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Decidability problem for exponential equations in finitely presented groups

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Abstract. We study the following decision problem: given an exponential equation $a_1 g_1^{x_1} a_2 g_2^{x_2} \ldots a_n g_n^{x_n} = 1$ over a recursively presented group G, decide if it has a solution with all x_i in $\mathbb{Z}$. We construct a finitely presented group G where this problem is decidable for equations with one variable and is undecidable for equations with two variables. We also study functions estimating possible solutions of such an equation through the lengths of its coefficients with respect to a given generating set of G. Another result concerns Turing degrees of some natural fragments of the above problem.

1 Introduction

An *exponential equation* over a group G is an equation of the form

$$a_1 g_1^{x_1} a_2 g_2^{x_2} \ldots a_n g_n^{x_n} = 1, \tag{1.1}$$

where $a_1, g_1, \ldots, a_n, g_n$ are elements from G and $x_1, \ldots, x_n$ are variables which take values in $\mathbb{Z}$. We always assume that G is given by a recursive presentation $\langle X \mid R \rangle$. In this paper, we study the *exponential equations problem* (briefly EE-problem), which is the following decision problem:

> *Given an exponential equation over* G, *decide if it has a solution, which is a tuple of integers.*

The study of exponential equations in groups was initiated by Myasnikov, Nikolaev, and Ushakov in [17], where they showed that the EE-problem is algorithmically decidable in any hyperbolic group G. According to [11], it is in **LogCFL**, a subclass of **P**. The study of problems related to the EE-problem and its complexity in various families of groups has become a very active area of investigations that uses methods of geometric and combinatorial group theory, automata, complexity theory, recursive functions, and logic (see [4–7, 9–13, 16]).

We mention results of Lohrey and Zetzsche on right-angled Artin groups and on virtually special groups [12], of Mishchenko and Treier on nilpotent groups [16], and of Dudkin, Treyer, Lohrey, and Zetzsche on Baumslag–Solitar groups $\mathrm{BS}(n, m)$ (see [4, 13]). König, Lohrey, and Zetzsche studied in [9] the EE-problem for the

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Heisenberg group $H_3(\mathbb{Z})$ and its direct products. Continuing the line of [17], Lohrey described solutions of exponential equations in hyperbolic groups (see [11]). Bier and Bogopolski showed in [1] that if G is a relatively hyperbolic group with respect to a finite collection of subgroups $\{H_1, \dots, H_n\}$, then the EE-problem for G reduces to the EE-problems for H_i's, provided some natural assumptions are satisfied.

In this paper, we study the EE-problem and its complexity in general, i.e., not focusing on a specific class of groups. For the forthcoming discussion, it is convenient to introduce the following definition.

Definition 1.1 Let G be a group, and let n be a fixed natural number. The $\mathrm{EE}[n]$-*problem* for G is the following problem: given an exponential equation over G with n variables, decide if it has a solution or not.

By $\mathrm{WP}(G)$ and $\mathrm{CP}(G)$, we denote the word and the conjugacy problems for G, respectively. Sometimes we omit G in these notations. We have the following relations among these decision problems:

$$\mathrm{WP} \Leftarrow \mathrm{EE}[1] \Leftarrow \mathrm{EE}[2] \Leftarrow \cdots \Leftarrow \bigcup_{i=1}^{\infty} \mathrm{EE}[i] \quad (= \mathrm{EE}).$$

The first implication follows from the equivalence $g = 1 \Leftrightarrow (\exists z \in \mathbb{Z})\,(g = 1^z)$; the other implications are obvious. Note that the $\mathrm{EE}\,[1]$-problem, i.e., the problem about the solvability of equations of kind $a^x = b$, is called the *power problem* (see [15, 20]). McCool proved in [15] that the implication $\mathrm{WP} \Rightarrow \mathrm{EE}[1]$ is not valid in general in the class of recursively presented groups. Ol'shanskii and Sapir have found a finitely presented example with decidable CP and undecidable $\mathrm{EE}[1]$ (see Theorem 1.3(2) in [21]). This motivated us to raise the following problem.

Problem 1 *For any $n \in \mathbb{N} \setminus \{0\}$, construct a finitely presented group G with decidable* $\mathrm{EE}[n]$ *and undecidable* $\mathrm{EE}[n+1]$.

The main result of this paper is the solution of Problem 1 for $n = 1$.

Theorem A *There exists a finitely presented group with decidable* $\mathrm{EE}[1]$ *and undecidable* $\mathrm{EE}[2]$. *Moreover, this group has decidable conjugacy problem.*

This theorem is proved in Sections 2 and 3.

The next issue concerns estimation of possible solutions of exponential equations over $G = \langle X \mid R \rangle$ by recursive functions on lengths of coefficients of these equations with respect to the generating set X. The motivation comes from the fact that this is a usual way to solve such equations. In Proposition 4.5, we show that primitive recursive functions are not sufficient for this aim. More information about the complexity of estimating functions for some interesting classes of groups can be found in Remarks 4.6 and 4.7.

In Section 5, we introduce decision problems $\mathrm{EE}[g, G^n]$ and $\mathrm{EE}[G, \overline{g}]$, which can be considered as fragments of $\mathrm{EE}\,[n]$ for G. We show that these fragments can have diverse recursively enumerable (r.e.) Turing degrees in the same finitely presented group. From a quite general Theorem 5.3, we deduce the following statement.

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Theorem B *There exists a finitely presented torsion-free group G with decidable conjugacy problem and undecidable* EE[1] *such that any r.e. Turing degree is realized as the Turing degree of the problem* $\mathrm{EE}[g, G]$ *for appropriate* $g \in G$.

We use methods from combinatorial group theory and some standard facts from computability theory. We also use variants of Higman embeddings developed by Ol'shanskii and Sapir in [18]–[20]. In the places where arguments are of computability theory flavor, we follow the terminology of [22] (in particular, we write "computable" instead of "recursive"). In the remaining parts of the paper, we keep the traditions of algorithmic group theory [14].

The following remark is not only a warning that some terminology used in this paper differs from that by other authors, but it also leads to an interesting mathematical problem.

Remark 1.2 Using conjugations, one can rewrite the exponential equation (1.1) in the equivalent form

$$f_1^{z_1} f_2^{z_2} \dots f_n^{z_n} = f_0. \tag{1.2}$$

In [17], the decision problem for these equations asking about solutions in $\mathbb{Z}$ is called the *integer knapsack problem* (IKP). The corresponding problem for $\mathbb{N}$ instead of $\mathbb{Z}$ is called *knapsack problem* (KP) in analogy with the optimization problem for natural numbers. Clearly, decidability of $\mathrm{KP}(G)$ implies decidability of IKP(G) (use inversions $f_i \mapsto f_i^{-1}$). We conjecture that the converse is not valid.

Problem 2 *Construct a recursively presented (finitely presented) group G for which there is an algorithm deciding if a given exponential equation over G has a solution with components in $\mathbb{Z}$, and there is no algorithm deciding the analogous question about solutions with components in $\mathbb{N}$.*

In our paper, we will often work with equations of the form (1.2) instead of (1.1).

