A CHARACTERIZATION OF THE GROUP Aut($PGL(3, 4)$)

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1. Introduction

In a recent paper, Z. Janko [5] announced the discovery of two new finite non-abelian simple groups and characterized these groups in terms of the centralizer of an involution. In fact, he proved the following result.

THEOREM. *Let G be a non-abelian finite simple group with the following properties*:

(i) *The centre $Z(T)$ of a Sylow 2-subgroup T of G is cyclic.*

(ii) *If z is the involution in $Z(T)$, then the centralizer H of z in G is an extension of a group E of order 2^5 by A_5. Then we have the following possibilities.*

If G has only one class of involutions, then G has order 50, 232, 960 *and a uniquely determined character table.*

If G has more than one class of involutions, then G has order 604, 800 *and is uniquely determined (up to isomorphism).*

It is proved in [5] that E is the central product of a dihedral group of order 8 and a quaternion group and that $C(E) = Z(E)$.

This result suggests studying finite groups of even order in which the centralizer of an involution has a structure similar to that described above. In this paper we shall prove the following result.

MAIN THEOREM. *Let G be a finite group of even order which contains an involution z such that the centralizer H of z in G has the following properties*:

(i) *H has a normal subgroup E of order* 32 *which is the central product of a dihedral group of order* 8 *and a quaternion group.*

(ii) *We have $C_H(E) \subseteqq E$.*

(iii) *The factor group H/E is isomorphic to the symmetric group S_4 in four letters.*

Then $O(G)$ is abelian and G is isomorphic to either

(a) *the group* $H \cdot O(G)$.
(b) *the group* $\mathrm{Aut}(PGL(3, 4))$, *or*
(c) *the normalizer in* $\mathrm{Aut}(PGL(3, 4))$ *of an* S_2*-subgroup of* $PGL(3, 4)$.
In particular, if G is non-soluble, then G is isomorphic to $\mathrm{Aut}(PGL(3, 4))$.

The group $\mathrm{Aut}(PGL(3, 4))$ is a split extension of $PGL(3, 4)$ by a four-group. The group $PGL(3, 4)$ has been described by Suzuki [7]. An S_2-subgroup T of $PGL(3, 4)$ is a special 2-group of order 64 whose centre is a four-group. The normalizer of T in $\mathrm{Aut}(PGL(3, 4))$ has order $2^8 \cdot 3^2$; it is a split extension of T by the group of those automorphisms of T which induce non-trivial automorphisms of $T/Z(T)$. It is now straightforward to verify that the groups described in (b) and (c) satisfy the conditions of the theorem.

Since the outer automorphism group of E is isomorphic to S_5, it follows from condition (ii) that H/E is isomorphic to a subgroup of S_5. The case in which H/E is isomorphic to S_5 has been studied by A. Struik in his M.Sc. thesis at Monash University. He proves that G must contain as a subgroup of index 2 one of the simple groups discovered by Janko [5]. If H/E is isomorphic to A_4, then a conclusion similar to that of the main theorem holds. The proof is almost identical with that of the main theorem.

Throughout this paper G will denote a finite group of even order which has an involution z such that the centralizer H of z in G satisfies the conditions (i), (ii) and (iii) above. From time to time we shall impose further conditions on G. For any subset X of G we shall put $N(X) = N_G(X)$ and $C(X) = C_G(X)$. If Y is a group, then we shall use $D(Y)$ to denote the Frattini subgroup of Y and $O(Y)$ to denote the largest normal subgroup of odd order in Y. The other notation follows [3].

Suppose that $O(G) \neq 1$ and set $\bar{G} = G/O(G)$. For a subset X of G, let $\bar{X}$ be the image of X in $\bar{G}$. By the Frattini argument we have $C_{\bar{G}}(\bar{z}) = \bar{H}$. Since $\bar{H}$ is isomorphic to H, $\bar{G}$ satisfies the conditions of the theorem together with $O(\bar{G}) = 1$. In order to prove the theorem we shall assume that $\bar{G}$ is not equal to $\bar{H}$ and then determine the structure of G. From Theorem 4 of [2] we see that $\langle z \rangle$ is not weakly closed in H. The fact that $O(G) = 1$ in cases (b) and (c) follows from Lemma 10 and the Brauer-Wielandt formula [4].

2. The structure of E

The group E has order 32 and is the central product of a dihedral group of order 8 and a quaternion group. Therefore E has 10 dihedral subgroups, of order 8, 10 quaternion subgroups and 15 subgroups which are abelian of type (4,2). These being all the subgroups of E of order 8. Furthermore, E has 10 non-central involutions and 20 elements of order 4. Since

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$C_H(E) \subsetneqq E$ it follows that $z \in E$ and we have $Z(E) = E' = D(E) = \langle z \rangle$. If T is an S_2-subgroup of H, then $Z(T) = Z(E) = \langle z \rangle$ and so T is an S_2-subgroup of G. Furthermore, we have $C(E) = \langle z \rangle$ and $N(E) = H$. Hence the factor group H/E is isomorphic to a subgroup of the outer automorphism group of E.

