by classical topological and algebraic methods as well as via mixed Hodge structures (see [21, 25]).

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For reduced homogeneous polynomials, Esnault [11] introduced a method to compute the Betti numbers, the rank and the signature of the intersection matrices of the singularity $(f, O)$, using mixed Hodge structures on cohomology groups of the Milnor fibre $M$ and the existence of spectral sequences converging to the cohomology groups, together with resolution of singularities. The work by Esnault inspired the study by Loeser and Vaquié [19] of the Alexander polynomial of a reduced complex projective plane curve, where they provided a formula for the Alexander polynomial which generalises the previous one by Libgober [17, 18]. The work of Libgober in [18] and Loeser and Vaquié in [19], as well as that of Nadel in [22], probably sparked the studies on multiplier ideals and local systems which were pursued by Esnault and Viehweg [12], Ein and Lazarsfeld [10], Demailly [7], Kollar [13], Budur [2, 4] and Budur and Saito [6].

Using the theory of (mixed) multiplier ideals and local systems, Budur [3] gave an explicit description of the local system of the complement in $\mathbb{P}^n$ of the divisor defined by a homogeneous polynomial $f$ *not necessarily reduced*. We use Budur's article [3] to study the characteristic polynomials, the Hodge spectrum and the monodromy zeta function of an arbitrary homogeneous hypersurface singularity.

Denote by $D$ the closed subscheme of $\mathbb{P}^n$ defined by the zero locus of a homogeneous polynomial $f$ of degree $d$ and by $U$ the complement of $D$ in $\mathbb{P}^n$. The homogeneity of $f$ gives rise to a natural action of $\mathbb{Z}/d\mathbb{Z}$ on $M$. Since this action is free we have a natural isomorphism $M/(\mathbb{Z}/d\mathbb{Z}) \cong U$, from which the quotient map $\sigma : M \to U$ is a cyclic cover of degree $d$. The automorphism $T : M \to M$ induces an obvious automorphism $\sigma_*\mathbb{C}_M \to \sigma_*\mathbb{C}_M$ of the $\mathcal{O}_U$-module sheaf $\sigma_*\mathbb{C}_M$ on $U$. From [3, 4], there is an eigensheaf decomposition of $\sigma_*\mathbb{C}_M$ into the unitary local systems $\mathcal{V}_k$ on $U$, with respect to the eigenvalues $e^{-2\pi ik/d}$ for $0 \le k \le d-1$. In the cohomology level, from the Leray spectral sequence, $H^l(U, \mathcal{V}_k)$ is the eigenspace of the monodromy $T^*|_{H^l(M,\mathbb{C})}$ with respect to the eigenvalue $e^{-2\pi ik/d}$, for any $l$ in $\mathbb{N}$ (see [3]). Assume that $D$ has $r$ distinct irreducible components $D_i$ and that, for each $i$, $m_i$ is the multiplicity of $D_i$ in $D$. By [3, Lemma 4.2], for each $k$, modulo the identification $RH$ in [4, Theorem 1.2], $\mathcal{V}_k$ is just the element $(\mathcal{O}_{\mathbb{P}^n}(\sum_{j=1}^r \{km_j/d\}d_j), (\{km_1/d\}, \dots, \{km_r/d\}))$ in the group $\mathrm{Pic}^\tau(\mathbb{P}^n, D)$ of *realisations of boundaries of* $\mathbb{P}^n$ *on* $D$ (see [4, Definition 1.1]). Here $\{\alpha\}$ denotes the fractional part of a rational number $\alpha$.

The computation of the complex dimension of $H^l(U, \mathcal{V}_k)$ can be solved completely using the work of Budur [2–5] in terms of resolution of singularities. Let $\pi : Y \to \mathbb{P}^n$ be a log-resolution of $D$, with normal crossing divisor $E$. We write

$$\mathcal{L}^{(k)} := \pi^*\mathcal{O}_{\mathbb{P}^n}\Big(\sum_{j=1}^r \Big\{\frac{km_j}{d}\Big\}d_j\Big) \otimes \mathcal{O}_Y\Big(-\Big\lfloor \sum_{j=1}^r \Big\{\frac{km_j}{d}\Big\}\pi^*D_j\Big\rfloor\Big),$$

an invertible sheaf on $Y$. As proved in Lemma 3.9, for $l \ge 0$ and $1 \le k \le d$,

$$\dim_{\mathbb{C}} H^l(U, \mathcal{V}_{d-k}) = \sum_{p\ge 0} \dim_{\mathbb{C}} H^{l-p}(Y, \Omega_Y^p(\log E) \otimes \mathcal{L}^{(k)^{-1}}),$$

from which the characteristic polynomial $\Delta_l(t)$ of $T^*|_{H^l(M,\mathbb{C})}$ follows.

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In particular, for $n = 2$ and $l = 1$, we give a description of $\Delta_C(t) = \Delta_1(t)$ via the multiplier ideal of $\sum_{j=1}^{r}\{km_j/d\}C_j$, where we write $C_j$ instead of $D_j$ when $D$ is a curve $C$. Let $m$ be the greatest common divisor of $m_1, \ldots, m_r$. By Remark 3.6, the set $\{[k] \in \mathbb{Z}/d\mathbb{Z} \mid d \text{ divides } km_j \text{ for every } j\}$ is a subgroup of $\mathbb{Z}/d\mathbb{Z}$, whose quotient is denoted by $G$. Identifying $k \in [0, d-1] \cap \mathbb{Z}$ with its class in $G$ gives the next result.

Theorem 1.1 (Theorem 4.3). *With the notation introduced above,*

$$\Delta_C(t) = (t^m - 1)^{r-1} \prod_{k \in G\setminus\{0\}} \Big(t^{2m} - 2t^m \cos\frac{2km\pi}{d} + 1\Big)^{\ell_k},$$

*where*

$$\ell_k := \dim_{\mathbb{C}} H^1\Big(\mathbb{P}^2, \mathcal{J}\Big(\mathbb{P}^2, \sum_{j=1}^{r}\Big\{\frac{km_j}{d}\Big\}C_j\Big)\Big(\sum_{j=1}^{r}\Big\{\frac{km_j}{d}\Big\}d_j - 3\Big)\Big).$$

Further, Theorem 4.4 discusses the relation between the Hodge spectrum and the monodromy zeta function of a homogeneous singularity.

Theorem 1.2 (Theorem 4.4). *The monodromy zeta function and the Hodge spectrum of $(f, O)$ are related as follows:*

$$\zeta_{f,O}(t)^{(-1)^{n+1}} = (1 - t^m)^{1+\sum_{p=1}^{n} n_{p,O}(f)} \prod_{k\in G\setminus\{0\}} (1 - e^{2\pi ikm/d}t^m)^{\sum_{p=0}^{n} n_{(d-k)/d+p,O}(f)},$$

*where $n_{\alpha,O}(f)$ are the spectrum multiplicities of $f$ at $O$ (see (2.1)).*

This result can also be deduced from [3, Proposition 4.3] and Proposition 3.7.

