Canad. J. Math. Vol. **61** (3), 2009 pp. 617–640

Square Integrable Representations and the Standard Module Conjecture for General Spin Groups

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Abstract. In this paper we study square integrable representations and L-functions for quasisplit general spin groups over a p-adic field. In the first part, the holomorphy of L-functions in a half plane is proved by using a variant form of Casselman's square integrability criterion and the Langlands–Shahidi method. The remaining part focuses on the proof of the standard module conjecture. We generalize Muić's idea via the Langlands–Shahidi method towards a proof of the conjecture. It is used in the work of M. Asgari and F. Shahidi on generic transfer for general spin groups.

Introduction

This paper concerns the representation theory of general spin groups over a p-adic field. We shall study square integrable representations and use them to prove the standard module conjecture for general spin groups which also implies the validity of standard module conjecture for spin groups (Proposition 5.6). The importance of the standard module conjecture for general spin groups lies in its application to the proof of a functorial lift from general spin groups to general linear groups [2].

To be specific, let $\mathbf{G}$ be a connected reductive quasisplit algebraic group defined over a p-adic field F. Fix a Borel subgroup $\mathbf{B}$ of $\mathbf{G}$, and write $\mathbf{B} = \mathbf{TU}$, where $\mathbf{U}$ is the unipotent radical of $\mathbf{B}$ and $\mathbf{T}$ is a fixed maximal torus of $\mathbf{G}$. Let $\mathbf{P} = \mathbf{MN}$ be a standard parabolic subgroup of $\mathbf{G}$, where $\mathbf{M}$ is a Levi subgroup and $\mathbf{N}$ is the unipotent radical of $\mathbf{P}$. Let π be an irreducible tempered representation of $M = \mathbf{M}(F)$ and $\nu \in \mathfrak{a}_{\mathbb{C}}^*$, where $\mathfrak{a}_{\mathbb{C}}^*$ is the complex dual of the real Lie algebra of the split component $\mathbf{A}$ of $\mathbf{M}$. An induced representation $I(\nu, \pi)$ is called a *standard module* if the parameter ν is in the positive Weyl chamber. A standard module $I(\nu, \pi)$ has a unique irreducible quotient $J(\nu, \pi)$, the Langlands quotient ([6, Theorem 2.11]). For an irreducible tempered generic representation π of $\mathbf{M}(F)$ (see [20, 22] or Section 1 for the definition of generic representations), the standard module conjecture states that a standard module $I(\nu, \pi)$ is irreducible if and only if the Langlands quotient $J(\nu, \pi)$ is generic (Conjecture 5.1). It was proved by Vogan in [28], using Kostant's characterization of large (generic) representations [13], when $F = \mathbb{R}$. In the p-adic case, Casselman and Shahidi proved the standard module conjecture when π is an irreducible unitary, generic, supercuspidal representation [8]. Since then, there has been progress towards proving some special cases of this conjecture. We refer to [8] for an extensive

Received by the editors April 20, 2006; revised October 16, 2006.

Partially supported by NSF Grant DMS-0200325. This paper is a part of the author's Ph.D. thesis submitted to Purdue University.

AMS subject classification: Primary: 11F70; secondary:11F85.

reference on this progress. Meanwhile Muić proved the standard module conjecture for quasisplit classical groups in full generality in [14]. His method depended upon the representation theory of classical groups developed by Tadić and others [3, 26] and the theory of L-functions developed by Shahidi [20, 22]. Our approach is based on Muić's idea and the Langlands–Shahidi method [17, 19–22].

Section 1 provides definitions and notations to be used in this paper. The multiplicative property of γ-factors and L-functions is reviewed in detail, since it proves to be important when we use the L-function theory of Shahidi. A general result of Silberger on square integrable representations via Plancherel measures (Theorem 1.3) is also a key to our approach.

General spin groups are to spin groups as the general linear groups are to special linear groups. As in the case of general linear groups, the derived group of a general spin group is a spin group. Furthermore, Levi subgroups of general spin groups are isomorphic to a product of general linear groups and a general spin group of smaller rank. Therefore the theory of representations of general spin groups are similar to that of classical groups. The definition and the structure theory of the general spin group GSpin_m are introduced in Section 2. An explicit description of Langlands L-functions associated to irreducible components of adjoint action of ${}^LM$ on ${}^L\mathfrak{n}$ is also given there, after discussing Galois action on quasisplit GSpin_{2n}.

In Section 3, we study square integrable representations of GSpin_m. First, we interpret Casselman's square integrability criterion [7] in the form of Proposition 3.2. It was Tadić who originally introduced this type of interpretation of Casselman's square integrability criterion to study square integrable representations of classical groups of type B_n or C_n in a series of papers [26, 27]. The same idea was used in Ban's work for SO_{2n} of type D_n [3]. Following these ideas, we prove Theorem 3.3 as they did for classical groups. Then we prove the holomorphy of L-functions for GSpin_m. For split GSpin_m, it was done by Asgari who followed the idea presented in [8], and we also use the idea of Casselman and Shahidi. It is interesting to observe that the representation theory is intimately related to L-function theory. The concepts of segments are used in the proof of holomorphy of L-functions. Due to possible cancellations between L-functions in the multiplicative property of γ-factors, it is important to know which representations of GL_n or GSpin_m contribute such cancellations. Segments are used in order to simplify situations in Theorem 3.10 as in [8].