2 A recursively presented group with decidable EE[1] and undecidable EE[2]

When G is a group given by a recursive presentation $\langle X \mid R \rangle$ and $w \in G$, we denote by $|w|_X$ the length of a shortest word in the alphabet $X \cup X^{-1}$ representing w. The free group generated by X is denoted by $\mathrm{F}(X)$. The length of $u \in \mathrm{F}(X)$ with respect to X will be often written as $|u|$. A syllable of u is a maximal subword of the form x^k, $k \in \mathbb{Z}$, where $x \in X$. When the word u is cyclically reduced, it can be viewed as a cyclic word, i.e., the set of all cyclic shifts of u.

The main purpose of this section is the following weaker version of Theorem A.

Proposition 2.1 *There exists a recursively presented group G such that* EE[1] *is decidable, but* EE[2] *is undecidable.*

In the proof of this proposition, we use the following lemmas. The first one is obvious.

Lemma 2.2 *Let w and u be two nontrivial elements of the free group* $F(X)$*. If* $w = u^z$ *for some* $z \in \mathbb{Z}$*, then* $|z| \leqslant |w|_X$*.*

Lemma 2.3 *Let* $w(a,b,c)$ *be a nonempty reduced cyclic word in* $F(a,b,c)$*, and let*

$$M = \max\{|z| : w \quad \text{has a subword of the form} \quad a^z \text{ or } b^z,\ z \in \mathbb{Z}\}.$$

Suppose that $m > M$*. Then* $w(a,b,a^m b^m) \neq 1$ *in* $F(a,b)$*. Moreover, if*

$$w(a,b,a^m b^m) = v(a,b)^z,$$

for some $v(a,b) \in F(a,b)$ *and* $z \in \mathbb{Z}$*, then* $|z| \leqslant |w(a,b,c)|$*.*

Proof We assume that w contains at least one c or c^{-1} (otherwise the statement is obvious).

After substitution $c \to a^m b^m$, the word w is uniquely factorized as $w_0 w_1 \dots w_k$ where every w_i with $i \notin \{0,k\}$ has one of the following forms for a reduced $u_i = u_i(a,b)$:

1. $b^m u_i a^m$,
2. $a^{-m} u_i b^{-m}$,
3. $b^m u_i b^{-m}$,
4. $a^{-m} u_i a^m$.

The word w_0 (resp. w_k) has the same form except that the initial (final) syllable b^m or a^{-m} (resp. a^m or b^{-m}) is missing. Note that in cases 3 and 4, the word u_i cannot be empty; otherwise, w would not be reduced.

Since u_i contains no exponent larger than M, the reduced normal form $\mathrm{red}(w_i)$ for $i \notin \{0,k\}$ is as follows. In case 1, it is of the form $b \dots a$; in case 2, it is of the form $a^{-1} \dots b^{-1}$; and in case 3, it is of the form $b \dots b^{-1}$ provided u_i contains an a-syllable. When u_i does not contain an a-syllable, $\mathrm{red}(w_i) = b^r$, where $1 \leq |r| < M$. Case 4 is similar to case 3 (with a instead of b).

Applying this analysis (with natural versions of it in the cases of w_0 and w_k) and using the observation that if w_i ends with a^m (resp. b^{-m}) then w_{i+1} starts with b^m (resp. a^{-m}), we see that

$$\mathrm{red}(w(a,b,a^m b^m)) = \mathrm{red}(w_0)\mathrm{red}(w_1) \cdot \dots \cdot \mathrm{red}(w_k).$$

Thus, $\mathrm{red}(w(a,b,a^m b^m))$ has at least two syllables, i.e., it is not empty.

For the second statement of the lemma, we may assume that $v(a,b)$ is cyclically reduced and when it starts with $a^{\pm 1}$ (resp. $b^{\pm 1}$), then it ends with $b^{\pm 1}$ (resp. $a^{\pm 1}$). This can be achieved using conjugations. Let n_1 be the number of syllables in $\mathrm{red}(w(a,b,a^m b^m))$, and let n_2 be the number of syllables in $v(a,b)$. Then $\min(n_1,n_2) \geqslant 2$ and $z = n_1/n_2 \leqslant n_1/2$. It remains to note that $n_1 \leqslant 2|w(a,b,c)|$. The latter is valid since, after substitution $c \to a^m b^m$ in w, the total number of a-syllables and b-syllables increases by at most $2k$, where k is the number of occurrences of $c^{\pm 1}$ in w. ∎

Proof of Proposition 2.1 Our construction resembles McCool's example from [15]. Let $f : \mathbb{N} \to \mathbb{N}$ be a one-to-one recursive function with nonrecursive range. Consider

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the following infinite presentation:

$$G = \Big\langle \bigcup_{i\in\mathbb{N}} \{a_i, b_i, c_i\} \mid c_{f(i)} = a^i_{f(i)} b^i_{f(i)} \ (i \in \mathbb{N})\Big\rangle. \tag{2.1}$$

Let $X = \bigcup_{i\in\mathbb{N}} X_i$, where $X_i = \{a_i, b_i, c_i\}$. Let H_i be the subgroup of G generated by X_i. Then

$$G = \mathop{*}_{j\in\mathbb{N}} H_j, \tag{2.2}$$

where each H_j is free and

$$\mathrm{rk}(H_j) = \begin{cases} 2, & \text{if} \quad j \in \mathrm{im} f, \\ 3, & \text{if} \quad j \notin \mathrm{im} f. \end{cases}$$

Claim 1 The word problem is decidable for the presentation (2.1).

Proof Using the normal form of an element of the free product (2.2), we reduce $\mathrm{WP}(G)$ to the following problem. Given $j \in \mathbb{N}$ and given a reduced nonempty word $w(a_j, b_j, c_j)$, decide whether the corresponding element of H_j is trivial or not. The difficulty is that we do not know whether $j \in \mathrm{im}(f)$ or not.

From now on, we consider $w(a_j, b_j, c_j)$ as a nonempty reduced cyclic word in $\mathrm{F}(a_j, b_j, c_j)$. Let M be the maximum of absolute values of exponents of a_j and b_j in the word $w(a_j, b_j, c_j)$.

First, we verify whether there exists $m \leqslant M$ with $j = f(m)$ or not. If such m exists, we substitute $a_j^m b_j^m$ for c_j in $w(a_j, b_j, c_j)$ and verify whether the resulting word is trivial in $\mathrm{F}(a_j, b_j)$ or not. This can be done effectively.

We claim that, in the remaining cases, the word w is nontrivial in G. Indeed, if $j \notin \mathrm{im} f$, then $H_j \cong \mathrm{F}(a_j, b_j, c_j)$, and hence $w(a_j, b_j, c_j)$ is nontrivial in H_j. If $j = f(m)$ for some $m > M$, then $w(a_j, b_j, c_j) = w(a_j, b_j, a_j^m b_j^m)$ is nontrivial in $\mathrm{F}(a_j, b_j)$ by Lemma 2.3. ■

Claim 2 The group G has undecidable $\mathrm{EE}[2]$.

Proof The equation $c_k = a_k^x b_k^y$ is solvable if and only if $k = f(i)$ for some i (in this case, $x = y = i$ is the unique solution). Since the set $\mathrm{im}(f)$ is not recursive, we cannot recognize whether such i exists or not. Therefore, we cannot recognize the existence of such x and y. ■

Claim 3 The group G has decidable $\mathrm{EE}[1]$.