## 2. Multiplier ideals and the Hodge spectrum

**2.1. Multiplier ideals.** Suppose that $X$ is a smooth complex algebraic variety and $D = \sum_{i=1}^{r} D_i$ a closed subscheme of $X$, with $D_i$ irreducible (not necessarily distinct). Let $\mathfrak{a}$ be the sheaf of ideals of definition of $D$. For any $\alpha = (\alpha_1, \ldots, \alpha_r)$ in $\mathbb{Q}^r_{>0}$, write $\alpha D$ for the effective $\mathbb{Q}$-divisor $\sum_{i=1}^{r} \alpha_i D_i$. Let $\pi : Y \to X$ be a log-resolution of $\mathfrak{a}$ (also called a log-resolution of $D$). Note that $\pi$ is also a common log-resolution of all the ideals of definition of $D_i$ for $1 \le i \le r$. Let $K_X, K_Y$ denote the canonical divisors of $X, Y$, respectively. Then the divisor $K_{Y/X} := K_Y - \pi^* K_X$ is called the canonical divisor of $\pi$. For $\alpha$ in $\mathbb{Q}^r_{>0}$ put

$$\mathcal{J}(X, \alpha D) := \pi_* \mathcal{O}_Y(K_{Y/X} - \lfloor \pi^*(\alpha D) \rfloor),$$

where $\lfloor \pi^*(\alpha D) \rfloor$ is the divisor whose coefficients are the round-downs of the corresponding coefficients of $\pi^*(\alpha D)$.

Theorem 2.1 (Lazarsfeld [14]). *For any $\alpha \in \mathbb{Q}^r_{>0}$, the sheaf of ideals $\mathcal{J}(X, \alpha D)$ is independent of the choice of $\pi$, and $R^i\pi_*\mathcal{O}_Y(K_{Y/X} - \lfloor \pi^*(\alpha D) \rfloor) = 0$ for $i \ge 1$. The sheaf of ideals $\mathcal{J}(X, \alpha D)$ is called the* (mixed) multiplier ideal *of $\alpha D$.*

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A *jumping number* of $D$ in $X$ is a rational number $\alpha \in \mathbb{Q}_{>0}$ such that $\mathcal{J}(X, \alpha D) \neq \mathcal{J}(X, (\alpha - \varepsilon)D)$ for every rational number $\varepsilon > 0$. The *log canonical threshold* $\mathrm{lct}(X, D)$ of $(X, D)$ is the smallest jumping number of $D$ in $X$. In [14], Lazarsfeld gives a formula for $\mathrm{lct}(X, D)$ in terms of the discrepancies and multiplicities of a log-resolution of $D$. To determine how a singular point affects a jumping number, Budur [2] introduces *inner jumping multiplicities*. By definition, the inner jumping multiplicity $m_{\alpha,\mathbf{p}}(D)$ of $\alpha$ at a closed point $\mathbf{p} \in D$ is the dimension of the complex vector space

$$\mathcal{K}_{\mathbf{p}}(X, \alpha D) := \mathcal{J}(X, (\alpha - \varepsilon)D)/\mathcal{J}(X, D_{\alpha,\varepsilon,\delta}),$$

for $0 < \varepsilon \ll \delta \ll 1$, where $D_{\alpha,\varepsilon,\delta}$ is the divisor whose sheaf of ideals of definition is $\mathfrak{a}^{\alpha-\varepsilon} \cdot \mathfrak{m}_{\mathbf{p}}^{\delta}$ and $\mathfrak{m}_{\mathbf{p}}$ is the ideal sheaf of $\mathbf{p}$ in $X$. If $m_{\alpha,\mathbf{p}}(D) \neq 0$, the number $\alpha$ is called an *inner jumping number* of $(X, D)$ at $\mathbf{p}$. It is proved by Budur in [2, Proposition 2.8] that if $\alpha$ is an inner jumping number of $(X, D)$ at $\mathbf{p}$, for some $\mathbf{p} \in D$, then $\alpha$ is a jumping number of $(X, D)$. Budur gives an explicit formula for the number $m_{\alpha,\mathbf{p}}(D)$. Let $\pi : Y \to X$ be a log-resolution of $D$, with $E = \pi^*(D) = \sum_{i\in A} N_i E_i$, $E_i$ irreducible components, and, for each $d \in \mathbb{N}_{>0}$, let $J_{d,\mathbf{p}} := \{i \in A \mid N_i \neq 0,\ d|N_i,\ \pi(E_i) = \mathbf{p}\}$ and $E_{d,\mathbf{p}} := \bigcup_{i\in J_{d,\mathbf{p}}} E_i$.

Proposition 2.2 (Budur [2]). *Assume $\alpha = k/d$, with $k, d$ coprime positive integers, and $0 < \varepsilon \ll 1$. Then $m_{\alpha,\mathbf{p}}(D) = \chi(Y, \mathcal{O}_{E_{d,\mathbf{p}}}(K_{Y/X} - \lfloor (1 - \varepsilon)\alpha\pi^* D \rfloor))$, where $\chi$ is the sheaf Euler characteristic.*

**2.2. Hodge spectrum.** Let $X$ be a smooth complex variety of pure dimension $n$, let $f$ be a regular function on $X$ with zero locus $D \neq \emptyset$, and let $\mathbf{p}$ be a closed point in $D_{\mathrm{red}}$. Fixing a smooth metric on $X$, we may define a closed ball $B(\mathbf{p}, \varepsilon)$ around $\mathbf{p}$ in $X$ and a punctured closed disc $D_\delta^*$ around the origin of $\mathbb{C}$. It is well known (see [20]) that, for $0 < \delta \ll \varepsilon \ll 1$, the map

$$f : B(\mathbf{p}, \varepsilon) \cap f^{-1}(D_\delta^*) \to D_\delta^*$$

is a smooth locally trivial fibration, called the Milnor fibration, whose diffeomorphism type is independent of $\varepsilon$ and $\delta$. Denote the Milnor fibre $B(\mathbf{p}, \varepsilon) \cap f^{-1}(\delta)$ by $M_{\mathbf{p}}$, the geometric monodromy by $T : M_{\mathbf{p}} \to M_{\mathbf{p}}$ and the induced map on cohomology by $T^* : H^*(M_{\mathbf{p}}, \mathbb{C}) \to H^*(M_{\mathbf{p}}, \mathbb{C})$.