In Section 4 we generalize a square integrability criterion, proved by Muić for classical groups, to one for general spin groups. It is an application of the Langlands–Shahidi method [17, 19–22] and the results proved in this paper for representations of GSpin_m.

In Section 5 we prove the standard module conjecture for GSpin_m and consider its application. The standard module conjecture implies an important result concerning holomorphy of normalized intertwining operators which is called *Assumption A* by H. Kim [10] (see Conjecture 5.7 for the statement). Assumption A is one of the key ingredients in establishing functioriality as H. Kim explained in his recent paper [12] where he proved many cases of holomorphy for local L-functions and related results on the normalized intertwining operators via the standard module conjecture in the general context. Using the standard module conjecture for GSpin_m, the Langlands–Shahidi method and the converse theorems of Cogdell and Piatetski-Shapiro [9], Asgari and Shahidi have proved the functoriality of a generic transfer from general spin groups to general linear groups in [2].

1 Preliminaries

Fix a nonarchimedean local field F of characteristic zero. Let $\mathcal{O}_F$ be the ring of integers in F, $\mathfrak{p}$ the maximal ideal, and ϖ_F a uniformizing element of $\mathfrak{p}$. The cardinality of the residue field $\mathcal{O}/\mathfrak{p}$ is denoted by $q = q_F$, and the absolute value of F is normalized so that $|\varpi|_F = q^{-1}$.

Let $\mathbf{G}$ be a connected reductive quasisplit algebraic group defined over F. Fix a Borel subgroup $\mathbf{B}$ of $\mathbf{G}$ and write $\mathbf{B} = \mathbf{TU}$, where $\mathbf{U}$ is the unipotent radical of $\mathbf{B}$ and $\mathbf{T}$ is a maximal torus of $\mathbf{G}$. We also fix a maximal torus $\mathbf{T}$. Let $\mathbf{T}_d$ be the maximal F-split subtorus of $\mathbf{T}$. Then the Weyl group $W = W(\mathbf{G}, \mathbf{T}_d)$ of $\mathbf{T}_d$ in $\mathbf{G}$ acts both on the character group $X^*(\mathbf{T})_F$ and the cocharacter group $X_*(\mathbf{T})_F$. Denote by $\chi^\vee$ the coroot of a root χ. Let $\mathbf{P} = \mathbf{P}_\theta$ be a standard parabolic subgroup corresponding to $\theta \subset \Delta$ where Δ is the set of simple roots. There is a unique Levi decomposition $\mathbf{P} = \mathbf{MN}$ such that $\mathbf{N} \supset \mathbf{B}$ and $\mathbf{M} \supset \mathbf{T}$.

For any algebraic group $\mathbf{H}$, let $H = \mathbf{H}(F)$ be the F-rational points of $\mathbf{H}$. For example, we have the corresponding F-points G, B, T, U, P, M, N of groups $\mathbf{G}, \mathbf{B}, \mathbf{T}, \mathbf{U}, \mathbf{P}, \mathbf{M}, \mathbf{N}$.

For the group $X(\mathbf{M})_F$ of F-rational characters, let

$$\mathfrak{a}^* = X(\mathbf{M}_F) \otimes_{\mathbb{Z}} \mathbb{R}.$$

Then $\mathfrak{a}^*$ is the dual of the real Lie algebra $\mathfrak{a}$ of the split component $\mathbf{A}$ of $\mathbf{M}$. Denote by $\mathfrak{a}^*_{\mathbb{C}}$ the complexification of $\mathfrak{a}^*$. There is a homomorphism

$$H_M \colon M \to \mathfrak{a} = \mathrm{Hom}(X(\mathbf{M})_F, \mathbb{R})$$

characterized by $q^{\langle \chi, H_M(m) \rangle} = |\chi(m)|_F$ for any $m \in M, \chi \in X(\mathbf{M})_F$.