Proof Consider an exponential equation

$$u = v^z, \tag{2.3}$$

where u and v are nontrivial words in the alphabet $X = \bigcup_{i\in\mathbb{N}} X_i$. In order to decide if it is solvable, we may assume that $u \neq 1$ and $v \neq 1$ in G. ■

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We write $v = v_1 v_2 \dots v_\ell$, where v_i is a word in the alphabet $X_{\lambda(i)}$ for some $\lambda(i)$, $i = 1, \dots, \ell$, and $\lambda(j) \neq \lambda(j+1)$ for $j = 1, \dots, \ell - 1$. Moreover (using decidability of $\mathrm{WP}(G)$), we assume that each v_i represents a nontrivial element of $H_{\lambda(i)}$. Using conjugation, we may additionally assume that $\lambda(1) \neq \lambda(\ell)$ if $\ell > 1$. Analogously, we write $u = u_1 u_2 \dots u_k$. Note that v_1 and u_1 (resp. v_ℓ and u_k) belong to the same subgroupH_j.

Suppose that $\ell > 1$. Then a necessary condition for solvability of equation (2.3) is $k > 1$. If this condition is fulfilled, then any possible solution z of equation (2.3) satisfies $|z| = k/\ell$, and the existence of a solution z can be verified using decidability of $\mathrm{WP}(G)$.

Let $\ell = 1$. Then a necessary condition for solvability of equation (2.3) is $k = 1$. Thus, we assume that u, v are words in the alphabet X_j for some j. We want to solve the equation

$$u(a_j, b_j, c_j) = v(a_j, b_j, c_j)^z. \tag{2.4}$$

Without loss of generality, we assume that $u(a_j, b_j, c_j)$ is a reduced cyclic word. Let M be the maximum of absolute values of exponents of a_j and b_j in $u(a_j, b_j, c_j)$.

First, we check whether some $m \in \{1, \dots, M\}$ satisfies $f(m) = j$. If such m is found, equation (2.4) takes the form

$$u(a_j, b_j, a_j^m b_j^m) = v(a_j, b_j, a_j^m b_j^m)^z,$$

and the solvability of this equation can be verified with the help of Lemma 2.2.

If no such m exists, then either $j \notin \operatorname{im} f$, or $j = f(m)$ for some $m > M$. We claim that, in these cases, the absolute value of a possible solution z of equation (2.4) does not exceed the length of the word $u(a_j, b_j, c_j)$ in $\mathrm{F}(a_j, b_j, c_j)$. Indeed, if $j \notin \operatorname{im} f$, then H_j is the free group with basis $\{a_j, b_j, c_j\}$, and the claim follows from Lemma 2.2. If $j \in \operatorname{im} f$, then the claim follows from Lemma 2.3.

Using the estimation for $|z|$ and decidability of $\mathrm{WP}(G)$, we can verify whether equation (2.4) has a solution. ■

Remark 2.4 One can show that the group G constructed in the proof of Proposition 2.1 has solvable conjugacy problem. However, we do not need this for the proof of Theorem A.

3 Proof of Theorem A

Below, we deduce Theorem A from Proposition 2.1 and the following result of Ol'shanskii and Sapir.

Theorem 3.1 (See [20, Theorem 1]) *Every countable group $G = \langle x_1, x_2, \dots \mid R \rangle$ with solvable power problem is embeddable into a 2-generated finitely presented group $\overline{G} = \langle y_1, y_2 \mid \overline{R} \rangle$ with solvable conjugacy and power problems.*

Remark 3.2 In this remark, we recall the main steps of the proof of Theorem 3.1. We do this to make clear that the embedding $\varphi : G \to \overline{G}$ constructed in the proof of this theorem is *computable*. This means that there exists an algorithm, which, given

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$i \in \mathbb{N}$, expresses x_i as a word in y_1 and y_2. Furthermore, these steps will be also used in arguments of Sections 4 and 5.

Four steps in the construction of Ol'shanskii and Sapir. Before we start, observe that any countable group $G = \langle x_1, x_2, \dots \mid R \rangle$ with solvable power problem has solvable word problem; hence, it admits a recursive presentation. Thus, we may assume that the given presentation of G is recursive. Moreover, the solvability of power problem implies the solvability of order problem (there exists an algorithm which computes orders of elements).

Step 1. In [3], Collins noticed that if H is a recursively presented group with solvable power problem and a, b are two elements in H of the same order, then the HNN extension $H_{a,b} = \langle H, t \mid t^{-1}at = b \rangle$ has solvable power problem.

Using a sequence of HNN extensions of this type, G can be embedded into a recursively presented group G_1 with solvable power problem where every two elements of the same order are conjugate. Thus, the conjugacy problem in G_1 is decidable. Moreover, the constructed embedding $\varphi_1 : G \to G_1$ (the identity map on the generators of G) is computable.

Step 2. In [18], Ol'shanskii suggested the following construction for embedding of countable groups into 2-generated groups. Let $H = \langle x_1, x_2, \dots \mid \mathcal{R} \rangle$ be any countable group. Denote by $\mathcal{R}_1$ the set of words in the alphabet $\{a, b\}$ obtained by substituting the word

$$A_i = a^{100} b^i a^{101} b^i \dots a^{199} b^i$$

for every x_i in every word from $\mathcal{R}$. It was shown in [18] that the map $x_i \mapsto A_i$, $i \in \mathbb{N}$, extends to an embedding of H into $H_1 = \langle a, b \mid \mathcal{R}_1 \rangle$. Lemmas 10 and 11 of [20] say that if the group H has decidable word or conjugacy problem or power problem, then the same problem is decidable for the group H_1.

Applying this construction, we obtain a computable embedding $\varphi_2 : G_1 \to G_2$, where $G_2 = \langle a, b \mid R_2 \rangle$ is 2-generated, recursively presented, and has solvable power and conjugacy problems.

Step 3. Lemma 12 of [20] says that this G_2 can be embedded into a finitely presented group $G_3 = \langle a, b, c_1, \dots, c_n \mid R_3 \rangle$ with solvable power and conjugacy problems. This embedding extends the identity map $a \mapsto a$, $b \mapsto b$, which is obviously computable.

The corresponding embedding was first described in [19]. We indicate that G_2 and G_3 play the roles of K and H in [19].

Step 4. Using the construction from Step 2 once more, we embed G_3 into a 2-generated finitely presented group $\overline{G} = \langle y_1, y_2 \mid \overline{R} \rangle$ with solvable conjugacy and power problems.

Since the embeddings at all steps are computable, their composition $\varphi : G \to \overline{G}$ is computable as well.

Proof of Theorem A By Proposition 2.1, there is a recursively presented group G with decidable EE[1] and undecidable EE[2]. Using Theorem 3.1 and Remark 3.2, we obtain a computable embedding $\varphi : G \to \overline{G}$, where $\overline{G}$ is finitely presented and has decidable EE[1]. Since φ is computable, undecidability of EE[2] for G implies undecidability of EE[2] for $\overline{G}$.