Let $\mathrm{MHS}_{\mathbb{C}}^{\mathrm{mon}}$ be the abelian category of complex mixed Hodge structures endowed with an automorphism of finite order. For an object $(H, T_H)$ of $\mathrm{MHS}_{\mathbb{C}}^{\mathrm{mon}}$, define its Hodge spectrum as

$$\mathrm{Hsp}(H, T_H) := \sum_{\alpha\in\mathbb{Q}} n_\alpha t^\alpha,$$

where $n_\alpha := \dim_{\mathbb{C}} Gr_F^{\lfloor \alpha \rfloor} H_{e^{2\pi i\alpha}}$, $H_{e^{2\pi i\alpha}}$ is the eigenspace of $T_H$ with respect to the eigenvalue $e^{2\pi i\alpha}$ and $F$ is the Hodge filtration. By [24] and [23], for any $l$, $H^l(M_{\mathbf{p}}, \mathbb{C})$ carries a canonical mixed Hodge structure, which is compatible with the semisimple part $T_s^*$ of $T^*$ so that $(H^l(M_{\mathbf{p}}, \mathbb{C}), T_s^*)$ is an object of $\mathrm{MHS}_{\mathbb{C}}^{\mathrm{mon}}$. As in [8, Section 4.3] and [2, Section 3], we set

$$\mathrm{Hsp}'(f, \mathbf{p}) := \sum_{j\in\mathbb{Z}} (-1)^j \mathrm{Hsp}(\widetilde{H}^{n-1+j}(M_{\mathbf{p}}, \mathbb{C}), T_s^*),$$

where we use the reduced cohomology $\widetilde{H}$ to present the vanishing cycle sheaf cohomology, since $\widetilde{H}^l(M_{\mathbf{p}}, \mathbb{C})_{e^{2\pi i\alpha}} = H^l(M_{\mathbf{p}}, \mathbb{C})_{e^{2\pi i\alpha}}$ if $l \neq 0$ or $\alpha \notin \mathbb{Z}$, and $\widetilde{H}^0(M_{\mathbf{p}}, \mathbb{C})_1 = \operatorname{coker}(H^0(*, \mathbb{C}) \to H^0(M_{\mathbf{p}}, \mathbb{C})_1)$ (see also [6, Section 5.1]). Then the Hodge spectrum of $f$ at $\mathbf{p}$, denoted by $\mathrm{Sp}(f, \mathbf{p})$, is

$$\mathrm{Sp}(f, \mathbf{p}) = t^n \iota(\mathrm{Hsp}'(f, \mathbf{p})),$$

where $\iota$ is given by $\iota(t^\alpha) = t^{-\alpha}$. Writing $\mathrm{Sp}(f, \mathbf{p}) = \sum_{\alpha \in \mathbb{Q}} n_{\alpha,\mathbf{p}}(f) t^\alpha$, one calls the coefficients $n_{\alpha,\mathbf{p}}(f)$ the *spectrum multiplicities* of $f$ at $\mathbf{p}$. By [6, Proposition 5.2], $n_{\alpha,\mathbf{p}}(f) = 0$ if $\alpha$ is a rational number with $\alpha \leq 0$ or $\alpha \geq n$. From [3, Corollary 2.3], for $\alpha \in (0, n) \cap \mathbb{Q}$,

$$n_{\alpha,\mathbf{p}}(f) = \sum_{j \in \mathbb{Z}} (-1)^j \dim_{\mathbb{C}} Gr_F^{\lfloor n-\alpha \rfloor} H^{n-1+j}(M_{\mathbf{p}}, \mathbb{C})_{e^{-2\pi i\alpha}}. \tag{2.1}$$

Using [8, Corollary 4.3.1] and important computations on multiplier ideals, Budur found the following effective way to compute $n_{\alpha,\mathbf{p}}(f)$, for $\alpha \in (0, 1] \cap \mathbb{Q}$.

Theorem 2.3 (Budur [2]). *Let $X$ be a smooth quasi-projective complex variety and $D$ an effective divisor on $X$. Assume that $\mathbf{p}$ is a closed point of $D_{\mathrm{red}}$ and $f$ is any local equation of $D$ at $\mathbf{p}$. Then $n_{\alpha,\mathbf{p}}(f) = m_{\alpha,\mathbf{p}}(D)$ for any $\alpha \in (0, 1] \cap \mathbb{Q}$.*

## 3. Local systems and Milnor fibres of homogeneous singularities

**3.1. Local systems and normal $G$-covers.** A $\mathbb{C}$-*local system* $\mathcal{V}$ on a complex manifold is a locally constant sheaf of finite-dimensional $\mathbb{C}$-vector spaces. As mentioned in Budur [4], rank-one local systems on a complex manifold $U$ correspond to group morphisms $H_1(U) \to \mathbb{C}^*$. In this correspondence, a rank-one local system is called *unitary* if it is sent to a morphism of groups $H_1(U) \to S^1$. The constant sheaf $\mathbb{C}_U$ and any local system of rank one of finite order are simple examples of unitary local systems.

Let $X$ be a smooth complex projective variety of dimension $n$ and $f$ a regular function on $X$ whose zero divisor $D := f^{-1}(0)$ has distinct irreducible components $D_1, \ldots, D_r$. Denote $U := X \setminus D$ and write $c_1(\mathcal{L})$ for the first Chern class of a line bundle $\mathcal{L}$. We consider the group

$$\mathrm{Pic}^\tau(X, D) := \left\{ (\mathcal{L}, \alpha) \in \mathrm{Pic}(X) \times [0, 1)^r \mid c_1(\mathcal{L}) = \sum_{j=1}^r \alpha_j \langle D_j \rangle \in H^2(X, \mathbb{R}) \right\}$$

in which the operation is given by

$$(\mathcal{L}, \alpha) \cdot (\mathcal{L}', \alpha') := (\mathcal{L} \otimes \mathcal{L}' \otimes \mathcal{O}_X(-\lfloor (\alpha + \alpha') D \rfloor), \{\alpha + \alpha'\}), \tag{3.1}$$

where $\langle D_j \rangle$ is the cohomology class of $D_j$ in $H^2(X, \mathbb{R})$, $\lfloor \alpha \rfloor := (\lfloor \alpha_1 \rfloor, \ldots, \lfloor \alpha_r \rfloor)$ and $\{\alpha\} := \alpha - \lfloor \alpha \rfloor$. By [4, Theorem 1.2], there is a canonical isomorphism of groups

$$RH : \mathrm{Pic}^\tau(X, D) \cong \mathrm{Hom}(H_1(U), S^1). \tag{3.2}$$

Hence one may identify a unitary local system of rank one on $U$ with an element of $\mathrm{Pic}^\tau(X, D)$. Let $\pi : Y \to X$ be a log-resolution of $D$ and $E := Y \setminus \pi^{-1}(U)$.

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Proposition 3.1 (Budur [4, Proposition 3.3]). *The map* $\pi^*_{\mathrm{par}} : \mathrm{Pic}^\tau(X, D) \to \mathrm{Pic}^\tau(Y, E)$ *which sends* $(\mathcal{L}, \alpha)$ *to* $(\pi^*\mathcal{L} \otimes \mathcal{O}_Y(-\lfloor \beta E \rfloor), \{\beta\})$ *with* $\beta$ *defined by* $\pi^*(\alpha D) = \beta E$ *is an isomorphism of groups.*

Theorem 3.2 (Budur [5, Theorem 4.6]). *Let* $\mathcal{V}$ *be a rank-one unitary local system on* $U$ *which corresponds to* $(\mathcal{L}, \alpha) \in \mathrm{Pic}^\tau(X, D)$*. Then, for all* $p, q \in \mathbb{N}$,

$$Gr_F^p H^{p+q}(U, \mathcal{V}^\vee) = H^{n-q}(Y, \Omega_Y^p(\log E)^\vee \otimes \omega_Y \otimes \pi^*\mathcal{L} \otimes \mathcal{O}_Y(-\lfloor \pi^*(\alpha D) \rfloor))^\vee.$$

*In particular,* $Gr_F^0 H^q(U, \mathcal{V}^\vee) = H^{n-q}(X, \omega_X \otimes \mathcal{L} \otimes \mathcal{J}(X, \alpha D))^\vee$.