Let $\mathbf{P}$ be a maximal parabolic subgroup generated by $\theta = \Delta \setminus \{\alpha\}$, and let $\rho_{\mathbf{P}}$ be half the sum of the positive roots in $\mathbf{N}$. Let $\widetilde{\alpha} = \langle \rho_{\mathbf{P}}, \alpha \rangle^{-1} \rho_{\mathbf{P}} \in \mathfrak{a}^*$. Since $\mathbf{P} = \mathbf{P}_{\Delta \setminus \{\alpha\}}$ is maximal, $\mathfrak{a}/\mathfrak{z} \cong \mathbb{R}$ where $\mathfrak{z}$ is the Lie algebra of the center of $\mathbf{G}$. Thus, we may identify $(\mathfrak{a}/\mathfrak{z})^*_{\mathbb{C}}$ with $\mathbb{C}$ under the correspondence $s\widetilde{\alpha} \longleftrightarrow s$. According to [20], the adjoint action of ${}^L M$ on ${}^L\mathfrak{n}$ decomposes to the irreducible submodules $V_i = \{X_{\beta^\vee} \in \mathfrak{n}^L : \langle \widetilde{\alpha}, \beta \rangle = i\}, 1 \le i \le m$, and it is denoted by $r = \bigoplus_{i=1}^m r_i$. Suppose $\psi = \psi_F$ is a generic character of $U = \mathbf{U}(F)$. Choose a generic character ψ_M of $U_M = U \cap M$ such that $\psi_M = \psi|_{U_M}$. Now let π be a ψ_M-generic tempered representation of M. Then there exist complex-valued functions $\gamma(s, \pi, r_i, \psi_F)$ such that

$$\gamma(s, \pi, r_i, \psi_F) = \epsilon(s, \pi, r_i, \psi_F) \frac{L(1-s, \pi, \widetilde{r_i})}{L(s, \pi, r_i)}$$

where $\epsilon(s, \pi, r_i, \psi_F)$ is a monomial in q^{-s}. The following conjecture is expected to be true in general [1, 8, 22].

Conjecture 1.1 If π is tempered, then $L(s, \pi, r_i)$ is holomorphic for $Re(s) > 0$.

https://doi.org/10.4153/CJM-2009-033-3 Published online by Cambridge University Press

Next, we describe the multiplicative property of γ-factors and L-functions. Suppose that $\pi \hookrightarrow \mathrm{Ind}_{P_1}^{M}(\pi_1)$ where $\mathbf{P}_1 = \mathbf{M}_1\mathbf{N}_1$ is a parabolic subgroup of $\mathbf{M}$ and π_1 is a representation of $M_1 = \mathbf{M}_1(F)$. Let w be the longest element in the Weyl group of $\mathbf{T}_d$ in $\mathbf{G}$ modulo that of $\mathbf{T}_d$ in $\mathbf{M}$. Fix a reduced decomposition $w = w_{n-1} \cdots w_1$. For each j, there exists a unique simple root α_j such that $w_j(\alpha_j) < 0$. Set $\overline{w}_1 = 1$ and $\overline{w}_j = w_{j-1} \cdots w_1$ for $2 \le j \le n-1$. Let $\theta_1 \subset \Delta$ such that $\mathbf{M}_1 = \mathbf{M}_{\theta_1}$, and define $\theta_{j+1} = w_j(\theta_j)$. For each j, set $\Omega_j = \theta_j \cup \{\alpha_j\}$. Then $\mathbf{M}_{\Omega_j}$ contains $\mathbf{M}_{\theta_j}$ as the Levi subgroup of a maximal parabolic subgroup and $\overline{w}_j(\pi)$ is a representation of M_{θ_j}. Each irreducible constituent of $r_i|_{{}^LM_1}$ is equivalent to, under $\overline{w}_j$, an irreducible constituent of the adjoint action of M_{θ_j} on ${}^L\mathfrak{n}_{\theta_j}$ for a unique j, $2 \le j \le n-1$. Denote by $i(j)$ the index of the irreducible component of M_{θ_j}. Let S_i be the set of all such j for a given i. The importance of the multiplicative property lies in the following theorem [21, 22].

Theorem 1.2 **(Multiplicativity of γ-factors and L-functions)** *For each $j \in S_i$, let $\gamma(s, \overline{w}_j(\pi_1), r_{i(j)}, \psi_F)$, $1 \le i \le m$, be the corresponding γ-factors. Then*

$$\gamma(s, \pi, r_i, \psi_F) = \prod_{j \in S_i} \gamma(s, \overline{w}_j(\pi_1), r_{i(j)}, \psi_F).$$

Furthermore, if Conjecture 1.1 *is true for every $L(s, \overline{w}_j(\pi_1), r_{i(j)})$, then we have the corresponding multiplicative property of L-functions*

$$L(s, \pi, r_i) = \prod_{j \in S_i} L(s, \overline{w}_j(\pi_1), r_{i(j)}).$$

Let π be an irreducible unitary representation of $\mathbf{M}(F)$. The (normalized) Plancherel measure $\mu(s, \pi)$ defined in [22] has the following property proved by Silberger [25].

Theorem 1.3 *If π is a square integrable representation (modulo center), then $\mu(s, \pi)$ has at most a double zero at $s = 0$.*

Furthermore, if π is a generic representation, there is a connection between the γ-factors and the Plancherel measures. More precisely, we have the following result by Shahidi [22].