We may suppose that E is generated by elements t_1, t_2, h_1 and h_2 which satisfy the following relations:

$$\begin{aligned} t_1^2 = t_2^2 &= 1, & (t_1 t_2)^2 &= z, \\ h_1^2 = h_2^2 &= z, & h_2^{h_1} &= h_2^{-1}, \\ [t_i, h_j] &= 1, & i, j &= 1, 2. \end{aligned}$$

Then $E_1 = \langle t_1, t_2 \rangle$ is a dihedral group of order 8, $E_2 = \langle h_1, h_2 \rangle$ is a quaternion group and E is the central product of E_1 and E_2. The 10 non-central involutions of E give rise to 5 cosets in $E/\langle z \rangle$ with representatives:

$$t_1, t_2, t_1 t_2 h_1, t_1 t_2 h_2, t_1 t_2 h_1 h_2.$$

We label these cosets with the numbers 1, 2, 3, 4 and 5 respectively. The outer automorphism group of E is isomorphic to S_5 and acts faithfully and transitively on the above 5 cosets by conjugation. Since H/E is isomorphic to S_4 we may suppose that each element of H fixes the coset 1 with representative t_1.

If x is any non-central element of E, then $|E : C_E(x)| = 2$ and the conjugates of x in E are x and xz.

If F_1 is a dihedral subgroup of E of order 8, then $F_2 = C_E(F_1)$ is a quaternion subgroup of E and E is the central product of F_1 and F_2. Conversely, if F_1 is a quaternion subgroup of E, then $F_2 = C_E(F_1)$ is a dihedral subgroup of E of orders 8 and again E is the central product of F_1 and F_2.

3. The involutions in H

We first determine the conjugacy classes of H which lie in E. As remarked in section 2, the coset of $E/\langle z \rangle$ with representative t_1 is fixed by H and H is faithfully represented as a transitive permutation group of the remaining four cosets of non-central involutions. Since any non-central element x of E is already conjugate to xz in E we have the following lemma.

LEMMA 1. *The group H has three classes of involutions which lie in E with representatives t_1, t_2 and z. The involution t_1 has two conjugates in H and we have $C_E(t_1) = \langle t_1 \rangle \times E_2$ and $C_H(t_1)/C_E(t_1) \cong S_4$. For t_2 we have $C_E(t_2) = \langle t_2 \rangle \times E_2$, $C_H(t_2)/C_E(t_2) \cong S_3$ and t_2 has 8 conjugates in H.*

The next lemma determines the action of an S_3-subgroup of H on E.

LEMMA 2. *Let P be an S_3-subgroup of H. Then $F_1 = C_E(P)$ is a dihedral group of order* 8 *and $F_2 = [E, P]$ is a quaternion group. The group E is the central product of F_1 and F_2.*

PROOF.[1] Since E has precisely 10 dihedral subgroups of order 8, P must normalize and hence centralize one of them. Let this be F_1. Since $C_H(E) \subseteqq E$ by assumption, we must have $C_E(P) = F_1$ and P must normalize the quaternion group $F_2 = C_E(F_1)$. Since E is the central product of F_1 and F_2 it follows that $[E, P] = F_2$.

For the rest of the paper we shall suppose that P is an S_3-subgroup of H such that $E_1 = C_E(P)$ and $E_2 = [E, P]$. Furthermore, let H_1 be the subgroup of H such that $E \subseteqq H$ and $H_1/E \cong \boldsymbol{A}_4$.

The involution t_1 is contained in precisely 4 dihedral subgroups of E of order 8 and the elements of order 3 of H permute these dihedral groups transitively. Therefore, the 8 elements of order 4 which are contained in these dihedral groups are conjugate in H to t_1t_2. The remaining 12 elements of order 4 in E are contained in $C_E(t_1)$ and are conjugate in H to h_1 since the 4 quaternion groups of $C_E(t_1)$ are permuted transitively by the elements of order 3 of H. The elements h_1 and t_1t_2 cannot be conjugate in H since the order of H is not divisible by 5. We have thus proved the following lemma.

LEMMA 3. *The group H has two classes of order* 4 *in E with representatives t_1t_2 and h_1. The element t_1t_2 has* 8 *conjugates in H and h_1 has* 12 *conjugates in H. Therefore $|C_H(t_1t_2)| = 2^5 \cdot 3$ and $|C_H(h_1)| = 2^6$.*

The next step is to determine the classes of involutions in $H-E$.

LEMMA 4. *There exists an involution in $H-H_1$. Also $C_H(P) = E_1 \times P$ and $N_H(P) = SP$ where we have the following two possibilities for S:*

(1) *The group S is the central product of E_1 and Y, where Y is a cyclic group of order* 4. *In this case H has precisely one class of involutions in $H-H_1$.*

(2) *The group S is equal to $E_1 \times \langle d \rangle$ where d is an involution in $H-H_1$ such that $d \in C_H(t_1)$. In this case there are either two or three classes of involutions in $H-H_1$.*

Let x be an involution in $H-H_1$. Then $\langle z \rangle$ is characteristic in $C_H(x)$ and $C_H(x)$ is an S_2-subgroup of $C(x)$.