Let $G$ be a finite abelian group such that its dual group $G^* = \mathrm{Hom}(G, \mathbb{C}^*)$ can be embedded into $\mathrm{Pic}^\tau(X, D)$. Then, by identifying $G^*$ with a subgroup $\{(\mathcal{L}_\eta, \alpha_\eta) \mid \eta \in G^*\}$ of $\mathrm{Pic}^\tau(X, D)$, we get the following normal $G$-cover of $X$ unramified above $U$,

$$\phi : \widetilde{X} = \mathrm{Spec}_{\mathcal{O}_X}\Big(\bigoplus_{\eta \in G^*} \mathcal{L}_\eta^{-1}\Big) \to X,$$

which is a morphism of varieties induced by the $\mathcal{O}_X$-module structural morphisms $\mathcal{O}_X \to \mathcal{L}_\eta$, for all $\eta \in G^*$. The group $G$ acts on $\mathcal{L}_\eta^{-1}$ via the character $\eta$, hence it acts on the $\mathcal{O}_X$-module sheaf $\phi_*\mathcal{O}_{\widetilde{X}}$. By [4, Corollary 1.11], $\phi_*\mathcal{O}_{\widetilde{X}}$ admits an eigensheaf decomposition

$$\phi_*\mathcal{O}_{\widetilde{X}} = \bigoplus_{\eta \in G^*} \mathcal{L}_\eta^{-1}, \tag{3.3}$$

where the eigensheaf $\mathcal{L}_\eta^{-1}$ is with respect to the eigenvalue $\eta$ of the action of $G$ on $\phi_*\mathcal{O}_{\widetilde{X}}$.

Now we consider the log-resolution $\pi$. By Proposition 3.1, since $\{(\mathcal{L}_\eta, \alpha_\eta) \mid \eta \in G^*\}$ is a finite subgroup of $\mathrm{Pic}^\tau(X, D)$, $\{(\pi^*\mathcal{L}_\eta \otimes \mathcal{O}_Y(-\lfloor \beta_\eta E \rfloor), \beta_\eta) \mid \eta \in G^*\}$, with $\beta_\eta$ defined by $\pi^*(\alpha_\eta D) = \beta_\eta E$, is a finite subgroup of $\mathrm{Pic}^\tau(Y, E)$. As before, we can construct the corresponding normal $G$-cover of $Y$ unramified above $\pi^{-1}(U) \cong U$,

$$\rho : \widetilde{Y} = \mathrm{Spec}_{\mathcal{O}_Y}\Big(\bigoplus_{\eta \in G^*} \pi^*\mathcal{L}_\eta^{-1} \otimes \mathcal{O}_Y(\lfloor \beta_\eta E \rfloor)\Big) \to Y,$$

where the group $G$ acts on $\widetilde{Y}$ and on $\rho_*\mathcal{O}_{\widetilde{Y}}$. The following result is similar to (3.3).

Proposition 3.3 (Budur [4, Corollary 1.12]). *There is an eigensheaf decomposition*

$$\rho_*\mathcal{O}_{\widetilde{Y}} = \bigoplus_{\eta \in G^*} \pi^*\mathcal{L}_\eta^{-1} \otimes \mathcal{O}_Y(\lfloor \beta_\eta E \rfloor),$$

*where the eigensheaf* $\pi^*\mathcal{L}_\eta^{-1} \otimes \mathcal{O}_Y(\lfloor \beta_\eta E \rfloor)$ *is with respect to the eigenvalue* $\eta$ *of the action of* $G$ *on* $\rho_*\mathcal{O}_{\widetilde{Y}}$.

**3.2. Milnor fibres of homogeneous singularity.** Let $f(x_0, \dots, x_n) \in \mathbb{C}[x_0, \dots, x_n]$ be a homogeneous polynomial of degree $d$. We associate to $f$ two closely related objects, a Milnor fibre at the origin of $\mathbb{C}^{n+1}$ and a complex projective hypersurface of $\mathbb{P}^n$. By [20, Lemma 9.4], the Minor fibre $M$ of $f$ at the origin of $\mathbb{C}^{n+1}$ is diffeomorphic to $\{(x_0, \dots, x_n) \in \mathbb{C}^{n+1} \mid f(x_0, \dots, x_n) = 1\}$. The geometric monodromy $T : M \to M$ corresponds to the multiplication of elements of $M$ by $e^{2\pi i/d}$ and induces an endomorphism $T^*$ of the complex vector space $H^*(M, \mathbb{C})$.

Following [3, Section 4], we consider the smooth complex projective variety $X = \mathbb{P}^n$ and the closed subscheme $D$ of $X$ defined by the zero locus of $f$. Put $U := X \setminus D$. Since the action of $\mathbb{Z}/d\mathbb{Z}$ on $M$ is free, we have a natural isomorphism $M/(\mathbb{Z}/d\mathbb{Z}) \cong U$. Denote by $\sigma$ the quotient map $M \to U$, which is the cyclic cover of degree $d$ of $U$. Then there is an eigensheaf decomposition of the $\mathcal{O}_U$-module sheaf $\sigma_* \mathbb{C}_M = \bigoplus_{k=0}^{d-1} \mathcal{V}_k$, where $\mathcal{V}_k$ is the rank-one unitary local system on $U$ given by the eigensheaf of $T$ with respect to the eigenvalue $e^{-2\pi ik/d}$. This implies that

$$H^l(U, \sigma_* \mathbb{C}_M) = \bigoplus_{k=0}^{d-1} H^l(U, \mathcal{V}_k).$$

Let us consider the Leray spectral sequence

$$E_2^{p,q} = H^q(U, R^p \sigma_* \mathbb{C}_M) \Rightarrow H^{p+q}(M, \mathbb{C}_M).$$

Since $\sigma$ is a finite morphism of schemes, $R^p \sigma_* \mathbb{C}_M = 0$ for all $p \geq 1$. Hence, by this spectral sequence, $H^l(U, \sigma_* \mathbb{C}_M) = H^l(M, \mathbb{C}_M) = H^l(M, \mathbb{C})$, for $l \in \mathbb{N}$.

Lemma 3.4 (Budur [3]). *If the $\mathbb{C}$-vector space $H^l(U, \mathcal{V}_k)$ is nontrivial, it is the eigenspace of $T^*|_{H^l(M,\mathbb{C})}$ with respect to the eigenvalue $e^{-2\pi ik/d}$.*

In fact, there are two commuting monodromy actions on $H^l(M, \mathbb{C})$. Besides $T^*$, the other is the monodromy of $\mathcal{V}_k$ for each $k$ around a generic point of $D_j$ and, by [3, Lemma 4.1], it is given by multiplication by $e^{2\pi ikm_j/d}$. Together with [4, Proposition 3.3], this leads to the following important lemma.