Indeed, consider an arbitrary equation $g_0 = g_1^x g_2^y$ with $g_0, g_1, g_2 \in G$ written as words in the generators of G. Using computability of φ, we can write $\varphi(g_0)$, $\varphi(g_1)$, $\varphi(g_2)$ as words in the generators of $\overline{G}$. The equation $\varphi(g_0) = \varphi(g_1)^x \varphi(g_2)^y$ has the same solutions as the original one. If we could decide whether this equation is solvable, we could decide whether the original equation is solvable. However, EE[2] is undecidable for G. Hence, it is undecidable for $\overline{G}$. ∎

Remark 3.3 The Knapsack counterpart of EE[2] is also undecidable in $\overline{G}$.

4 Estimating functions for solutions of exponential equations

Let G be a group generated by a set X. For any finite tuple $\bar{g} = (g_0, \dots, g_n)$ of elements of G, the ∞-*norm* of this tuple is the number

$$\|\bar{g}\|_X = \max\{|g_0|_X, \dots, |g_n|_X\}.$$

In the case where $G = \mathbb{Z}$ and $X = \{1\}$, we omit X and write $\|\bar{g}\|$.

Definition 4.1 Let G be a group generated by a set X. A function $f : \mathbb{N} \to \mathbb{N}$ is called an EE[n]-*estimating function* for G (with respect to X) if for any exponential equation $g_0 = g_1^{z_1} \cdot \dots \cdot g_n^{z_n}$ over G with nonempty set of solutions, there exists a solution $\bar{k} = (k_1, \dots, k_n)$ with

$$\|\bar{k}\| \leqslant f(\|(g_0, \dots, g_n)\|_X).$$

Remark 4.2 Let G be a group, and let X and Y be two generating sets of G. Suppose that

$$\sup_{y \in Y} |y|_X < \infty.$$

If there exists a (recursive) EE[n]-estimating function for G with respect to X, then there exists a (recursive) EE[n]-estimating function for G with respect to Y.

The following lemma relates decidability of EE[n] in G and existence of a *total recursive* EE[n]-estimating function. It is a counterpart of the fact that a group G with a finite generating set X has solvable WP if and only if the Dehn function of G with respect to X is total recursive.

Lemma 4.3 *Let G be a group generated by a finite set X. For any $n \in \mathbb{N}$, the following two conditions are equivalent.*

(1) EE[n] *is decidable in G.*
(2) WP(G) *is decidable, and there exists a total recursive* EE[n]*-estimating function for G with respect to X.*

Proof (1) ⇒ (2). Suppose that EE[n] is decidable in G. Then, clearly, WP(G) is decidable. Now, we define the desired function $f : \mathbb{N} \to \mathbb{N}$ at arbitrary point $m \in \mathbb{N}$ in four steps.

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(1) Let $B(m)$ be the set of all tuples $\bar{g} = (g_0, g_1, \dots, g_n)$ of words in the alphabet X satisfying $\|\bar{g}\|_X \leqslant m$. Since X is finite, the set $B(m)$ is finite and we can compute it.

(2) Let $B(m)'$ be the subset of $B(m)$ consisting of the tuples $\bar{g} = (g_0, g_1, \dots, g_n)$ for which the equation $g_0 = g_1^{x_1} \cdot \dots \cdot g_n^{x_n}$ has a solution. We can compute $B(m)'$ using decidability of EE$[n]$ in G. Note $(1, \dots, 1) \in B(m)'$.

(3) For each tuple $\bar{g} \in B(m)'$, we can find some solution $\bar{k} = (k_1, \dots, k_n)$ of the equation $g_0 = g_1^{z_1} \cdot \dots \cdot g_n^{z_n}$ using an effective enumeration of n-tuples of integers and the decidability of WP(G). We denote this solution by $\bar{k}(\bar{g})$. When $\bar{g} = \bar{1}$, we put $\bar{k}(\bar{g}) = \bar{1}$.

(4) Finally, we set $f(m)$ to be the maximum of $\|\bar{k}(\bar{g})\|$ over all $\bar{g} \in B(m)'$.

The function f is total recursive and satisfies Definition 4.1.

(2) $\Rightarrow$ (1). Consider an exponential equation $g_0 = g_1^{z_1} \cdot \dots \cdot g_n^{z_n}$ over G. To decide whether this equation has a solution, we verify whether the equality $g_0 = g_1^{k_1} \cdot \dots \cdot g_n^{k_n}$ holds for at least one tuple $\bar{k} = (k_1, \dots, k_n) \in \mathbb{Z}^n$ with $\|\bar{k}\| \leqslant f(\|\bar{g}\|_X)$. The verification for a concrete tuple $\bar{k}$ can be done using WP(G). ■

Remark 4.4 In [8], Kharlampovich constructed a group G which is finitely presented in the variety $x^m = 1$ and has undecidable word problem. By Lemma 4.3, EE$[n]$ is undecidable for each n. On the other hand, the constant function $f(k) = m$, $k \in \mathbb{N}$, is a total recursive EE$[n]$-estimating function for G.

The following proposition shows that there is a finitely presented group with decidable EE[1] which does not have a primitive recursive EE[1]-estimating function.

Proposition 4.5 *There exists a finitely presented group $G = \langle X | \mathcal{R} \rangle$ with decidable* EE[1]*, and there exists a collection of elements $(c_n)_{n\in\mathbb{N}}$ of G such that the following holds.*

(1) *For any n, the equation $c_1 = c_n^x$ has a unique solution, say k_n; this solution is positive.*

(2) *There is no primitive recursive function f such that $k_n \leqslant f(\max\{|c_n|_X, |c_1|_X\})$.*

Proof We enumerate all primitive recursive functions $g_1, g_2, \dots$ and, for any $n \in \mathbb{N}$, we define a function $f_n : \mathbb{N} \to \mathbb{N}$ by the rule

$$f_n(x) = \sum_{i=1}^{n} \sum_{j=1}^{x} g_i(j), \quad x \in \mathbb{N}.$$

Clearly, f_n is primitive recursive, nondecreasing, $g_n \leqslant f_n$, and $f_n \leqslant f_{n+1}$. Finally, we define a function $F : \mathbb{N} \to \mathbb{N}$ by the rule

$$F(n) = n!(f_n(100n + 14950) + 1).$$

Clearly, F is recursive. We also define rational numbers $c_1 = 1$ and $c_n = \frac{1}{F(n)}$ for $n \geqslant 2$. Then we fix a recursive presentation (written multiplicatively) for the group $(\mathbb{Q}, +)$:

$$\langle \{q : q \in \mathbb{Q}\} \,|\, \mathcal{C}_{\mathbb{Q}} \rangle,$$

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where $\mathcal{C}_{\mathbb{Q}}$ is the Cayley table for $\mathbb{Q}$. Note that $c_1, c_2, \dots$ appear in this presentation and the equalities $c_n^{F(n)} = c_1 (n, m \in \mathbb{N})$ follow from $\mathcal{C}_{\mathbb{Q}}$. We may assume that they are in $\mathcal{C}_{\mathbb{Q}}$. It is clear that EE[1] is decidable for this presentation. By the choice of $F(n)$, the elements $c_1, c_2, \dots$ generate $\mathbb{Q}$. By some obvious transformations, we obtain a recursive presentation of $\mathbb{Q}$ in the form

$$\langle c_1, c_2, \dots \mid R_0 \rangle,$$

so that EE[1] is decidable. We embed $\mathbb{Q}$ into a finitely presented group G_3 by the Ol'shanskii–Sapir construction, which we described in Steps 1–4 in Section 3. Note that since $\mathbb{Q}$ has decidable conjugacy problem, we do not need to do Step 1. Thus, we start with Step 2, where we use the following map.