PROOF. Since an S_3-subgroup of $\boldsymbol{S}_4$ is selfcentralizing, it follows that $C_H(P) = E_1 \times P$. By a Frattini argument and a theorem of Burnside P is inverted by a 2-element of $H-H_1$. Hence the order of $N_H(P)$ is $2^4 \cdot 3$. Now let S be an S_2-subgroup of $N_H(P)$. Then $E_1 \lhd S$ and $|S : E_1| = 2$.

[1] The author owes a simplification in the proof of Lemma 2 to the referee.

The group $\langle t_1 t_2\rangle$ is the unique cyclic subgroup of order 4 of E_1 and so $\langle t_1 t_2\rangle \lhd S$. Since $t_1 t_2$ does not centralize E_1 we have $|C_S(t_1 t_2)| = 8$. It follows that $C_S(t_1 t_2)$ is abelian. If $C_S(t_1 t_2)$ were cyclic, then an element of order 8 would induce an outer automorphism of E_1. This is impossible since t_1 and t_2 are not conjugate in H. Therefore, $C_S(t_1 t_2)$ is abelian of type (4, 2) and so there exists an involution d in $C_S(t_1 t_2) - \langle t_1 t_2\rangle$. We have $d \in H - H_1$, d inverts P and $E_2\langle d\rangle$ is a semi-dihedral group of order 16. Therefore d is conjugate in H to dz. We may choose the notation so that $h_1^d = h_1 z$ and $h_2^d = h_1 h_2 z$. For the action of d on E we have the following two cases:

(1) $$t_1^d = t_1 z \quad \text{and} \quad t_2^d = t_2 z$$

(2) $$t_1^d = t_1 \quad \text{and} \quad t_2^d = t_2.$$

If x is an involution in $H - H_1$, then $x \notin O_2(H)$. By Theorem 3.8.2 of [3], x inverts an S_3-subgroup of H and so x is conjugate in H to an involution in $S - E_1$.

In Case (1) it follows that S is the central product of E_1 and $Y = \langle t_1 t_2 d\rangle$. Since in this case d and dz are the only involutions in $S - E_1$, all involutions in $H - H_1$ are conjugate. For the centralizer of d in E we have $C_E(d) = \langle t_1 h_1, t_2 h_1\rangle$, which is a quaternion group. The coset Ed contains 16 elements of order 8, 12 elements of order 4 and 4 involutions. From the structure of S_4 we have $|C_H(d)/C_E(d)| \leqq 4$ whence $|C_H(d)| = 2^5$ and $C_H(d)/C_E(d)$ is a four-group.

Now suppose we are in Case (2). Then $C_E(d) = \langle t_1, t_2\rangle = E_1$ and $S = E_1 \times \langle d\rangle$. It follows that the involution x is conjugate to one of d, $t_1 d$ or $t_2 d$. We have $C_E(t_1 d) = \langle t_2 h_1, t_1\rangle$ and $C_E(t_2 d) = \langle t_1 h_1, t_2\rangle$ whence $C_E(x)$ is always a dihedral group of order 8 and so $|E : C_E(x)| = 4$. The coset Ed contains 16 elements of order 8, 4 elements of order 4 and 12 involutions. Again from the structure of S_4 we have $|C_H(x)/C_E(x)| \leqq 4$. Since $2^4 \leqq |C_H(x)|$, it follows that there are either two or three classes of involutions in $H - H_1$.

In either case $C_H(x)' \subseteqq C_E(x)$ and so $C_H(x)' \cap Z(C_H(x)) = \langle z\rangle$. It now follows that $\langle z\rangle$ is characteristic in $C_H(x)$ and $C_H(x)$ is an S_2-subgroup of $C(x)$.

LEMMA 5. *If u is an element of $O_2(H) - E$, then $|C_E(u)| \leqq 8$. If u is an involution of $O_2(H) - E$, then $C_E(u) = \langle t_1, z\rangle$ and $|C_H(u)| = 2^5$. Furthermore, H has at most one class of involutions in $O_2(H) - E$.*

PROOF. Any element of $O_2(H) - E$ is conjugate to an element whose action on the cosets of non-central involutions in $E/\langle z\rangle$ is represented by the permutation (23)(45). It now follows that $C_E(u) \subseteqq \langle h_1, t_1\rangle$ and so $|C_E(u)| \leqq 8$.

Suppose that u is an involution. We have $h_2^u = t_1 h_2$ or $h_2^u = t_1 h_2 z$.