Lemma 3.5 (Budur [3, Lemma 4.2]). *Assume $D = \sum_{j=1}^r m_j D_j$, with $D_j$ irreducible of degree $d_j$. Then the element in $\mathrm{Pic}^\tau(X, D)$ corresponding via the isomorphism RH (see (3.2)) to the unitary local system $\mathcal{V}_k$ is $(\mathcal{O}_{\mathbb{P}^n}(\sum_{j=1}^r \{km_j/d\} d_j), (\{km_1/d\}, \dots, \{km_r/d\}))$.*

Notice that $\sum_{j=1}^r \{km_j/d\} d_j$ is an integer because, if $km_j = dn_j + s_j$ for $1 \leq j \leq r$, with $n_j, s_j \in \mathbb{N}$ and $0 \leq s_j < d$, then

$$\sum_{j=1}^r \left\{\frac{km_j}{d}\right\} d_j = \sum_{j=1}^r \frac{s_j d_j}{d} = \sum_{j=1}^r \frac{km_j d_j - dn_j d_j}{d} = k - \sum_{j=1}^r n_j d_j.$$

Fix a log-resolution $\pi : Y \to \mathbb{P}^n$ of $D$. Let $E = Y \setminus \pi^{-1}(U)$ and let $E_j$ (for all $j$ in some finite set $A$) be the irreducible components of $E$. Let

$$\mathcal{L}^{(k)} := \pi^* \mathcal{O}_{\mathbb{P}^n}\Big(\sum_{j=1}^r \left\{\frac{km_j}{d}\right\} d_j\Big) \otimes \mathcal{O}_Y\Big(-\Big\lfloor \sum_{j=1}^r \left\{\frac{km_j}{d}\right\} \pi^* D_j \Big\rfloor\Big). \tag{3.4}$$

Let $B$ denote the set of integers $k$ such that $0 \le k \le d-1$ and $d$ divides $km_j$ for $1 \le j \le r$, and let $\overline{B}$ be the complement of $B$ in $[0, d-1] \cap \mathbb{Z}$.

Remark 3.6. If $k$ is in $B$, then $\mathcal{L}^{(k)} = \mathcal{O}_Y$. Furthermore, if $k$ is in $B$ and $k \neq 0$, so is $d-k$; if $k$ and $k'$ are in $B$, so is either $k+k'$ or $k+k'-d$; hence we can consider $B$ as a subgroup of $\mathbb{Z}/d\mathbb{Z}$. Let $m = \gcd(m_1, \dots, m_r)$ and choose $u_j \in \mathbb{N}_{>0}$ with $m_j = mu_j$ for $1 \le j \le r$. Then $k \in B$ if and only if $0 \le k \le d-1$ and $ku_s$ is divisible by $\sum_{j=1}^r d_j u_j$ for any $1 \le s \le r$. Since $u_1, \dots, u_r$ are coprime, the latter means that $k$ is divisible by $\sum_{j=1}^r d_j u_j$. Hence $|B| = m$.

For simplicity of notation, from now on, if $\mathcal{A}$ is a sheaf on $\mathbb{P}^n$ and $l \in \mathbb{Z}$, we shall write $\mathcal{A}(l)$ instead of $\mathcal{A} \otimes \mathcal{O}_{\mathbb{P}^n}(l)$.

Proposition 3.7. *With the notation as in Lemma 3.5,*

(i) $\dim_{\mathbb{C}} Gr_F^p H^{p+q}(U, \mathcal{V}_k) = \dim_{\mathbb{C}} H^q(Y, \Omega_Y^p(\log E))$, *for* $k \in B$;

(ii) $\dim_{\mathbb{C}} Gr_F^p H^{p+q}(U, \mathcal{V}_{d-k}) = \dim_{\mathbb{C}} H^q(Y, \Omega_Y^p(\log E) \otimes {\mathcal{L}^{(k)}}^{-1})$, *for* $k \in \overline{B}$.

*In particular, for* $k \in \overline{B}$, $\dim_{\mathbb{C}} Gr_F^0 H^q(U, \mathcal{V}_{d-k})$ *is equal to*

$$\dim_{\mathbb{C}} H^{n-q}\Big(\mathbb{P}^n, \mathcal{J}\Big(\mathbb{P}^n, \sum_{j=1}^r \Big\{\frac{km_j}{d}\Big\} D_j\Big)\Big(\sum_{j=1}^r \Big\{\frac{km_j}{d}\Big\} d_j - n - 1\Big)\Big).$$

Proof. From the group law (3.1) of $\mathrm{Pic}^\tau(X, D)$ and definition of $\mathcal{V}_k$, it is obvious that $\mathcal{V}_k = \mathcal{V}_k^\vee = \mathcal{V}_0$ for $k \in B$ and $\mathcal{V}_{d-k} = \mathcal{V}_k^\vee$ for $k \in \overline{B}$. By Lemma 3.5 and Theorem 3.2,

$$Gr_F^p H^{p+q}(U, \mathcal{V}_k) = H^{n-q}(Y, \Omega_Y^p(\log E)^\vee \otimes \omega_Y)^\vee$$

for $k \in B$, and

$$\begin{aligned} Gr_F^p H^{p+q}(U, \mathcal{V}_{d-k}) &= H^{n-q}(Y, \Omega_Y^p(\log E)^\vee \otimes \omega_Y \otimes \mathcal{L}^{(k)})^\vee \\ &= H^{n-q}(Y, (\Omega_Y^p(\log E) \otimes {\mathcal{L}^{(k)}}^{-1})^\vee \otimes \omega_Y)^\vee \end{aligned}$$

for $k \in \overline{B}$. Serre duality gives (i) and (ii). For the rest, we again apply Lemma 3.5 and the particular case in Theorem 3.2, together with the definition of multiplier ideal. □

Denote $\mathcal{L}_{\mathrm{red}}^{(k)} := \pi^* \mathcal{O}_{\mathbb{P}^n}(k) \otimes \mathcal{O}_Y(-\lfloor \frac{k}{d} E \rfloor)$, for $0 \le k \le d-1$.

Corollary 3.8. *With the notation as in Lemma 3.5 and D reduced, for* $1 \le k \le d$,

(i) $\dim_{\mathbb{C}} Gr_F^p H^{p+q}(U, \mathcal{V}_{d-k}) = \dim_{\mathbb{C}} H^q(Y, \Omega_Y^p(\log E) \otimes {\mathcal{L}_{\mathrm{red}}^{(k)}}^{-1})$;

(ii) $\dim_{\mathbb{C}} Gr_F^0 H^q(U, \mathcal{V}_{d-k}) = \dim_{\mathbb{C}} H^{n-q}(\mathbb{P}^n, \mathcal{J}(\mathbb{P}^n, (k/d)D)(k-n-1))$.

Proof. Applying Proposition 3.7 to the special case $m_1 = \cdots = m_r = 1$ gives the statements. Note that, in this case, $B = \{0\}$ and $\overline{B} = \{1, \dots, d-1\}$. □

Lemma 3.9. *With the notation as in Lemma 3.5, and noting that* $\mathcal{L}^{(d)} = \mathcal{L}^{(0)}$,

(i) $\dim_{\mathbb{C}} H^1(U, \mathcal{V}_k) = r - 1$, *if* $n = 2$ *and* $k \in B$;

(ii) $\dim_{\mathbb{C}} H^l(U, \mathcal{V}_{d-k}) = \sum_{p \ge 0} \dim_{\mathbb{C}} H^{l-p}(Y, \Omega_Y^p(\log E) \otimes {\mathcal{L}^{(k)}}^{-1})$ *for* $l \ge 0$, $1 \le k \le d$.