- Let φ_2 map each c_i to the word $a^{100} b^i a^{101} b^i \dots a^{199} b^i$ (of length $100i + 14950$), $i \in \mathbb{N}$.

By Step 2, φ_2 extends to an embedding $\varphi_2 : \mathbb{Q} \to G_2$, where the group $G_2 = \langle a, b \mid R_2 \rangle$ is 2-generated, recursively presented, and has solvable power and conjugacy problems. Then we only apply Step 3. By this step, the map $a \mapsto a$, $b \mapsto b$ extends to an embedding $\varphi_3 : G_2 \to G_3$, where $G_3 = \langle X \mid R_3 \rangle$ is a finite presentation with solvable power and conjugacy problems, and $\{a, b\} \subseteq X$. We set $G = G_3$.

The statement (1) is valid: for any n, the equation $c_n^x = c_1$ has a unique solution, namely $k_n = F(n)$. To prove statement (2), we first observe that

$$\max\{|c_n|_X, |c_1|_X\} \leqslant \max\{|c_n|_{\{a,b\}}, |c_1|_{\{a,b\}}\} \leqslant 100n + 14950. \tag{4.1}$$

Suppose that statement (2) is not valid, i.e., there exists a primitive recursive function g_m such that

$$k_n \leqslant g_m(\max\{|c_n|_X, |c_1|_X\}) \tag{4.2}$$

for any n. Using that $g_m \leqslant f_m$ and that f_m is nondecreasing, we deduce from (4.1) and (4.2) that

$$F(n) \leqslant f_m(100n + 14950)$$

for any n. In particular, $F(m) \leqslant f_m(100m + 14950)$. This contradicts the definition of F. ∎

Remark 4.6 Theorem 4.2 in [13] states that the KP in the Baumslag–Solitar group $\mathrm{BS}(1,2)$ is **NP**-computable, but the EE[3]-estimation function for this group cannot be essentially smaller than the doubly exponential function. This statement can be considered as a counterpart of Proposition 4.5 at the level of polynomial computability.

Remark 4.7 However, for hyperbolic groups, EE[n]-estimating functions can be chosen to be linear for any n (a polynomial estimation was known earlier; see [17]). This follows from the preprint [1] of the first-named author and Bier. It is proved in [1] that similar linearity result holds for acylindrically hyperbolic groups in the case of loxodromic coefficients g_i.

5 Restricted versions of EE$[n]$

We introduce two new algorithmic problems which can be considered as fragments of EE$[n]$. Informally we call them the *left and the right fragments* of EE$[n]$. We show that these fragments can take diverse computational complexities for the same finitely presented group (see Theorem B).

5.1 Definitions and observations

Below, we assume that G is given by a recursive presentation and X is the corresponding set of generators.

Definition 5.1 (1) Let $g_1, \dots, g_n \in G$. By EE$[G, g_1, \dots, g_n]$, we denote the set of all $g \in G$ such that the equation $g = g_1^{z_1} \cdot \dots \cdot g_n^{z_n}$ has a solution which is a tuple of integers. (2) For a fixed $g \in G$, let EE$[g, G^n]$ be the set consisting of all tuples $(g_1, \dots, g_n) \in G^n$ such that the equation $g = g_1^{z_1} \cdot \dots \cdot g_n^{z_n}$ has a solution which is a tuple of integers.

Note that for a tuple of units $\bar{1}$, the membership problem for EE$[G, \bar{1}]$ is equivalent to the word problem. Decidability of the problem EE$[n]$ is a uniform form of decidability of all EE$[G, \bar{g}]$ (resp. EE$[g, G^n]$). Indeed, if for each $g \in G$ there is an algorithm (provided by the word g in an effective way) which decides the membership problem for EE$[g, G^n]$, then EE$[n]$ is decidable. The similar statement holds for problems EE$[G, g_1, \dots, g_n]$.

Remark 5.2 Suppose that G is a group given by a recursive presentation. Let g be a nontrivial element of G. Suppose that EE$[G, g]$ is decidable in G and the order of g is known. Then WP is decidable in G.

Indeed, in order to determine whether a given h is trivial in G, we first verify whether h is a power of g. If h is not a power of g, then $h \neq 1$. If h is a power of g, we start a diagonal computation for verification of the following equalities: $h = 1$, $h = g$, $\dots, h = g^k, \dots$. Here, we use the recursive presentation of G. At some stage, we will find a number k with $h = g^k$. Since the order of g is known, we can check whether $h = 1$ or not.

> Given a group G and a natural number $n \geqslant 1$, how diverse can be algorithmic complexities of the problems EE$[g, G^n]$, and EE$[G, g_1, \dots, g_n]$, where $g, g_1, \dots, g_n$ run over G? How these complexities are related to the complexity of the problem EE$[n]$?

A partial answer to these problems (in the case where G is finitely presented) is given in Theorem B.

5.2 Example

Let p_n denote the nth prime number. For any function $F : \mathbb{N} \to \mathbb{N}^2$ and any $n \in \mathbb{N}$, we denote $(\mathrm{im}F)_n = \{m \in \mathbb{N} \mid (n, m) \in \mathrm{im}(F)\}$ and write $F = (F_1, F_2)$.

Let $F : \mathbb{N} \to \mathbb{N}^2$ be a total, injective, computable function such that, for any $n \in \mathbb{N}$, we have either $(\mathrm{im}F)_n = \varnothing$ or

$$p_n \in (\mathrm{im}F)_n \subseteq \{p_n^k \mid k \in \mathbb{N} \setminus \{0\}\}.$$

Thus, all sets $(\mathrm{im}F)_n$, $n \in \mathbb{N}$, are pairwise disjoint. We put

$$X = \{a_n \mid n \in \mathbb{N}\} \cup \{b_m \mid m \text{ is a power of a prime number}, m \neq 1\}$$

and consider the group with the following recursive presentation:

$$G = \big\langle X \mid \mathcal{R}_1 \cup \mathcal{R}_2 \big\rangle, \tag{5.1}$$

where

$$\mathcal{R}_1 = \{[a_n, b_m] = 1 \mid n, m \in \mathbb{N},\ m = p_n^k \ \text{ for some } \ k \in \mathbb{N} \setminus \{0\}\},$$
$$\mathcal{R}_2 = \{a_{F_1(n)} = b^n_{F_2(n)} \mid n \in \mathbb{N}\}.$$

Theorem 5.3 *For the above defined group G, the following statements are valid.*

(1) $\mathrm{CP}(G)$, $\mathrm{EE}[1, G]$, *and* $\mathrm{EE}[G, 1]$ *are decidable.*
(2) $\mathrm{EE}[1]$ *is undecidable for G if the set* $\mathrm{im}\,(F_1)$ *is not computable.*
(3) *For any fixed* $g_0 \in G$, *the problem* $\mathrm{EE}[g_0, G]$ *(resp.* $\mathrm{EE}[G, g_0]$*) is decidable or there is a number n such that* $\mathrm{EE}[g_0, G]$ *(resp.* $\mathrm{EE}[G, g_0]$*) is Turing reducible to* $(\mathrm{im}\,F)_n$. *Each of these possibilities can be effectively recognized, and the corresponding number n can be computed.*
(4) *If* $n \in \mathrm{im}\,F_1$, *then the problem* $\mathrm{EE}[a_n, G]$ *is computably equivalent to the membership problem for* $(\mathrm{im}F)_n$.