Theorem 1.4 *If π is a ψ-generic representation, then*

$$\mu(s, \pi) = \prod_{i=1}^{m} |\gamma(is, \pi, r_i, \psi_F)|^2 \quad \textit{for } Re(s) = 0.$$

2 Structure Theory and L-Functions of GSpin_m

Spin groups are the simple simply connected algebraic groups of type B_n or D_n. We define the general spin group to be

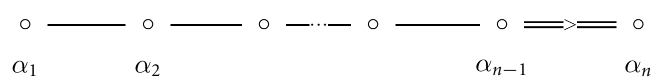
$$\mathrm{GSpin}_m = \frac{GL_1 \times Spin_m}{\{(1, 1), (-1, c)\}}, \quad m \ge 3$$

where $c = \alpha_{n-1}^\vee(-1)\alpha_n^\vee(-1)$ if $m = 2n$, and $c = \alpha_n^\vee(-1)$ if $m = 2n+1$. Here α_{n-1}, α_n are the roots of $Spin_m$. For $m \le 1$, $\mathrm{GSpin}_0 = \mathrm{GSpin}_1 = GL_1$. It is clear from the definition that the derived group of GSpin_m is $Spin_m$. To get precise information on GSpin_m, we need to look at its root datum. We follow Asgari's description ([1, Proposition 2.4]).

Proposition 2.1 *Let $X = \mathbb{Z}e_0 \oplus \mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_n$ and $X^\vee = \mathbb{Z}e_0^* \oplus \mathbb{Z}e_1^* \oplus \cdots \oplus \mathbb{Z}e_n^*$ be free abelian groups of rank n equipped with the standard $\mathbb{Z}$-pairing on $X \times X^\vee$. Then in the root datum $(X, R, X^\vee, R^\vee)$ the set of simple roots Δ and that of coroots $\Delta^\vee$ are as follows:*

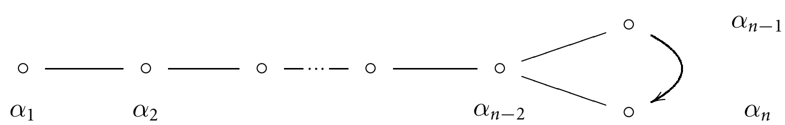
(i) GSpin_{2n+1} *of type B_n*

$$\Delta = \{\alpha_1 = e_1 - e_2, \dots, \alpha_{n-1} = e_{n-1} - e_n, \alpha_n = e_n\}$$
$$\Delta^\vee = \{\alpha_1^\vee = e_1^* - e_2^*, \dots, \alpha_{n-1}^\vee = e_{n-1}^* - e_n^*, \alpha_n^\vee = 2e_n^* - e_0^*\}$$



(ii) GSpin_{2n} *of type D_n*

$$\Delta = \{\alpha_1 = e_1 - e_2, \dots, \alpha_{n-1} = e_{n-1} - e_n, \alpha_n = e_{n-1} + e_n\}$$
$$\Delta^\vee = \{\alpha_1^\vee = e_1^* - e_2^*, \dots, \alpha_{n-1}^\vee = e_{n-1}^* - e_n^*, \alpha_n^\vee = e_n^* + e_n^* - e_0^*\}.$$



The dual groups of GSpin_{2n+1} and GSpin_{2n} are the similitude groups GSp_{2n} and GSO_{2n}, respectively. The groups GSp_{2n} and GSO_{2n} can be described by their matrix representations. More precisely,

$$GSp_{2n} = \{g \in GL_{2n} : {}^t g J g = \mu(g) J\}$$

and

$$GSO_{2n} = \{g \in GL_{2n} : {}^t g J' g = \mu(g) J'\}^\circ,$$

where

$$J = \begin{pmatrix} & & & & & 1 \\ & & & & \iddots & \\ & & & 1 & & \\ & & -1 & & & \\ & \iddots & & & & \\ -1 & & & & & \end{pmatrix}, \quad J' = \begin{pmatrix} & & & & & 1 \\ & & & & \iddots & \\ & & & 1 & & \\ & & 1 & & & \\ & \iddots & & & & \\ 1 & & & & & \end{pmatrix}$$

and $\mu(g)$'s are the similitude characters of groups GSp_{2n} and GSO_{2n}. The corresponding root data for these groups are as follows.

https://doi.org/10.4153/CJM-2009-033-3 Published online by Cambridge University Press

Proposition 2.2 *Let $X = \mathbb{Z}e_0 \oplus \mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_n$ and $X^\vee = \mathbb{Z}e_0^* \oplus \mathbb{Z}e_1^* \oplus \cdots \oplus \mathbb{Z}e_n^*$ be free abelian groups of rank n equipped with the standard $\mathbb{Z}$-pairing on $X \times X^\vee$. Then in the root datum $(X, R, X^\vee, R^\vee)$ the set of simple roots Δ and that of coroots $\Delta^\vee$ are as follows:*