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In either case it follows that $t_1^u = t_1$. Again, $t_2^u = t_1 t_2 h_1$ or $t_2^u = t_1 t_2 h_1 z$ and in either case it follows that $h_1^u = h_1 z$. Therefore, $C_E(u) = \langle t_1, z\rangle$ and since $\langle t_1, z\rangle$ is normal in H it follows that $\langle t_1, z\rangle$ is the centralizer in E of any involution in $O_2(H)-E$. The coset Eu contains 16 elements of order 8, 8 elements of order 4 and 8 involutions. Since $|E : C_E(u)| = 8$, it follows that all involutions in Eu are conjugate in $E\langle u\rangle$. Therefore, $|C_H(u)| = 2^5$ and $C_H(u)/C_E(u)$ is a dihedral group of order 8.

The following information about the centralizer of t_2 will be needed later.

Lemma 6. *The group $\langle z\rangle$ is a characteristic subgroup of $C_H(t_2)$. An S_2-subgroup T_0 of $C_H(t_2)$ is an S_2-subgroup of $C(t_2)$ and we have $Z(T_0) = \langle t_2, z\rangle$.*

Proof. Since $|C_H(t_2)| = 2^5 \cdot 3$ we have $|T_0| = 2^5$. From Lemma 4 we may suppose that either d or $t_1 d$ lies in T_0. We have

$$T_0 \cap E = C_E(t_2) = \langle t_2\rangle \times E_2$$

and $E_2 = \langle h_1, h_2\rangle$ is a quaternion group. Thus $T_0' = \langle h_1\rangle$ and $\mho^1(T_0') = \langle z\rangle$. Hence $\langle z\rangle$ is characteristic in T_0 and T_0 is an S_2-subgroup of $C(t_2)$.

Suppose that $Z(T_0) \nsubseteq C_E(t_2)$ and let x be an element of $Z(T_0)-C_E(t_2)$. Then we have $C_E(x) = \langle t_2\rangle \times E_2$ and so x centralizes $E/\langle z\rangle$. This contradicts the fact that x permutes some cosets of $E/\langle z\rangle$. Thus we must have $Z(T_0) \subseteqq C_E(t_2)$, whence $Z(T_0) = \langle t_2, z\rangle$.

Lemma 7. *The involution t_2 cannot be fused in G with any involution in H.*

Proof. By Lemma 6 an S_2-subgroup of $C(t_2)$ has order 2^5. Therefore t_2 cannot be conjugate to either z or t_1. Suppose that t_2 is conjugate in G to an involution x in $H-H_1$. An S_2-subgroup T_0 of $C_H(t_2)$ is an S_2-subgroup of $C(t_2)$ and $C_H(x)$ is an S_2-subgroup of $C(x)$. Thus, there exists $g \in G$ such that $T_0^g = C_H(x)$ and then $Z(T_0)^g = Z(C_H(x))$. But then $z^g \in \langle x, z\rangle$, which is impossible, since $g \notin H$ and z is not conjugate to t_2. Thus t_2 is not conjugate to any involution in $H-H_1$.

Suppose that t_2 is conjugate to an element u in H_1-E. By Lemma 5 and Lemma 6, $C_H(u)$ is an S_2-subgroup of $C(u)$. By the above argument we again get a contradiction. The lemma is proved.

We now use the assumption that $\langle z\rangle$ is not weakly closed in H to prove the existence of involutions in H_1-E.

Lemma 8. *There exists precisely one class of involutions in H_1-E.*

Proof. Suppose that there are no involutions in H_1-E. By Lemma 4, Lemma 7 and the fact that $\langle z\rangle$ is not weakly closed in H it follows that z is conjugate to t_1. We have $|E : C_E(t_1)| = 2$ and $C_H(t_1)/C_E(t_1) \cong S_4$. Let

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T_1 be an S_2-subgroup of $C_H(t_1)$. Then $T_1 \cap E = C_E(t_1) = \langle t_1 \rangle \times E_2$, $|T_1| = 2^7$ and $T = T_1 E$ is an S_2-subgroup of H. Furthermore, $T_1/C_E(t_1)$ is a dihedral group of order 8. Suppose at first that $Z(T_1) \nsubseteq C_E(t_1)$ and let u be an element of $Z(T_1) - C_E(t_1)$. By Lemma 5 we have $|C_E(u)| \leqq 8$, which contradicts $|T_1 \cap E| = 16$. Therefore, $Z(T_1) = \langle t_1, z \rangle$. Let T^* be an S_2-subgroup of $C(t_1)$ which contains T_1. Then $T_1 = T^* \cap H$, $|T^* : T_1| = 2$, $|T : T_1| = 2$ and $Z(T_1) \lhd \langle T, T^* \rangle$. Also we have $C(Z(T_1)) = C_H(t_1)$. All this shows that $N(Z(T_1))/C_H(t_1)$ is a non-abelian group of order 6. Let us put $T_2 = O_2(C_H(t_1))$. Then $T_2 \lhd N(Z(T_1))$ and T_2 has order 2^6. We have $C(T_2) \subseteqq T_2$ since an S_3-subgroup of $C_H(t_1)$ cannot centralize T_2. An S_3-subgroup Q of $N(Z(T_1))$ has order 9 and acts faithfully on $T_2/D(T_2)$. Since $|T_2/D(T_2)| \leqq 2^5$, Q is elementary. If t_1 were not a square in T_2, then $\langle z \rangle$ would be characteristic in T_2, which is not possible. Hence we have $|T_2/D(T_2)| = 2^4$ and $D(T_2) = Z(T_1)$. Let $\tilde{Q}$ be a subgroup of order 3 of Q such that $\tilde{Q} \nsubseteq H$ and $\tilde{Q}$ fixes a non-trivial element in $T_2/D(T_2)$. Then $\tilde{Q}$ fixes an element y in $T_2 - D(T_2)$ and so $\tilde{Q}$ fixes y^2. But $T_2 \subseteqq H_1$ and so $y^2 \neq 1$, $y^2 \in \langle t_1, z \rangle$, whence $\tilde{Q}$ centralizes $\langle t_1, z \rangle$ which is a contradiction. We have proved that there exists an involution in $H_1 - E$. By Lemma 5 there is precisely one class of involutions in $H_1 - E$.