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PROOF. By Proposition 3.7(i), $\dim_{\mathbb{C}} Gr_F^p H^{p+q}(U, \mathcal{V}_k) = \dim_{\mathbb{C}} H^q(Y, \Omega_Y^p(\log E))$ for $k$ in $B$. Thus

$$\dim_{\mathbb{C}} H^1(U, \mathcal{V}_0) = \dim_{\mathbb{C}} H^1(Y, \mathcal{O}_Y) + \dim_{\mathbb{C}} H^0(Y, \Omega_Y^1(\log E)).$$

Assume that $n = 2$. Then $\dim_{\mathbb{C}} H^1(Y, \mathcal{O}_Y) = 0$, because $Y$ is birationally equivalent to $\mathbb{P}^2$, and $\dim_{\mathbb{C}} H^0(Y, \Omega_Y^1(\log E)) = r - 1$, from the proof of [11, Théorème 6], which proves (i). Statement (ii) is a consequence of Proposition 3.7(ii). □

## 4. Monodromy characteristic polynomials and zeta function

**4.1. Characteristic polynomials.** The Milnor fibre $M$ of the singularity $f(x_0, \ldots, x_n)$ at the origin of $\mathbb{C}^{n+1}$ is diffeomorphic to $\{(x_0, \ldots, x_n) \in \mathbb{C}^{n+1} \mid f(x_0, \ldots, x_n) = 1\}$, and the monodromy $T^*$ is induced by $e^{2\pi i/d} \cdot (x_0, \ldots, x_n) = (e^{2\pi i/d} x_0, \ldots, e^{2\pi i/d} x_n)$ (see Section 3.2). By definition, the (monodromy) characteristic polynomial $\Delta_l(t)$ of $T^*|_{H^l(M,\mathbb{C})}$ is the monic polynomial

$$\Delta_l(t) = \det(t\mathrm{Id} - T|_{H^l(M,\mathbb{C})}).$$

Let $f_j(x_0, \ldots, x_n)$ be distinct irreducible homogeneous polynomials of degree $d_j$ and $D_j = \{(x_0 : \cdots : x_n) \in \mathbb{P}^n \mid f_j(x_0, \ldots, x_n) = 0\}$, for $1 \leq j \leq r$, and set

$$f(x_0, \ldots, x_n) = \prod_{j=1}^{r} f_j(x_0, \ldots, x_n)^{m_j}.$$

Fix a log-resolution $\pi : Y \to \mathbb{P}^n$ of $D = \{(x_0 : \cdots : x_n) \in \mathbb{P}^n \mid f(x_0, \ldots, x_n) = 0\}$, with normal crossing divisor $E$. As mentioned in Section 3, there is an isomorphism $M/(\mathbb{Z}/d\mathbb{Z}) \cong U = \mathbb{P}^n \setminus D$ so that the canonical projection $\sigma : M \to U$ induces an eigensheaf decomposition $\sigma_* \mathbb{C}_M = \bigoplus_{k=0}^{d-1} \mathcal{V}_k$, where $\mathcal{V}_k$ are the rank-one unitary local systems on $U$ given in Lemma 3.5. By Lemma 3.4, for $1 \leq k \leq d$ and $l \in \mathbb{N}$, the vector space $H^l(U, \mathcal{V}_{d-k})$ if nontrivial is the eigenspace of $T^*|_{H^l(M,\mathbb{C})}$ with respect to the eigenvalue $e^{2\pi ik/d}$. This, together with Lemma 3.9 and Remark 3.6, proves the following lemma.

LEMMA 4.1. *The characteristic polynomial $\Delta_l(t)$ of $T^*|_{H^l(M,\mathbb{C})}$ is $\prod_{k=0}^{d-1}(t - e^{2\pi ik/d})^{h_l^{(k)}}$, where*

$$h_l^{(k)} := \dim_{\mathbb{C}} H^l(U, \mathcal{V}_{d-k}) = \sum_{p+q=l} h^q(\Omega_Y^p(\log E) \otimes \mathcal{L}^{(k)^{-1}}),$$

*with $h^q(\Omega_Y^p(\log E) \otimes \mathcal{L}^{(k)^{-1}}) = \dim_{\mathbb{C}} H^q(Y, \Omega_Y^p(\log E) \otimes \mathcal{L}^{(k)^{-1}})$ and*

$$\mathcal{L}^{(k)} = \pi^* \mathcal{O}_{\mathbb{P}^n}\Big(\sum_{j=1}^{r} \Big\{\frac{km_j}{d}\Big\} d_j\Big) \otimes \mathcal{O}_Y\Big(-\Big\lfloor \sum_{j=1}^{r} \Big\{\frac{km_j}{d}\Big\} \pi^* D_j \Big\rfloor\Big).$$

As above, $B$ denotes the set of $k$ in $\mathbb{Z}$ such that $0 \leq k \leq d - 1$ and $d$ divides $km_j$ for $1 \leq j \leq r$, $\overline{B}$ is the complement of $B$ in $[0, d-1] \cap \mathbb{Z}$ and $m = \gcd(m_1, \ldots, m_r)$. From Remark 3.6, $B$ may be considered as a subgroup of $\mathbb{Z}/d\mathbb{Z}$. Let $G$ be the quotient group $(\mathbb{Z}/d\mathbb{Z})/B$. For convenience, we shall identify $k \in [0, d-1] \cap \mathbb{Z}$ with its class in $G$.

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Lemma 4.2. $\Delta_l(t) = \prod_{k\in G}(t^m - e^{2\pi ikm/d})^{h_l^{(k)}}$ *for* $l \in \mathbb{N}$; *in particular,* $\Delta_0(t) = t^m - 1$.

Proof. If $k$ and $k'$ belong to the same class in $G$, we have $h_l^{(k)} = h_l^{(k')}$. This, together with Lemma 4.1, implies the first statement. Since $h^0(\mathcal{O}_Y) = 1$, it remains to check that $h^0(\mathcal{L}^{(k)^{-1}}) = 0$ for $k \in G \setminus \{0\}$. By Lemmas 3.4 and 3.9,

$$\dim_{\mathbb{C}} H^0(M, \mathbb{C}) = \sum_{k\in B} h^0(\mathcal{L}^{(k)^{-1}}) + \sum_{k\in \overline{B}} h^0(\mathcal{L}^{(k)^{-1}}). \tag{4.1}$$

It is known that $\dim_{\mathbb{C}} H^0(M, \mathbb{C}) = m$ (see [9, Proposition 2.3]). Note that $|B| = m$ (see Remark 3.6), $\mathcal{L}^{(k)} = \mathcal{O}_Y$ for $k \in B$ and $h^0(\mathcal{O}_Y) = 1$. Then (4.1) is equivalent to $\sum_{k\in\overline{B}} h^0(\mathcal{L}^{(k)^{-1}}) = 0$, which implies that $h^0(\mathcal{L}^{(k)^{-1}}) = 0$ for $k \in \overline{B}$; in particular, $h^0(\mathcal{L}^{(k)^{-1}}) = 0$ for $k \in G \setminus \{0\}$. □

In the case $n = 2$, we write $C$, $C_j$, $\Delta_C(t)$ instead of $D$, $D_j$, $\Delta_1(t)$, respectively. Then $\Delta_C(t)$ is an important invariant of the singularity $f$, considered as the *global Alexander polynomial* of the nonreduced nonirreducible complex projective plane curve $C$ (see, for instance, [16, Section 3]). The following theorem is one of our main results.