Proof Before we start to prove these statements, we establish the structure of G. We decompose $X = \bigcup_{i \in \mathbb{N}} X_i$, where

$$X_i = \{a_i\} \cup \{b_j \mid j \text{ is a power of } p_i\}.$$

Let H_i be the subgroup of G generated by X_i. Then

$$G = \mathop{*}_{i \in \mathbb{N}} H_i. \tag{5.2}$$

To describe the structure of H_i, we first introduce the following subgroups of H_i:

$$H_i^- = \langle b_j \mid j \text{ is a power of } p_i \text{ satisfying } j \notin (\mathrm{im}F)_i \rangle,$$

$$H_i^+ = \langle b_j \mid j \text{ is a power of } p_i \text{ satisfying } j \in (\mathrm{im}F)_i \rangle.$$

Then H_i^- is the free product of all its subgroups $\langle b_j \rangle$, and H_i^+ is the amalgamated product over $\langle a_i \rangle$ of all its subgroups $\langle b_j \rangle$. Moreover, we have

$$H_i = (\langle a_i \rangle \times H_i^-) \mathop{*}_{\langle a_i \rangle} H_i^+. \tag{5.3}$$

Note that $\langle a_i \rangle$ is the center of H_i.

Before we start the proof of statement (1), we make the following important observation.

Observation. Let $a_i, b_j \in X$ and $k \in \mathbb{Z} \setminus \{0\}$. Then b_j^k is a power of a_i if and only if j is a power of p_i and there exists a positive divisor d of k such that $F(d) = (i, j)$. We can recognize the existence of such d since F is computable. If such d exists, then $a_i = b_j^d$ and hence $a_i^{k/d} = b_j^k$. ∎

Proof of statement (1) First, we prove that $\mathrm{WP}(G)$ is decidable. Using the normal form of an element of the free product (5.2), we reduce this problem to the following one. Given $i \in \mathbb{N}$ and given a cyclically reduced nonempty word $a_i^s w(\bar{b})$, where $w(\bar{b})$ is over $X_i \setminus \{a_i\}$, decide whether the corresponding element of H_i is trivial or not.

We may assume that the word $w(\bar{b})$ is nonempty. Indeed, otherwise $a_i^s w(\bar{b})$ lies in the cyclic subgroup $\langle a_i \rangle$ of H_i and therefore is trivial exactly when $s = 0$.

Using the above observation, we verify whether some subword b_j^k of $w(\bar{b})$ is a power of a_i or not. If no one such subword is a power of a_i, then the element $a_i^s w(\bar{b})$ is nontrivial in the amalgamated product (5.3). Suppose that some subword b_j^k of $w(\bar{b})$ is a power of a_i, say $a_i^\ell = b_j^k$. Since a_i lies in the center of H_i, we can move this subword to the left and adjoin to a^s. After this operation, $|w(\bar{b})|_{X_i \setminus \{a_i\}}$ decreases and we can proceed by induction. ∎

Now, we show that the conjugacy problem in G is decidable. Using (5.2), we reduce this problem to the conjugacy problem in the groups H_i, $i \in \mathbb{N}$. By (5.3), each H_i is an amalgamated product over the center of H_i. This fact, the decidability of $\mathrm{WP}(G)$, and a criterion for conjugacy of elements in amalgamated products (see [14, Chapter IV, Theorem 2.8]), imply that there is a universal algorithm deciding the conjugacy problem in each H_i and hence in G.

Decidability of $\mathrm{EE}[G,1]$ follows from decidability of the word problem, and for decidability of $\mathrm{EE}[1,G]$, observe that G is torsion-free.

Proof of statement (2) This statement easily follows from the equivalence

$$n \in \mathrm{im} F_1 \iff a_n \text{ is a power of } b_{p_n}.$$

Indeed, if $\mathrm{im} F_1$ is not computable, we cannot decide, given $n \in \mathbb{N}$, whether the equation $a_n = b_{p_n}^x$ has a solution or not. ∎

Proof of statement (3) For a fixed element $g_0 \in G$, we study the problem $\mathrm{EE}[g_0, G]$. Given another element $g_1 \in G$, we shall consider the exponential equation

$$g_0 = g_1^z.$$

We may assume that $g_0 \neq 1$; otherwise, $\mathrm{EE}[g_0, G]$ is decidable since G is torsion-free and $\mathrm{WP}(G)$ is decidable. Having $g_0 \neq 1$, we may assume that $g_1 \neq 1$. Standardly, we assume that g_0 and g_1 are represented by some words u and v in the alphabet $X = \bigcup_{i \in \mathbb{N}} X_i$. Thus, we consider the following exponential equation in G:

$$u = v^z. \tag{5.4}$$

We write $v = v_1 v_2 \dots v_\ell$, where v_i is a word in the alphabet $X_{\lambda(i)}$ for some $\lambda(i)$, $i = 1, \dots, \ell$, and $\lambda(j) \neq \lambda(j+1)$ for $j = 1, \dots, \ell - 1$. Moreover, we assume that each v_i represents a nontrivial element of $H_{\lambda(i)}$. This can be recognized by decidability of WP(G). Using conjugation, we may additionally assume that $\lambda(1) \neq \lambda(\ell)$ if $\ell > 1$. Analogously, we write $u = u_1 u_2 \dots u_k$. Note that v_1 and u_1 (resp. v_ℓ and u_k) belong to the same subgroup H_j.

Suppose that $\ell > 1$. Then a necessary condition for solvability of equation (5.4) is $k > 1$. If this condition is fulfilled, then any possible solution z of equation (5.4) satisfies $|z| = k/\ell$, and the existence of a solution z can be verified using decidability of WP(G).

Note that until this moment the corresponding algorithm is uniform on u and v. In the following case, we will call some oracle depending on u.

Suppose that $\ell = 1$. Then $u \in H_n$ for $n = \lambda(1)$, and this n can be determined using the definition of X_n. Using the procedure described in the proof of decidability of WP(G), we write u in the normal form with respect to the amalgamated product (5.3), i.e., we write $u = a_n^s b_{i_1}^{s_1} \dots b_{i_r}^{s_r}$ where, in particular, each $b_{i_1}^{s_1}, \dots, b_{i_r}^{s_r}$ does not have a subword which is a power of a_n. By using conjugation, we may additionally assume that u is cyclically reduced in the sense that $i_1 \neq i_r$ if $r > 1$.

Furthermore, we may now assume that v belongs to H_n too. We also write v in the normal form with respect to the amalgamated product (5.3), $v = a_n^t b_{j_1}^{t_1} \dots b_{j_q}^{t_q}$.