(i) *GSp_{2n} of type C_n*

$$\Delta = \{\alpha_1 = e_1 - e_2, \ldots, \alpha_{n-1} = e_{n-1} - e_n, \alpha_n = 2e_n - e_0\}$$
$$\Delta^\vee = \{\alpha_1^\vee = e_1^* - e_2^*, \ldots, \alpha_{n-1}^\vee = e_{n-1}^* - e_n^*, \alpha_n^\vee = e_n^*\}$$

(ii) *GSO_{2n} of type D_n*

$$\Delta = \{\alpha_1 = e_1 - e_2, \ldots, \alpha_{n-1} = e_{n-1} - e_n, \alpha_n = e_{n-1} + e_n - e_0\}$$
$$\Delta^\vee = \{\alpha_1^\vee = e_1^* - e_2^*, \ldots, \alpha_{n-1}^\vee = e_{n-1}^* - e_n^*, \alpha_n^\vee = e_{n-1}^* + e_n^*\}.$$

Next we construct an explicit outer automorphism of GSpin_{2n}. In [3], Ban defines an explicit outer automorphism of SO_{2n} by $s(g) = sgs^{-1}, g \in SO_{2n}$, where

$$s = \begin{pmatrix} I_{n-1} & & \\ & J_2 & \\ & & I_{n-1} \end{pmatrix}, \quad J_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The outer automorphism s of SO_{2n} can be lifted to that of $Spin_{2n}$ by the universal property of the covering map $p\colon Spin_{2n} \to SO_{2n}$, in such a way that the following diagram commutes

$$\begin{tikzcd} Spin_{2n} \arrow[r, dotted, "s'"] \arrow[d, "p"] & Spin_{2n} \arrow[d, "p"] \\ SO_{2n} \arrow[r, "s"] & SO_{2n}. \end{tikzcd}$$

Fix a lifting s' of $Spin_{2n}$. The automorphism defined by

$$GL_1 \times Spin_{2n} \to GL_1 \times Spin_{2n}\colon (a, x) \mapsto (a, s'(x)), \quad a \in GL_1, x \in Spin_{2n}$$

induces an outer automorphism $\widetilde{s}\colon \mathrm{GSpin}_{2n} \to \mathrm{GSpin}_{2n}$. This automorphism will be used to describe parabolic subgroups.

For $\mathbf{G} = \mathrm{GSpin}_m$, a Levi subgroup $\mathbf{M}$ of a maximal standard parabolic group $\mathbf{P}_\theta = \mathbf{P} = \mathbf{MN}$ corresponding to $\theta = \Delta \setminus \{\alpha_k\}$ is isomorphic to $GL_k \times \mathrm{GSpin}_{m'}$ $(2k + m' = m)$ if $m = 2n + 1$, or if $k \leq n - 2$ and $n = 2m$. In the case $k = n - 1$ and $n = 2m$, $\mathbf{M}$ is isomorphic to $GL_k \times \mathrm{GSpin}_0$ via an outer automorphism $\widetilde{s}$. It follows from the maximal case that

$$\mathbf{M} \cong G_{n_1} \times \cdots \times GL_{n_k} \times \mathrm{GSpin}_{m'} \subset \mathrm{GSpin}_m, \quad 2(n_1 + \cdot + n_k) + m' = m.$$

Now we consider the Galois action on GSpin_m. The Galois group acts trivially on GSpin_{2n+1}, and therefore GSpin_{2n+1} splits over F. For quasisplit $\mathrm{GSpin}_{2n}(n \neq 4)$

which is not split over F, the absolute Galois group $\mathrm{Gal}(\overline{F}/F)$ acts non-trivially and permutes the last two roots α_{n-1}, α_n. Let E/F be a field extension over which GSpin_{2n} splits. By considering the restrictions to E of automorphisms of $\overline{F}$ we may assume that Galois group is $\Gamma = \mathrm{Gal}(E/F)$. We leave out the triality case for quasisplit GSpin_8 for a future work, and we assume $[E : F] = 2$ throughout this paper for quasisplit GSpin_{2n} defined over F. We refer the reader to [16] for more information on the classification of quasisplit groups over a perfect field, and to [20] for the Galois action on the groups of type D_n.

Fix a maximal torus $\mathbf{T}$ in quasisplit GSpin_{2n}. The torus $\mathbf{T}$ can be described by

$$\mathbf{T} = \{t = \prod_{i=0}^{n} e_i^*(t_i) \mid t_i \in GL_1, i = 0, 1, \ldots, n\}.$$