Theorem 4.3. *For* $n = 2$, $\Delta_C(t) = (t^m - 1)^{r-1} \prod_{k\in G\setminus\{0\}}(t^{2m} - 2t^m \cos(2km\pi/d) + 1)^{\ell_k}$, *where*

$$\ell_k := \dim_{\mathbb{C}} H^1\Big(\mathbb{P}^2, \mathcal{J}\Big(\mathbb{P}^2, \sum_{j=1}^r \Big\{\frac{km_j}{d}\Big\} C_j\Big)\Big(\sum_{j=1}^r \Big\{\frac{km_j}{d}\Big\} d_j - 3\Big)\Big).$$

Proof. According to Lemma 4.2, it suffices to prove that $h^1(\mathcal{L}^{(k)^{-1}}) = \ell_k$ and that

$$h^0(\Omega^1_Y(\log E) \otimes \mathcal{L}^{(k)^{-1}}) = \ell_{d-k}, \tag{4.2}$$

for $k \in G \setminus \{0\}$. The former is a direct corollary of Proposition 3.7 and Lemma 3.9. To prove (4.2) we consider a common $G$-equivariant desingularisation of $\widetilde{X}$ and $\widetilde{Y}$, say, $\theta : Z \to \widetilde{X}$ and $\nu : Z \to \widetilde{Y}$, in the sense of [1], such that $\pi \circ \rho \circ \nu = \phi \circ \theta =: u$. Here, we use the notation in Section 3.1 with $X = \mathbb{P}^2$, and, in particular, the normal $G$-cover of $\mathbb{P}^2$,

$$\phi : \widetilde{X} = \mathrm{Spec}_{\mathcal{O}_{\mathbb{P}^2}}\Big(\bigoplus_{k\in G} \mathcal{O}_{\mathbb{P}^2}\Big(-\sum_{j=1}^r \Big\{\frac{km_j}{d}\Big\} d_j\Big)\Big) \to \mathbb{P}^2,$$

and the normal $G$-cover of $Y$,

$$\rho : \widetilde{Y} = \mathrm{Spec}_{\mathcal{O}_Y}\Big(\bigoplus_{k\in G} \mathcal{L}^{(k)^{-1}}\Big) \to Y,$$

where, as mentioned previously, we identify $k \in [0, d-1] \cap \mathbb{Z}$ with its class in $G$.

Note that

$$G^* = \Big\{\Big(\mathcal{O}_{\mathbb{P}^2}\Big(\sum_{j=1}^r \Big\{\frac{km_j}{d}\Big\} d_j\Big), \Big(\Big\{\frac{km_1}{d}\Big\}, \ldots, \Big\{\frac{km_r}{d}\Big\}\Big)\Big)\Big\}_{0\le k\le d-1},$$

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which is by Remark 3.6 a subgroup of order $d/m$ of the group $\mathrm{Pic}^{\tau}(\mathbb{P}^2, C)$. We may choose $Z$ such that $\Delta := Z \setminus u^{-1}(U)$ is normal crossing. An analogue of [11, Corollaire 4] shows that, for any $q \in \mathbb{N}$,

$$\begin{aligned} (\rho \circ \nu)_* \Omega^q_Z(\log \Delta) &\cong \Omega^q_Y(\log E) \otimes (\rho \circ \nu)_* \mathcal{O}_Z, \\ R^p(\rho \circ \nu)_* \Omega^q_Z(\log \Delta) &= 0 \quad \text{if } p > 0 \end{aligned} \tag{4.3}$$

(see also [12, Lemma 3.22]). By the Leray spectral sequence

$$E_2^{p,q} = H^q(Y, R^p(\rho \circ \nu)_* \Omega^1_Z(\log \Delta)) \Rightarrow H^{p+q}(Z, \Omega^1_Z(\log \Delta))$$

and by (4.3), in particular,

$$H^0(Y, \Omega^1_Y(\log E) \otimes (\rho \circ \nu)_* \mathcal{O}_Z) = H^0(Z, \Omega^1_Z(\log \Delta)). \tag{4.4}$$

By Proposition 3.3, $(\rho \circ \nu)_* \mathcal{O}_Z = \rho_* \mathcal{O}_{\widetilde{Y}} = \bigoplus_{k \in G} \mathcal{L}^{(k)^{-1}}$, which yields the decomposition

$$H^0(Y, \Omega^1_Y(\log E) \otimes (\rho \circ \nu)_* \mathcal{O}_Z) = \bigoplus_{k \in G} H^0(Y, \Omega^1_Y(\log E) \otimes \mathcal{L}^{(k)^{-1}}). \tag{4.5}$$

From the proof of Lemma 3.9, the direct summand of (4.5) corresponding to $k = 0$ has complex dimension $r - 1$.

Now we compute the dimension of the complex vector space on the right-hand side of (4.4). As in the proof of [11, Lemma 7],

$$\dim_{\mathbb{C}} H^0(Z, \Omega^1_Z(\log \Delta)) = \dim_{\mathbb{C}} H^0(Z, \Omega^1_Z) + (r - 1). \tag{4.6}$$

On the other hand, by [4, Corollary 1.13],

$$H^0(Z, \Omega^1_Z) \cong \bigoplus_{k \in G} H^1\Big(\mathbb{P}^2, \mathcal{J}\Big(\mathbb{P}^2, \sum_{j=1}^{r} \Big\{\frac{km_j}{d}\Big\} C_j\Big)\Big(\sum_{j=1}^{r} \Big\{\frac{km_j}{d}\Big\} d_j - 3\Big)\Big). \tag{4.7}$$

In (4.7), the direct summand corresponding to $k = 0$ is $H^1(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-3)) = H^1(\mathbb{P}^2, \omega_{\mathbb{P}^2})$. By Serre duality, $\dim_{\mathbb{C}} H^1(\mathbb{P}^2, \omega_{\mathbb{P}^2}) = \dim_{\mathbb{C}} H^1(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}) = 0$. Therefore, from (4.4)–(4.7),

$$\sum_{k \in G \setminus \{0\}} h^0(\Omega^1_Y(\log E) \otimes \mathcal{L}^{(k)^{-1}}) = \sum_{k \in G \setminus \{0\}} \ell_k. \tag{4.8}$$

Repeating the proof of [19, Proposition 4.6] and using $h^1(\mathcal{L}^{(k)^{-1}}) = \ell_k$, for $k \in G \setminus \{0\}$,

$$h^0(\Omega^1_Y(\log E) \otimes \mathcal{L}^{(k)^{-1}}) \geq \ell_{d-k}.$$

This, together with (4.8), implies $h^0(\Omega^1_Y(\log E) \otimes \mathcal{L}^{(k)^{-1}}) = \ell_{d-k}$, thus (4.2) is proved. □