If $r > 1$, then a necessary condition for solvability of equation (5.4) is $q > 1$. In this case, any possible solution z of (5.4) satisfies $|z| = r/q$, and the existence of the corresponding z can be verified using decidability of WP(G).

Suppose that $r = 1$. Then a necessary condition for solvability of (5.4) is $q = 1$ and $i_1 = j_1$. In this case, we only need to verify the existence of z satisfying (5.4) in the group $\langle a_n, b_{i_1} \rangle$. This is the only place where we need the oracle for $(\mathrm{im}F)_n$. Verifying whether $p_n \in (\mathrm{im}F)_n$, we decide if $n \in \mathrm{im}F_1$. If this happens, we easily compute in the oracle $(\mathrm{im}F)_n$ the relation from the presentation (5.1) of the form $a_n = b_{i_1}^m$ if it exists, and if it does not exist we recognize this. In the latter case, any possible solution z of equation (5.4) satisfies $|z| \leqslant |u|_{X_n}$. In the former case, $\langle a_n, b_{i_1} \rangle = \langle b_{i_1} \rangle$. Substituting $b_{i_1}^m$ instead of a_n both in u and v, we obtain an equation in the cyclic group $\langle b_{i_1} \rangle$, and the number z can be computed. This gives an algorithm which is computable with respect to $(\mathrm{im}\, F)_n$.

The case $r = 0$ is trivial, and we leave it to the reader.

This completes the proof of statement (3) for EE$[g_0, G]$. The argument for EE$[G, g_0]$ is analogous. ■

Proof of statement (4) Let $n \in \mathrm{im}\, F_1$. The following equivalence recognizes $j \in (\mathrm{im}\, F)_n$ under the oracle for EE$[a_n, G]$:

$$j \in (\mathrm{im}\, F)_n \iff j \text{ is of the form } p_n^k \text{ and } a_n \text{ is a power of } b_j. \qquad ■$$

Remark 5.4 Statements (1) and (4) also hold for the corresponding versions of the KP (where one looks for solutions in $\mathbb{N}$).

Theorem B *There exists a finitely presented torsion-free group G with decidable conjugacy problem and undecidable* EE[1] *such that any r.e. Turing degree is realized as the Turing degree of the problem* EE$[g, G]$ *for appropriate $g \in G$.*

Proof First, we construct a recursively presented group G with these properties. Let $\varphi(x, y)$ be Kleene's universal computable function, i.e., it finds the output (if it

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exists) of the Turing machine with index x on input y. Let

$$W = \{(x,z) \mid \exists y\,(z = \varphi(x,y))\},$$

and let $\Phi : \mathbb{N} \to \mathbb{N}^2$ be a total, injective, computable function with $\operatorname{im}\Phi = W$. Below, we use notations introduced at the beginning of this subsection. Obviously, the sets $(\operatorname{im}\Phi)_n$, $n \in \mathbb{N}$, have all possible r.e. Turing degrees. Now, we extend the set W as follows:

$$\widehat{W} = (W \cup \{(x,1) \mid \exists z : (x,z) \in W\}) \setminus \{(x,0) \mid \exists z : (x,z) \in W\}.$$

Let $\widehat{\Phi} : \mathbb{N} \to \mathbb{N}^2$ be a total, injective, computable function with $\operatorname{im}\widehat{\Phi} = \widehat{W}$. We have $(\operatorname{im}\widehat{\Phi})_n = ((\operatorname{im}\Phi)_n \setminus \{0\}) \cup \{1\}$ for nonempty $(\operatorname{im}\Phi)_n$ and $n \in \mathbb{N}$. Therefore, the sets $(\operatorname{im}\widehat{\Phi})_n$, $n \in \mathbb{N}$, have all possible r.e. Turing degrees as well.

Now, we define a function $F : \mathbb{N} \to \mathbb{N}^2$ by the formula

$$F = f \circ \widehat{\Phi},$$

where $f : \mathbb{N}^2 \to \mathbb{N}^2$ is the function sending each (n,m) to (n, p_n^m). The function F satisfies the conditions formulated at the beginning of this subsection, since it is total, injective, computable, and for any $n \in \mathbb{N}$, we have either $(\operatorname{im}\Phi)_n = \varnothing$ or

$$p_n \in (\operatorname{im} F)_n \subseteq \{p_n^k \mid k \in \mathbb{N} \setminus \{0\}\}.$$

Finally, we define a recursively presented group G by formula (5.1) and apply Theorem 5.3. By statements (1) and (2) of this theorem, $\mathrm{CP}(G)$ is decidable and $\mathrm{EE}[1]$ is undecidable for G.

The statement of Theorem B about Turing degrees follows from statement (4) of Theorem 5.3, which says that, for any $n \in \mathbb{N}$, the problem $\mathrm{EE}[a_n, G]$ is computably equivalent to the membership problem for $(\operatorname{im} F)_n$. It remains to note that

$$(\operatorname{im} F)_n = \{p_n^m \mid m \in (\operatorname{im}\widehat{\Phi})_n\};$$

therefore, these sets have all possible r.e. Turing degrees.

In the second part of the proof, we embed the group G into a finitely presented group $\overline{G}$ using Ol'shanskii–Sapir construction explained in Remark 3.2. Note that we do not need to do Step 1 there since G already has solvable conjugacy problem, and we do not need to do Step 4 since we do not specially want $\overline{G}$ to be 2-generated.

Thus, using notations of this remark, we may assume that $G = G_1$ and that we have embeddings $G_1 \to G_2 \to G_3$, where $G_3 = \overline{G}$. Simplifying notation, we assume $G_1 \leqslant G_2 \leqslant G_3$. By this construction, G_3 has solvable conjugacy problem if G_1 has solvable conjugacy problem. The latter is valid, and hence $\overline{G}$ has solvable conjugacy problem. It remains to prove the following claim. ∎

Claim For any $g \in G$, the problems $\mathrm{EE}[g, G_1]$ and $\mathrm{EE}[g, G_3]$ are computationally equivalent.

Proof The computational equivalence of $\mathrm{EE}[g, G_1]$ and $\mathrm{EE}[g, G_2]$ follows from the proof of Lemma 11 in [20]. (We stress that we use the proof and not the formulation of this lemma which requires solvability of power problem in G_1.) Indeed, given

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an exponential equation $g = u^z$ with $u \in G_2$, the proof (depending on u) either recursively reduces this equation to $\mathrm{EE}[g, G_1]$ or gives a linear upper bound for $|z|$ in terms of g and u.

The computational equivalence of $\mathrm{EE}[g, G_2]$ and $\mathrm{EE}[g, G_3]$ analogously follows from the proof of Lemma 12 in [20]. ∎

6 Complexity of the problem $\mathrm{EE}[n]$

Applying the approach of [2], we obtain the following proposition.

Proposition 6.1 (1) *Detecting a group with decidable* $\mathrm{EE}[n]$ *is* Σ_3^0 *in the class of recursively presented groups.*

(2) *The same statement holds for the* EE*-problem and the KP.*

Proof We will use standard terminology from [22]. Kleene's universal computable function $\varphi(x, y)$ will be applied to several families of objects. As usual, these objects are coded by natural numbers.