4. The structure of $N(\langle t_1, z \rangle)$

We first establish some notation. Let a be an involution in $H_1 - E$. We choose a so that its action on the cosets of non-central involutions in $E/\langle z \rangle$ is represented by (23) (45). Replacing a by $t_1 a$, $h_1 a$ or $t_1 h_1 a$ if necessary, we may suppose that a has the following action on E:

$$t_1^a = t_1, \qquad t_2^a = t_1 t_2 h_1$$
$$h_1^a = h_1 z, \qquad h_2^a = t_1 h_2 z.$$

We put $T_2 = O_2(C_H(t_1))$ as in Lemma 8. It is easily seen that the 8 involutions in Ea lie in T_2. Let $P = \langle \sigma \rangle$, where P is the S_3-subgroup of H chosen after Lemma 2. We may choose σ so that it has the following action on E.

$$t_1^\sigma = t_1, \qquad t_2^\sigma = t_2$$
$$h_1^\sigma = h_2, \qquad h_2^\sigma = h_1 h_2.$$

We next set $b = a^\sigma$. It follows that $b^\sigma = abe$, where $e \in E$ and $(abe)^\sigma = a$. By calculation we see that $e = 1$ or $e = z$. In either case it follows that $ab = ba$. Replacing a by az and b by bz if necessary, we may suppose that $b = a^\sigma$ and $ab = b^\sigma$. The group $A = \langle t_1, z, a, b \rangle$ is elementary of order 16 and if $t \in A - \langle t_1, z \rangle$, then $C_{H_1}(t) = A$. Since there are precisely 24 involutions in $T_2 - C_E(t_1)$ it follows that T_2 has exactly one more elementary subgroup B of order 16 and we have $B = \langle t_1, z, h_1 a, h_2 b \rangle$. The groups A and B are both normal in T_2 and are the only elementary subgroups of order 16 of H. Both A and B are P-admissible and we have

$$N_H(A)/A \cong N_H(A)/B \cong \boldsymbol{S}_4.$$

Furthermore, we have $A^{t_2} = B$.

LEMMA 9. *The group* $T_2 = O_2(C_H(t_1))$ *is a special 2-group of order* 64 *such that* $T_2' = D(T_2) = Z(T_2) = \mho^1(T_2) = \langle t_1, z \rangle$. *Furthermore,* T_2 *has precisely two elementary subgroups* A *and* B *of order* 16 *such that* $A \cap B = A \cap E = B \cap E = \langle t_1, z \rangle$. *These groups* A *and* B *are the only elementary groups of order* 16 *of* H. *The group* T_2 *is generated by* A *and* B. *We also have* $N_H(A)/A \cong N_H(B)/B \cong \boldsymbol{S}_4$ *and* $C(A) = A$, $C(B) = B$. *Finally, any involution* t *in* $H_1 - E$ *lies either in* A *or in* B *and so we have either* $C_{H_1}(t) = A$ *or* $C_{H_1}(t) = B$.

PROOF. We have only to prove that T_2 is a special 2-group. To this end we consider the action of P on T_2. Since $T_2 = \langle h_1, h_2, t_1, a, b \rangle$ we see that $[P, T_2] = T_2$ and $C_{T_2}(P) = \langle t_1, z \rangle$. If $Z(T_2) \supset \langle t_1, z \rangle$, then $Z(T_2)$ would be elementary of order 16. This contradicts the fact that $|C_{T_2}(t)| = 2^4$ for any involution t in $T_2 - \langle t_1, z \rangle$. Hence $Z(T_2) = \langle t_1, z \rangle$ and $T_2/Z(T_2)$ is elementary of order 16. Since both z and t_1 are squares it follows that $D(T_2) = \mho^1(T_2) = \langle t_1, z \rangle$. Again, both t_1 and z are commutators so that $T_2' = \langle t_1, z \rangle$. The lemma is proved.

LEMMA 10. *The involution* z *is conjugate in* G *to* t_1.