The Galois group $\Gamma = \mathrm{Gal(E/F)}$ fixes e_i's for $i = 0, 1, \ldots, n-1$, and sends e_n to $-e_n$. It follows that the maximal F-split torus $\mathbf{T}_d$ in $\mathbf{T}$ consists of elements of the form $t_d = \prod_{i=0}^{n-1} e_i^*(t_i), t_i \in GL_1, i = 0, 1, \ldots, n-1$. The restricted root datum of GSpin_{2n} is of type B_{n-1}, and the set Δ_F of F-simple roots consists of $\beta_i = \alpha_i = e_i - e_{i+1}$ for $1 \leq i \leq n-2$ and $\beta_{n-1} = e_{n-1} = \alpha_{n-1}|_{\mathbf{T}_d} = \alpha_n|_{\mathbf{T}_d}$. The L-group of quasisplit GSpin_{2n} is $GSO_{2n}(\mathbb{C}) \rtimes \Gamma$. The nontrivial element ϵ in Γ acts on $GSO_{2n}(\mathbb{C})$ by

$$\epsilon(g) = w_n {}^t g^{-1} w_n^{-1}, \quad g \in GSO_{2n}(\mathbb{C}),$$

where

$$w_n = \begin{pmatrix} & & J_{n-1} \\ & I_2 & \\ J_{n-1} & & \end{pmatrix}, \quad J_{n-1} = \begin{pmatrix} & & & & 1 \\ & & & \iddots & \\ & & 1 & & \\ & \iddots & & & \\ 1 & & & & \end{pmatrix}$$

and

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

We now describe L-functions of $\mathrm{GSpin}_m(F)$ explicitly. To accomplish such a goal, we need to determine the adjoint action of ${}^L M$ on ${}^L\mathfrak{n}$. In [21], Shahidi determined the irreducible components $r_i (1 \leq i \leq m)$ in terms of highest weight vectors for quasisplit reductive algebraic groups. Then Asgari computed those representations $r_i, i = 1, 2$, explicitly for split general spin groups. We review split cases, and determine $r_i, i = 1, 2$ explicitly for quasisplit GSpin_m via the general method of Shahidi.

Let $\mathbf{M} = \mathbf{M}_\theta$ be the Levi subgroup of a standard parabolic subgroup $\mathbf{P}_\theta = \mathbf{M}_\theta \mathbf{N}_\theta$ of $\mathbf{G} = \mathrm{GSpin}_m$ for $\theta = \Delta \setminus \{\alpha_k\}$. The adjoint action of ${}^L M$ on ${}^L\mathfrak{n}$ is denoted by $r = \bigoplus_{i=1}^m r_i$. Then each $r_i, 1 \leq i \leq m$, is irreducible by [20, Proposition 4.1]. Let ρ_k, R_{2n}^1 and R_{2n}^2 be the standard matrix representations of the groups $GL_k(\mathbb{C}), GSp_{2n}(\mathbb{C})$, and $GSO_{2n}(\mathbb{C})$, respectively, and let μ be the similitude character appearing in the definition of $GSp_{2n}(\mathbb{C})$ or $GSO_{2n}(\mathbb{C})$. We have the following cases.

(a) Suppose that

$$M = \mathbf{M}(F) \cong GL_k(F) \times \mathrm{GSpin}_{2n+1}(F) \subset \mathrm{GSpin}_{2n+2k+1}(F).$$

Then $r = r_1 \oplus r_2$ with $r_1 = \rho_k \otimes \widetilde{R^1_{2n}}$, $r_2 = \mathrm{Sym}^2 \rho_k \otimes \mu^{-1}$ if $n \geq 1$ and $r = r_1 = \mathrm{Sym}^2 \rho_k \otimes \mu^{-1}$ if $n = 0$.

(b) Suppose that GSpin_{2n} splits over F, and that

$$M = \mathbf{M}(F) \cong GL_k(F) \times \mathrm{GSpin}_{2n}(F) \subset \mathrm{GSpin}_{2n+2k}(F)(n \neq 1).$$

Then $r = r_1 \oplus r_2$ with $r_1 = \rho_k \otimes \widetilde{R^2_{2n}}$, $r_2 = \bigwedge^2 \rho_k \otimes \mu^{-1}$ if $n \geq 2$ and $r = r_1 = \bigwedge^2 \rho_k \otimes \mu^{-1}$ if $n = 0$.

(c) Suppose that GSpin_{2n} does not split over F, but splits over E. Let r_i° be the restriction of r_i on ${}^L M^\circ (i = 1, 2)$. For $\theta = \Delta_F \setminus \{\beta_k\}, k \leq n - 2$,

$$M = \mathbf{M}(F) \cong GL_k(F) \times \mathrm{GSpin}_{2n}(F) \subset \mathrm{GSpin}_{2n+2k}(F).$$

Then $r = r_1 \oplus r_2$ with $r_1^\circ = \rho_k \otimes \widetilde{R^2_{2n}}$, $r_2^\circ = \bigwedge^2 \rho_k \otimes \mu^{-1}$. Each r_i° is irreducible for $i = 1, 2$. For $\theta = \Delta_F \setminus \{\beta_{n-1}\}$,

$$M = \mathbf{M}(F) = GL_k(F) \times E^1 \times GL_1(F) \subset \mathrm{GSpin}_{2n}(F)$$

where E^1 is the multiplicative group of norm one elements in E. Then $r = r_1 \oplus r_2$ with $r_1^\circ = \rho_k \otimes \widetilde{R^2_2}, r_2^\circ = \bigwedge^2 \rho_k \otimes \mu^{-1}$. We note that r_1° is reducible since the standard matrix representation $\widetilde{R^2_2}$ of $GSO_2(\mathbb{C})$ is reducible, but r_2° is irreducible.