Take a computable indexation $G_i = \langle X \,|\, \mathcal{R}_i \rangle$, $i \in \omega$, of all recursively presented groups with respect to generators $X = \{x_1, x_2, \ldots\}$. Fix an algorithm which for the input (i, s) outputs the sth equality of the form $w = 1$ satisfied in G_i. We see that the set of pairs (G_i, w), where $G_i \vDash (w = 1)$ with $w \in \mathrm{F}(X)$, is computably enumerable. We use the notation

$$\mathrm{EE}[n](G_i) = \{(w_0, w_1, \ldots, w_n) \in G_i^{n+1} \,|\, \exists z_1 \ldots z_n \in \mathbb{Z}\,(w_0 = w_1^{z_1} \cdot \cdots \cdot w_n^{z_n})\}.$$

There exists a computable enumeration of the set of pair $(G_i, \bar{w})$ where $\bar{w} = (w_0, w_1, \ldots, w_n)$ belongs to $\mathrm{EE}[n](G_i)$. Thus, the set

$$I_{\bar{w}} = \{G_i \,|\, (w_0, w_1, \ldots, w_n) \in \mathrm{EE}[n](G_i)\}$$

is computably enumerable. These sets belong to Σ_1^0. On the other hand, the set

$$\bar{I}_{\bar{w}} = \{G_i \,|\, (w_0, w_1, \ldots, w_n) \notin \mathrm{EE}[n](G_i)\}$$

belongs to Π_1^0. The property $(w_0, w_1, \ldots, w_n) \notin \mathrm{EE}[n](G_i)$ exactly means that, for any $(s_0, s_1, \ldots, s_n)$, the equality $w_0 = w_1^{s_1} \cdot \cdots \cdot w_n^{s_n}$ is not recognized in G_i at step $|s_0|$. Based on these observations, we formulate decidability of $\mathrm{EE}[n]$ for G_i as follows.

> There is a number $m \in \mathbb{N}$ such that, for any tuple $w_0, w_1, \ldots, w_n \in \mathrm{F}(X)$ and any $(s_0, s_1, \ldots, s_n) \in \mathbb{Z}^{n+1}$, there exist numbers $\ell, k \in \mathbb{N}$ such that the following properties hold:
>
> - The algorithm $\varphi(m, .)$ applied to the code of $\bar{w}$ gives the value 0 or 1 at step ℓ.
> - The algorithm $\varphi(m, .)$ applied to the code of $\bar{w}$ gives the value 0 at step ℓ or the membership $G_i \in I_{\bar{w}}$ is confirmed at step k of computation.
> - The algorithm $\varphi(m, .)$ applied to the code of $\bar{w}$ gives the value 1 at step ℓ or the equality $w_0 = w_1^{s_1} \cdot \cdots \cdot w_n^{s_n}$ is not recognized at step $|s_0|$.

The second statement of the proposition is similar. ∎

Problem 3 *Are* EE[n] *and the KP* Σ^0_3*-complete in the class of recursively presented groups?*

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References

[1] A. Bier, and O. Bogopolski, *Exponential equations in acylindrically hyperbolic groups*. Preprint, 2021. arXiv:2106.11385v1

[2] I. Bilanovic, J. Chubb, and S. Roven, *Detecting properties from presentations of groups*. Arch. Math. Log. **59**(2020), 293–312.

[3] J. Donald, *Collins, on embedding groups and the conjugacy problem*. J. Lond. Math. Soc. (2) **1**(1996), 674–682.

[4] F. Dudkin and A. Treyer, *Knapsack problem for Baumslag–Solitar groups*. Siberian J. Pure Appl. Math. **18**(2018), no. 4, 43–55.

[5] M. Figelius, M. Ganardi, M. Lohrey, and G. Zetzsche, *The complexity of knapsack problems in wreath products*. In: *Proceedings of the 47th international colloquium on automata, languages, and programming, ICALP 2020*, Leibniz International Proceedings in Informatics, 168, Schloss Dagstuhl—Leibniz-Zentrum für Informatik, Germany, 2020, pp. 126:1–126:18.

[6] E. Frenkel, A. Nikolaev, and A. Ushakov, *Knapsack problems in products of groups*. J. Symb. Comput. **76**(2016), 96–108.

[7] M. Ganardi, D. König, M. Lohrey, and G. Zetzsche, *Knapsack problems for wreath products*. In: *Proceedings of STACS (2018)*, Leibniz International Proceedings in Informatics, 96, Schloss Dagstuhl—Leibniz-Zentrum für Informatik, Germany, 2018, pp. 1–13.

[8] O. Kharlampovich, *The word problem for the Burnside varieties*. J. Algebra **173**(1995), 613–621.

[9] D. König, M. Lohrey, and G. Zetzsche, *Knapsack and subset sum problems for nilpotent, polycyclic, and co-context-free groups*. In: *Algebra and computer science*, Contemporary Mathematics, 677, American Mathematical Society, Providence, RI, 2016, pp. 138–153.

[10] M. Lohrey, *Rational subsets of unitriangular groups*. Int. J. Algebra Comput. **25**(2015), nos. 1–2, 113–121.

[11] M. Lohrey, *Knapsack in hyperbolic groups*. J. Algebra **545**(2020), no. 1, 390–415.

[12] M. Lohrey and G. Zetzsche, *Knapsack in graph groups, HNN-extensions and amalgamated products*. Theory Comput. Syst. **62**(2018), no. 1, 192–246.

[13] M. Lohrey and G. Zetzsche, *Knapsack and the power word problem in solvable Baumslag–Solitar groups*. In: *Proceedings of the 45th international symposium on mathematical foundations of computer science, MFCS 2020*, Leibniz International Proceedings in Informatics, 170, Schloss Dagstuhl—Leibniz-Zentrum für Informatik, Germany, 2020, pp. 67:1–67:15.

[14] R. C. Lyndon and P. E. Schupp, *Combinatorial group theory*, Springer, Berlin, 1977.

[15] J. McCool, *Unsolvable problems in groups with solvable word problem*. Can. J. Math. **22**(1970), 836–838.

[16] A. Mishchenko and A. Treier, *Knapsack problem for nilpotent groups*. Groups Complex. Cryptol. **9**(2017), no. 1, 87–98.

[17] A. Myasnikov, A. Nikolaev, and A. Ushakov, *Knapsack problems in groups*. Math. Comp. **84**(2015), no. 292, 987–1016.

[18] A. Yu. Ol'shanskii, *SQ-universality of hyperbolic groups*. Mat. Sb. **186**(1995), no. 8, 119–132.

[19] A. Y. Ol'shanskii and M. V. Sapir, *The conjugacy problem and Higman embeddings*. Mem. Amer. Math. Soc. **170**(2004), no. 804, vii + 131.

[20] A. Y. Ol'shanskii and M. V. Sapir, *Subgroups of finitely presented groups with solvable conjugacy problem*. Int. J. Algebra Comput. **15**(2005), nos. 5–6, 1–10.

[21] A. Yu. Ol'shanskii and M. V. Sapir, *Algorithmic problems in groups with quadratic Dehn function*. Preprint, 2020. arXiv:2012.10417v1

[22] R. I. Soare, *Turing computability. Theory and applications*, Springer, Berlin, 2016.

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