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**4.2. A formula for the monodromy zeta function.** By definition, the monodromy zeta function of the homogeneous singularity $f(x_0, \ldots, x_n)$ at the origin $O$ of $\mathbb{C}^{n+1}$ is

$$\zeta_{f,O}(t) = \prod_{l \geq 0} \det(\mathrm{Id} - tT^*|_{H^l(M,\mathbb{C})})^{(-1)^{l+1}}.$$

This function may be expressed via the polynomials $\Delta_l(t)$ and then, by Lemma 4.2, we obtain

$$\zeta_{f,O}(t) = \prod_{l \geq 0} \Big( t^{\dim_{\mathbb{C}} H^l(M,\mathbb{C})} \Delta_l\Big(\frac{1}{t}\Big)\Big)^{(-1)^{l+1}} = \prod_{k \in G} (1 - e^{2\pi i k m/d} t^m)^{\sum_{l \geq 0} (-1)^{l+1} h_l^{(k)}}. \tag{4.9}$$

As explained in [3], the only numbers $\alpha \in (0, n+1) \cap \mathbb{Q}$ for which $n_{\alpha,O}(f)$, the coefficient of $t^\alpha$ in $\mathrm{Sp}(f, O)$, may be nonzero are of the form $(k/d) + p$, with $k, p \in \mathbb{Z}$, $1 \leq k \leq d$ and $0 \leq p \leq n$. From (2.1) and Lemma 3.4,

$$n_{(k/d)+p,O}(f) = \sum_{j \in \mathbb{Z}} (-1)^j \dim_{\mathbb{C}} Gr_F^{n-p} H^{n+j}(U, \mathcal{V}_k), \tag{4.10}$$

for integers $k, p$ with $1 \leq k \leq d$ and $0 \leq p \leq n$, where $\mathcal{V}_k$ is the rank-one local system corresponding to the element $(\mathcal{O}_{\mathbb{P}^2}(\sum_{j=1}^r \{km_j/d\} d_j), (\{km_1/d\}, \ldots, \{km_r/d\}))$ in $\mathrm{Pic}^\tau(X, D)$ via $RH$ in (3.2) (see Lemma 3.5). Note that $\mathcal{V}_d = \mathcal{V}_0$. By Proposition 3.7 and (4.10),

$$n_{(d-k)/d+p,O}(f) = \sum_{j \in \mathbb{Z}} (-1)^j h^{p+j}(\Omega_Y^{n-p}(\log E) \otimes \mathcal{L}^{(k)^{-1}}), \tag{4.11}$$

for $k \in G$ when $p < n$, and $k \in G \setminus \{0\}$ when $p = n$, where the quantities $\mathcal{L}^{(k)}$ and $h^q(\Omega_Y^p(\log E) \otimes \mathcal{L}^{(k)^{-1}})$ are as in Lemma 4.1 (see also (3.4)).

Theorem 4.4. *The invariants $\zeta_{f,O}(t)$ and $\mathrm{Sp}(f, O)$ are related by*

$$\zeta_{f,O}(t)^{(-1)^{n+1}} = (1 - t^m)^{1 + \sum_{p=1}^n n_{p,O}(f)} \prod_{k \in G \setminus \{0\}} (1 - e^{2\pi i k m/d} t^m)^{\sum_{p=0}^n n_{(d-k)/d+p,O}(f)}.$$

Proof. Recall from Lemma 4.1 that $h_l^{(k)} = \sum_{p+q=l} h^q(\Omega_Y^p(\log E) \otimes \mathcal{L}^{(k)^{-1}})$. Since $h^0(\mathcal{O}_Y) = 1$ and $h^q(\mathcal{O}_Y) = 0$ for all $q \geq 1$, formula (4.11) gives

$$(-1)^{n+1} + (-1)^{n+1} \sum_{p=0}^{n-1} n_{p+1,O}(f) = \sum_{j \in \mathbb{Z}} (-1)^{n+j+1} h_{n+j}^{(0)}.$$

As in the proof of Lemma 4.2, if $k \in G \setminus \{0\}$, then $h^0(\mathcal{L}^{(k)^{-1}}) = 0$, so by (4.11),

$$(-1)^{n+1} \sum_{p=0}^{n} n_{(d-k)/d+p,O}(f) = \sum_{j \in \mathbb{Z}} (-1)^{n+j+1} h_{n+j}^{(k)}.$$

Now applying (4.9) gives the statement of the theorem. □

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REMARK 4.5. Assume that $f(x_0, \ldots, x_n)$ is a homogeneous polynomial of degree $d$ and has an isolated singularity at the origin $O$ of $\mathbb{C}^{n+1}$. Then, by [24, Example 5.11],

$$\begin{aligned}\mathrm{Sp}(f, O) &= t^{(n+1)/d}(1 + t^{1/d} + t^{2/d} + \cdots + t^{(d-2)/d})^{n+1} \\ &= \sum_{k=1}^{d}\sum_{p=0}^{n}\Bigg(\sum_{\substack{\sum_{j=0}^{d-2} k_j = n+1 \\ \sum_{j=0}^{d-2}(j+1)k_j = dp+k}} \frac{(n+1)!}{k_0!k_1!\cdots k_{d-2}!}\Bigg) t^{(k/d)+p}.\end{aligned}$$

This implies that, for $1 \le k \le d$ and $0 \le p \le n$,

$$n_{(k/d)+p,O}(f) = \sum_{\substack{\sum_{j=0}^{d-2} k_j = n+1 \\ \sum_{j=0}^{d-2}(j+1)k_j = dp+k}} \frac{(n+1)!}{k_0!k_1!\cdots k_{d-2}!}.$$

Since $f$ has an isolated singularity $O$, it must be reduced, so by [9, Proposition 4.1.21],

$$\zeta_{f,O}(t) = (t^d - 1)^{-\chi(U)}.$$

On the other hand, from Theorem 4.4,

$$\zeta_{f,O}(t)^{(-1)^{n+1}} = (1-t)^{1+\sum_{p=1}^{n} n_{p,O}(f)} \prod_{1 \le k \le d-1} (1 - e^{2\pi i k/d} t)^{\sum_{p=0}^{n} n_{(d-k)/d+p,O}(f)}.$$

It follows that

$$1 + \sum_{p=1}^{n} n_{p,O}(f) = \sum_{p=0}^{n} n_{(d-k)/d+p,O}(f) = (-1)^n \chi(U),$$

In particular,

$$(-1)^n\chi(U) = 1 + \sum_{p=1}^{n} \sum_{\substack{\sum_{j=0}^{d-2} k_j = n+1 \\ \sum_{j=0}^{d-2}(j+1)k_j = dp}} \frac{(n+1)!}{k_0!k_1!\cdots k_{d-2}!}.$$

For example, the homogeneous polynomial $f(x, y, z) = x^4 + y^4 + z^4$ has an isolated singularity at the origin $O$ of $\mathbb{C}^3$ and it defines a projective curve in $\mathbb{P}^2$. The Euler characteristic of the complement $U$ of this curve in $\mathbb{P}^2$ is

$$\chi(U) = 1 + \frac{3!}{2!1!0!} + \frac{3!}{0!1!2!} = 1 + 3 + 3 = 7.$$

The monodromy zeta function of the singularity of $f$ at $O$ is

$$\zeta_{f,O}(t) = \frac{1}{(t^4 - 1)^7}.$$

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