Student Problems

Students up to the age of 19 are invited to send solutions to either or both of the following problems to Agnes Bokanyi-Toth, School of Science Reception, Schofield Building, Loughborough University, Loughborough, LE11 3TU.

Two prizes will be awarded – a first prize of £25, and a second prize of £20 – to the senders of the most elegant solutions for either problem. It is not necessary to submit solutions to both. Solutions should arrive by 13th May 2025 and will be published in the July 2025 issue of the *Gazette*.

The Mathematical Association and the *Gazette* comply fully with the provisions of the 2018 GDPR legislation. Submissions **must** be accompanied by the SPC permission form which is available on the Mathematical Association website

https:www.m-a.org.uk/the-mathematical-gazette

*Note that if permission is not given, a pupil **may still participate and will be eligible for a prize in the same way as others**.*

***Problem* 2025.1** (S. N. Maitra)

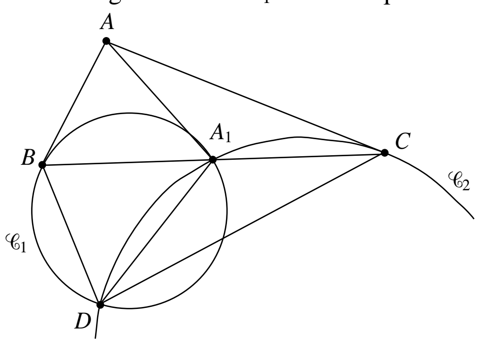
Given that

$$5a^2 + 5b^2 + 10c^2 + d^2 - 4ab - 4bc - 6cd = 4.$$

Determine the solutions where a, b, c, d are all distinct positive integers.

***Problem* 2025.2** (Gerry Leversha)

Let ΔABC be a triangle in which A_1 is the midpoint of BC.



Circle $\mathscr{C}_1$ through A_1 is tangential to AB at B, and circle $\mathscr{C}_2$ through A_1 is tangential to AC at C. Circles $\mathscr{C}_1$ and $\mathscr{C}_2$ meet again at D. Prove that

$$A_1A \times A_1D = A_1B \times A_1C.$$

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Solutions to 2024.3 and 2024.4

Both problems were solved by Chenyu Xie and Mohib Murtaza. Problem 2024.3 was also solved by Rajavel Kutralan and Matthew Alderman.

Problem 2024.3 (Paul Stephenson)

General Lucas sequences are defined by the recursion $l_{n+2} = l_{n+1} + l_n$. Two particular ones are the Fibonacci sequence, starting 0, 1, 1, 2, ... and the Lucas sequence, starting 2, 1, 3, 4, ... Show that, after the third term, the two sequences have no terms in common.

Solution (Rajavel Kutralan)

Note that Matthew Alderman provided the same solution for the problem 2024.3.

Let F_n denote the nth Fibonacci number, where $F_1 = 0$ and $F_2 = 1$, and L_n denote the nth Lucas number, where $L_1 = 2$ and $L_2 = 1$. First, we prove the following relationship between them.

Claim: For $n \geqslant 3$, $L_n = F_{n-1} + F_{n+1}$.

Proof by induction: Assume the relationship is true for $n = i$:

$$L_i = F_{i-1} + F_{i+1}.$$

Then it can be shown that the relationship holds for $n = i + 1$,

$$L_{i+1} = L_i + L_{i-1} = F_{i-1} + F_{i+1} + F_{i-2} + F_i = F_i + F_{i+2.}$$

Consider $i = 3$ as the base case:

$$L_3 = 3 = F_2 + F_4.$$

Hence, the relationship between the Fibonacci and Lucas numbers, $L_n = F_{n-1} + F_{n+1}$ is proved by induction.

Claim: If $n \geqslant 4$, there exists no $m \in \mathbb{N}$ such that

$$L_n = F_m.$$

Proof: By induction we have $L_n = F_{n-1} + F_{n+1}$. Since Fibonacci numbers are positive,

$$F_{n+1} < L_n.$$

As F_n is strictly increasing for $n \geqslant 4$, hence

$$L_n < F_n + F_{n+1}$$

$$L_n < F_{n+2}.$$

$$F_{n+1} < L_n < F_{n+2}.$$

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Suppose that there exists an m such that:

$$F_{n+1} < F_m < F_{n+2}.$$

As the Fibonacci sequence is strictly increasing for $n \geqslant 4$,

$$F_{n+1} < F_m < F_{n+2} \Rightarrow n + 1 < m < n + 2.$$

But since n is an integer, then there cannot be an integer m that satisfies

$$n + 1 < m < n + 2,$$

producing a contradiction. Hence, the Fibonacci and Lucas sequences do not share terms after the third.

Problem 2024.4 (Paul Stephenson)

Consider a convex hexagon whose vertices are grid points on a square lattice of unit edge and which has no boundary points other than vertices. By Pick's theorem, its area is given by $A = i + \frac{1}{2}b - 1$, where i is the number of interior points and b is the number of boundary points. Show that its area must be at least 4 units.

Editor's Note: The proposer intended to exclude the case where the edges of the convex hexagon are the main diagonals on the grid, hence excluding the case where the area is 3 units. The solution below, by Mohib Murtaza and Chenyu Xie, captures the essence of proposer's approach.

Solution (Mohib Murtaza and Chenyu Xie)

By Pick's theorem, the area of the convex hexagon is

$$A = i + \tfrac{1}{2}b - 1 = i + 3 - 1 = i + 2.$$

It must enclose some area and contain some interior lattice points.

If $i = 0$, then $A = 2$, which is impossible for a convex hexagon with no interior lattice points. If $i = 1$, then $A = 3$, so the hexagon has only one interior point. This is illustrated in the diagram below. It follows that the minimum area of a convex hexagon drawn on a lattice is 3 units.

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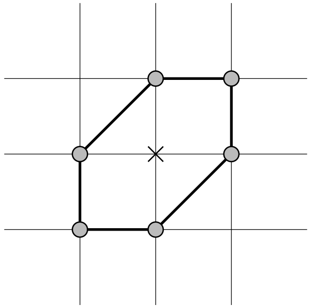

Prize Winners

The first prize of £25 is awarded to Chenyu Xie. Two second prizes of £20 are awarded to Rajavel Kutralan and Matthew Alderman.

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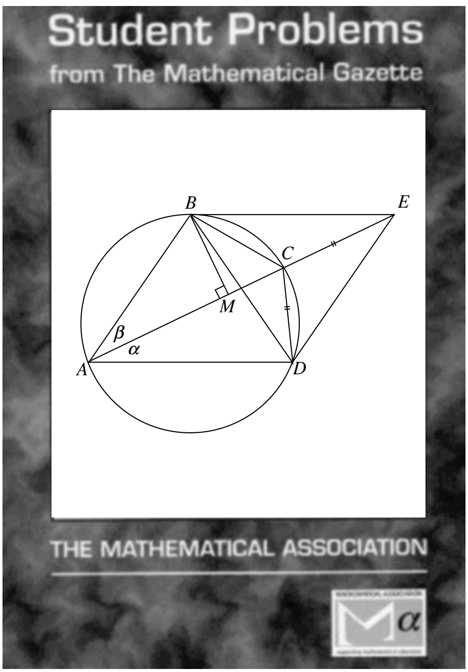

There are many

good problems

in here.