PROOF. Suppose that z is not conjugate to t_1. Since we assume that $\langle z \rangle$ is not weakly closed in H it follows that z is conjugate to a. But now the 13 involutions in $A - \{t_1, t_1 z\}$ are conjugate in G, whence $\{t_1, t_1 z\}$ is $N(A)$-invariant. Hence $\langle t_1, z \rangle \lhd N(A)$ and so $\langle z \rangle \lhd N(A)$, a contradiction since $N(A)$ is not contained in H. The lemma is proved.

Proceeding as in Lemma 8 we see that $N(Z(T_2))$ is a group of order $2^8 \cdot 3^2$ and an S_3-subgroup of $N(Z(T_2))$ is elementary of order 9.

LEMMA 11. *Let* P *be the* S_3*-subgroup of* H *chosen at the beginning of this section. Then* $C(P)/P$ *is isomorphic to* $\boldsymbol{S}_4$ *or* $\boldsymbol{S}_5$ *and* $N(P)$ *is isomorphic to* $\boldsymbol{S}_3 \times \boldsymbol{S}_4$ *or* $\boldsymbol{S}_3 \times \boldsymbol{S}_5$. *An* S_2*-subgroup of* $N(P)$ *is the direct product of a dihedral group of order* 8 *and a group of order* 2. *Hence Case* (1) *of Lemma* 4 *does not occur and we have* $d \in C_H(t_1)$, *where* d *is the involution chosen in the proof of Lemma* 4.

PROOF. We have $C_H(P) = E_1 \times P$. Since $\langle z \rangle = Z(E_1)$ is characteristic in E_1 it follows that E_1 is an S_2-subgroup of $C(P)$. The involutions in $C(P)$ are conjugate either to z or to t_2. Acting on $O(C(P))$ with the four-group

$\langle t_1, z\rangle$ we see from the Brauer-Wielandt formula [(4], Lemma 3) that $O(C(P)) = P$. Since $9 \mid |C(P)|$ it follows that $C(P)$ has no normal 2-complement. It now follows from a result of Gorenstein and Walter [4] that $C(P)/P$ is isomorphic to $\boldsymbol{S}_4$ or $\boldsymbol{S}_5$ (see the last two lines of p. 592 of [4]).

Since P splits in $C(P)$ we may put $C(P) = P\times V$, where V is isomorphic to $\boldsymbol{S}_4$ or $\boldsymbol{S}_5$. Now V is the 3-commutator subgroup of $C(P)$ so that V is normal in $N(P)$. By Lemma 4, P is inverted by an involution. This involution induces an automorphism of V. But V is a complete group (Burnside [1], p. 209) so that $N(P) = V\times V_1$, where $P \subseteqq V_1$ and $V_1 \cong \boldsymbol{S}_3$. The lemma now follows.

We are now able to determine all the classes of H. Let d be the involution chosen in the proof of Lemma 4. Then d inverts P and by Lemma 11 and Lemma 4 d centralizes E_1. Therefore, $C(P) = P\langle d\rangle \times V$ and $P\langle d\rangle \cong \boldsymbol{S}_3$. Let P_0 be the S_3-subgroup of V which normalizes $Z(T_2)$ and is inverted by t_2. We put $P_0 = \langle\tau\rangle$. Then $Q = P\times P_0$ is an S_3-subgroup of $N(T_2) = N(Z(T_2))$. As in Lemma 8 we have $C(T_2) = Z(T_2)$. The special 2-group T_2 is characteristic in $N(T_2)$ since A and B are the only elementary groups of order 16 in $N(T_2)$. Since $T_2 \cap N(Q) = 1$, it follows that $N(T_2) = T_2 N_{N(T_2)}(Q)$ and $T_2 \cap N_{N(T_2)}(Q) = 1$. By choice of Q we have $N_{N(T_2)}(Q) = Q\langle t_2, d\rangle$. Let P_1 and P_2 be the other two subgroups of order 3 of Q. Since $Q \cap H = P$, it follows that P_0, P_1 and P_2 act faithfully on $Z(T_2)$. By Maschke's theorem Q normalizes a complement of $Z(T_2)$ in A. Since $\langle a, b\rangle$ is the unique complement of $Z(T_2)$ in A which is P-admissible, it follows that $\langle a, b\rangle$ is Q-admissible. We have $A^{t_2} = B$, $P_1^{t_2} = P_2$, t_2 inverts P_0 and $C_Q(t_2) = P$. Since P_0 does not centralize $T_2/Z(T_2)$, it follows that one of P_1 or P_2 must centralize $\langle a, b\rangle$. We choose the notation so that $C_{T_2}(P_1) = \langle a, b\rangle$. Then we have

$$A = Z(T_2)\times C_{T_2}(P_1) \quad\text{and}\quad B = Z(T_2)\times C_{T_2}(P_2).$$