Finally, we consider L-functions for quasisplit generalized spin groups. Let π be an unramified representation of $\mathbf{M}(F) = GL_k(F) \times \mathrm{GSpin}_{2n}(F)$. Then there is a unique semisimple conjugacy class $(t \rtimes \epsilon)$ in ${}^L T^\circ \rtimes \Gamma$ determined by the Satake isomorphism. Furthermore, t can be chosen as an element fixed by the Galois action ([5, Proposition 6.7]). So we may assume $t \in {}^L T_d^\circ$. For each irreducible component r_i, the Langlands L-function can be described by

$$L(s, \pi, r_i) = det(1 - r_i^\circ (t \rtimes \epsilon) q^{-s})^{-1}.$$

For a ramified representation, the L-functions are defined via the Langlands–Shahidi method (See Section 1).

3 Square Integrable Representations of General Spin Groups

Let $\sigma \otimes \tau$ be an irreducible admissible representation of

$$\mathbf{M}(F) \cong GL_k(F) \times \mathrm{GSpin}_{m'}(F).$$

We write $\sigma = \nu^s \sigma^u$ where

$$\nu^s(h) = |\det(h)|_F^s, \quad s \in \mathbb{R}, \quad h \in GL_k(F)$$

and σ^u is a unitary representation of $GL_k(F)$. The (normalized) induced representation $\mathrm{Ind}_P^G(\sigma \otimes \tau) = I(s, \sigma^u \otimes \tau)$ is denoted by $\sigma \rtimes \tau$. We shall use similar notations for parabolic inductions induced from nonmaximal parabolic subgroups. There is a general result concerning irreducibility of induced representations induced from generic ones. In terms of our notation for GSpin_m, Theorem 8.1 of [22] can be stated as follows.

Theorem 3.1 *Let σ^u be an irreducible unitary supercuspidal representation of $GL_k(F)$ and let τ_0 be an irreducible generic supercuspidal representation of $\mathrm{GSpin}_m(F)$. If $\sigma^u \ncong \widetilde{\sigma^u}$, then $\nu^s\sigma^u \rtimes \tau_0$ is irreducible. If $\sigma^u \cong \widetilde{\sigma^u}$, then there exists $s_0 \in \{0, 1/2, 1\}$ such that $\nu^{\pm s_0}\sigma^u \rtimes \tau_0$ reduces, and $\nu^s\sigma^u \rtimes \tau_0$ is irreducible for any real number $s \neq s_0$.*

To state Casselman's square integrability criterion for GSpin_m, we need to introduce some notations. Let $\pi = \sigma_1 \otimes \cdots \otimes \sigma_l \otimes \tau$ be an irreducible representation of $M = GL_{n_1}(F) \times \cdots \times GL_{n_l}(F) \times \mathrm{GSpin}_{m'}(F) \subset \mathrm{GSpin}_m(F)$ with $2(n_1 + \cdots + n_l) + m' = m$. We write $\sigma_i = \nu^{e(\sigma_i)}\sigma_i^u$, where σ_i^u is unitary and $e(\sigma_i) \in \mathbb{R}$, $1 \leq i \leq l$. We define $\xi_1, \ldots, \xi_n, e_*(\pi) \in \mathbb{R}^n$ as follows:

$$\xi_i = (\underbrace{1, \ldots, 1}_{i \text{ times}}, 0, \ldots, 0), \quad 1 \leq i \leq n-2 \text{ or } i = n,$$

$$\xi_{n-1} = \begin{cases} (1, \ldots, 1, 0) & \text{if } \mathbf{G} = \mathrm{GSpin}_{2n+1}, \\ (1, \ldots, 1, -1) & \text{if } \mathbf{G} = \mathrm{GSpin}_{2n}, \end{cases}$$

and

$$e_*(\pi) = (\underbrace{e(\sigma_1), \ldots, e(\sigma_1)}_{n_1 \text{ times}}, \ldots, \underbrace{e(\sigma_k), \ldots, e(\sigma_k)}_{n_l \text{ times}} \underbrace{0, \ldots, 0}_{[\frac{m'}{2}] \text{ times}}).$$

In the following proposition, (v, w) denotes the standard inner product on $\mathbb{R}^n$.