In fact,

$$C_{T_2}(P_2) = C_{T_2}(P_1)^{t_2} = \langle t_1 h_1 az, t_1 h_2 bz\rangle$$

Replacing τ by τ^{-1} if necessary, we may suppose that $a^\tau = b$ and $b^\tau = ab$. Since t_2 inverts τ and σ commutes with τ, we calculate that $h_1^\tau = h_1 h_2 az$ and $h_2^\tau = h_1 bz$. It now follows that $t_1^\tau = z$ and $z^\tau = t_1 z$. From the action of d and a on E we calculate that $a^d = t_1 h_1 a$ or $a^d = t_1 h_1 az$. Suppose that $a^d = t_1 h_1 a$. Then $b^d = a^{\tau d} = a^{d\tau} = h_1 h_2 ab$, whence $b^{d^2} = (h_1 h_2 ab)^d = bz$, a contradiction. Therefore, we have $a^d = t_1 h_1 az$ and $b^d = t_1 h_1 h_2 abz$. It now follows that $a^{t_2 d} = a$ and $b^{t_2 d} = ab$. Thus $|C_H(d)| = 2^4$ and $|C_H(t_2 d)| = 2^5$. It follows from Lemma 4 that $H - H_1$ has precisely two classes of involutions with representatives d and $t_2 d$. We have proved the following result about $N(T_2)$.

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LEMMA 12. *The group* $N(T_2) = N(Z(T_2))$ *has order* $2^8 \cdot 3^2$ *and an* S_3-*subgroup* $Q = P \times P_0$ *of* $N(T_2)$ *is elementary of order* 9, *where* P_0 *is centralized by* d *and inverted by* t_2. *We have* $N(T_2) = T_2Q\langle t_2, d\rangle$, $T_2 \cap Q\langle t_2, d\rangle = 1$ *and* $N(Q) \cap N(T_2) = Q\langle t_2, d\rangle$. *The groups* A *and* B *are the only elementary groups of order* 16 *in* $N(T_2)$ *and we have* $A = Z(T_2) \times C_{T_2}(P_1)$, $B = Z(T_2) \times C_{T_2}(P_2)$ *where* $Q = P_1 \times P_2$ *and* P_1 *and* P_2 *act faithfully on* $Z(T_2)$. *We have* $P_1^{t_2} = P_2$, $P_1^d = P_2$, $A^{t_2} = B$ *and* $A^d = B$. *The group* $N(T_2)$ *has precisely five classes of involutions with representatives* z, a, t_2, d *and* t_2d. *Here* $a \in T_2$ *and* $C_{N(T_2)}(a) = AP_1\langle t_2d\rangle$ *has order* $2^5 \cdot 3$. *For* t_2 *we have*

$$C_{N(T_2)}(t_2) = \langle t_2\rangle \times E_2P\langle d\rangle \text{ and } E_2P\langle d\rangle \cong GL(2, 3).$$

For d *we have* $C_{N(T_2)}(d) = \langle d\rangle \times E_1P_0$ *and* $E_1P_0 \cong \mathbf{S}_4$. *For* t_2d *we have* $C_{N(T_2)}(t_2d) = \langle t_2d\rangle \times \langle t_2a, a\rangle$, *and* $\langle t_2a, a\rangle$ *is a dihedral group of order* 16. *The group* $N(T_2)$ *has precisely three classes of elements of order* 3 *with representatives* σ, τ *and* $\sigma\tau^{-1}$, *where* $P = \langle\sigma\rangle$, $P_0 = \langle\tau\rangle$ *and* $P_1 = \langle\sigma\tau^{-1}\rangle$. *For* σ *we have* $C_{N(T_2)}(\sigma) = E_1P_0 \times P$, *and* $E_1P_0 \cong \mathbf{S}_4$. *For* τ *we have* $C_{N(T_2)}(\tau) = P\langle d\rangle \times P_0$ *and* $P_0\langle d\rangle \cong \mathbf{S}_3$.

Finally, $C_{N(T_2)}(\sigma\tau^{-1}) = \langle a, b\rangle P_2 \times P_1$ *and* $\langle a, b\rangle P_2 \cong \mathbf{A}_4$.

5. The structure of G

LEMMA 13. *The group* G *has a normal subgroup* G_2 *of index* 4 *in* G *such that* $N(T_2) \cap G_2 = T_2Q$.

PROOF. We continue to use the notation developed in the preceding sections.

The group $T = T_2\langle t_2, d\rangle$ is an S_2-subgroup of G, and we have $T' = Z(T_2)E_2\langle a\rangle$ and $N(T) = T$. From a theorem of Grün ([3], Theorem 7.4.2) and our knowledge of the possible fusion of involutions we see that the focal group of T in G is equal to T_2. Therefore G has a normal subgroup G_2 of index 4 such that $G_2 \cap T = T_2$, and $G = TG_2$. It is clear that $N_{G_2}(T_2) = T_2Q$. The lemma is proved.

We now turn to the investigation of $N(A)$. Since A and B generate T_2, it follows that

$$N(A) \cap N(B) \subseteqq N(T_2) \text{ and } N(A) \cap N(B) = N_{N(T_2)}(A) = N_{N(T_2)}(B).$$

Furthermore, we have $N(A) \cap N(B) = T_2Q\langle t_2d\rangle$. Let us put $X = C_A(P_1)$ and $Y = C_B(P_2)$. Both X and Y are normalized by Q and we have $X^{t_2} = Y$.

LEMMA 14. *We have the following two possibilities*:

(1) *The group* $N(A)/A$ *equals* S_1L, *where* $|S_1| = 3$, $L \cong \mathbf{S}_5$, $S_1 \lhd S_1L$

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and $S_1 \cap L = 1$. *In this case the involution* z *is conjugate to* a *and* G_2 *has precisely one class of involutions.*

(2) *The group* $N(A)$ *is contained in* $N(T_2)$ *and* G *has precisely five classes of involutions.*

PROOF. Since A is not normal in an S_2-subgroup of G, an S_2-subgroup of $N(A)/A$ is dihedral of order 8. Since $N(A)/A$ is isomorphic to a subgroup of $GL(4, 2) \cong \boldsymbol{A}_8$ and from the structure of $N(A) \cap N(T_2)/A$ it follows that either $N(A) = T_2Q\langle t_2 d\rangle \subseteqq N(T_2)$ or $N(A)/A = S_1L$, where $|S_1| = 3$, $L \cong \boldsymbol{S}_5$, $S_1 \lhd S_1L$ and $S_1 \cap L = 1$. If $N(A)/A = S_1L$, then $N(A)$ acts transitively on the involutions in A and so z is conjugate to a. We see that $N_{G_2}(A)/A = S_1 \times L_1$, where $L_1 \subseteqq L$, and $L_1 \cong \boldsymbol{A}_5$. Thus z is conjugate in G_2 to a. Since $N_{G_2}(B) = N_{G_2}(A)^{t_2}$, it follows that $N_{G_2}(B)$ has the same structure as $N_{G_2}(A)$ whence all involutions in B are conjugate in $N_{G_2}(B)$. Thus G_2 has precisely one class of involutions in this case.

Now suppose that z is conjugate to a. By considering $C(a) \cap N(A)$ we see that $N(A) \nsubseteq N(T_2)$, whence $N(A)/A = S_1L$. From previous lemmas we see that no further fusion can occur in either case. The lemma is proved.

LEMMA 15. *In Case* (1) *of Lemma* 14 G *is isomorphic to the group* Aut$(PGL(3, 4))$.

PROOF. By a theorem of Suzuki [7] we see that G_2 is isomorphic to the group $PGL(3, 4)$. Since $C(G_2) = 1$ and from a comparison of orders, it follows that $G \cong \mathrm{Aut}(PGL(3, 4))$. The lemma is proved.

Because of Lemma 15 we henceforth assume that we are in Case (2) of Lemma 14. It follows that G_2 has precisely three classes of involutions with representatives z, a and t_1h_1a. We have

$$N_{G_2}(T_2) = N_{G_2}(A) = N_{G_2}(B) = T_2Q.$$

LEMMA 16. *In Case* (2) *of Lemma* 14 *the group* $N_{G_2}(T_2)$ *contains the centralizer in* G_2 *of each of its involutions.*

PROOF. Suppose the lemma to be false. Since $C_{G_2}(z) \subseteqq T_2Q$ and $a^{t_2} = t_1h_1a$ it follows that $C_{G_2}(a) \nsubseteq T_2Q$ and $C_{G_2}(t_1h_1a) \nsubseteq T_2Q$. Since $N_{G_2}(A) = T_2Q$, A is an S_2-subgroup of $C_{G_2}(a)$. The focal group A^* of A in $C_{G_2}(a)$ is equal to $Z(T_2)$ and so $C_{G_2}(a)$ has a normal subgroup M of index 4 such that $C_{G_2}(a) = AM$ and $M \cap A = Z(T_2)$ is an S_2-subgroup of M. We have $C_M(z) = Z(T_2)$, $N_M(Z(T_2)) = Z(T_2)P_1$ and $M \supset Z(T_2)P_1$. By a result of Suzuki [6], we get $M \cong \boldsymbol{A}_5$. Thus $C_{G_2}(a) = X \times M$ and we have $C_{G_2}(a) = C_{G_2}(b)$. Therefore, we have $C_{G_2}(x) = X \times M$ and both X and M are characteristic in $X \times M$. It follows that $N_{G_2}(X)/C_{G_2}(X)$ has order 3. An S_3-subgroup of $D = N_{G_2}(X)$ has order 9 and so $K = C_D(M) \cong \boldsymbol{A}_4$. Since $Z(T_2)P_1 \subseteqq M$, it follows that $K = XP$, and so $M \subseteqq V$ where V is

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the group defined in Lemma 11 such that $C(P) = P \times V$. But P_0 is contained in V, whence P_0 is contained in M. This contradiction proves the Lemma.

We can now apply Theorem 9.2.1 of [3] to the subgroup T_2Q of G_2 to conclude that $G_2 = T_2Q$. Thus in Case (2) of Lemma 14 we have $G = N(Z(T_2))$. This completes the proof of the Main Theorem of the Introduction.

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