Proposition 3.2 *Let $G = \mathrm{GSpin}_m(F)$ with $m = 2n + 1$ or $m = 2n$. Suppose that τ is an irreducible representation of G. Assume that $\mathbf{P} = \mathbf{MN}$ is a standard parabolic subgroup that is minimal among all standard parabolic subgroups which satisfy $r_{M,G}(\tau) \neq 0$, where $r_{M,G}(\tau)$ denotes the (normalized) Jaquet module of τ with respect to P. Let $M = GL_{n_1}(F) \times \cdots \times GL_{n_l}(F) \times \mathrm{GSpin}_{m_0}(F)$ and let $\pi = \sigma_1 \otimes \cdots \otimes \sigma_l \otimes \tau_0$ be any irreducible supercuspidal subquotient of $r_{M,G}(\tau)$. If τ is a square integrable representation of G, then the central character of τ is unitary and*

$$\begin{aligned} &(e_*(\pi), \xi_{n_1}) > 0, \\ &(e_*(\pi), \xi_{n_1+n_2}) > 0, \\ &\qquad \vdots \\ &(e_*(\pi), \xi_{n_1+n_2+\cdots+n_l}) > 0. \end{aligned}$$

Conversely, if the central character of τ is unitary and the above inequalities hold for any subquotient π of $r_{M,G}(\tau)$ such that $r_{M,G}(\tau) \neq 0$, then τ is square integrable.

Proof The proof follows immediately from [7, Theorem 6.5.1]. ■

Theorem 3.3 *Suppose $\sigma_1 \otimes \cdots \otimes \sigma_k \otimes \tau_0$ is an irreducible generic supercupidal representation of $M = GL_{n_1}(F) \times \cdots \times GL_{n_l}(F) \times \mathrm{GSpin}_{m_0}(F) \subset \mathrm{GSpin}_m(F)$. Write $\sigma_i = \nu^{e(\sigma_i)}\sigma_i^u$ where $e(\sigma_i) \in \mathbb{R}$ and σ_i^u is unitary supercuspidal for each $i = 1, \dots, l$.*

(i) *If $\sigma_1 \times \cdots \times \sigma_l \rtimes \tau_0$ contains a square integrable subrepresentation or a square integrable subquotient, then $\sigma_i^u \cong \widetilde{\sigma_i^u}$ for $i = 1, \dots, l$.*
(ii) *If $\sigma_1 \times \cdots \times \sigma_l \rtimes \tau_0$ contains a square integrable subrepresentation or a square integrable subquotient, then $2e(\sigma_i) \in \mathbb{Z}$ for $i = 1, \dots, l$.*

Proof Since Asgari gave a detailed proof for GSpin_{2n+1} in [1], we give only a proof for GSpin_{2n}. If GSpin_{2n} is quasisplit but not split over F, then the F-root system is of type B_{n-1} and the proof is similar to that of GSpin_{2n+1}. So we may assume that $\mathbf{G} = \mathrm{GSpin}_{2n}$ is split. First, assume that τ is an irreducible square integrable subrepresentation of $\sigma_1 \times \cdots \times \sigma_l \rtimes \tau_0$ where $\pi = \sigma_1 \otimes \cdots \otimes \sigma_l \otimes \tau_0$ is a supercuspidal representation of $M = GL_{n_1}(F) \times \cdots \times GL_{n_l}(F) \times \mathrm{GSpin}_{2n'}(F)$. Fix $i_0 \in \{1, \dots, l\}$ and set

$$\begin{aligned}
Y_{i_0}^0 &= \{i \in \{1, \dots, l\} : \exists k \in \mathbb{Z} \text{ such that } \sigma_{i_0} \cong \nu^k \sigma_i\},\\
Y_{i_0}^1 &= \{i \in \{1, \dots, l\} : \exists k \in \mathbb{Z} \text{ such that } \widetilde{\sigma_{i_0}} \cong \nu^k \sigma_i\},\\
Y_{i_0} &= Y_{i_0}^0 \cup Y_{i_0}^1,\\
Y_{i_0}^c &= \{1, \dots, k\} \setminus Y_{i_0}.
\end{aligned}$$

Suppose that $\sigma_{i_0}^u \ncong \widetilde{\sigma_{i_0}^u}$. For $j_0, j_0' \in Y_{i_0}^0$, $j_1, j_1' \in Y_{i_0}^1$ and $j_c \in Y_{i_0}^c$, we have the following isomorphisms due to Bernstein and Zelevinsky [4, 29]

$$\begin{aligned}
&\sigma_{j_0} \times \widetilde{\sigma_{j_0'}} \cong \widetilde{\sigma_{j_0'}} \times \sigma_{j_0}, \quad && \sigma_{j_1} \times \widetilde{\sigma_{j_1'}} \cong \widetilde{\sigma_{j_1'}} \times \sigma_{j_1},\\
&\sigma_{j_0} \times \sigma_{j_1} \cong \sigma_{j_1} \times \sigma_{j_0}, && \widetilde{\sigma_{j_0}} \times \widetilde{\sigma_{j_1}} \cong \widetilde{\sigma_{j_1}} \times \widetilde{\sigma_{j_0}},\\
&\sigma_{j_0} \times \sigma_{j_c} \cong \sigma_{j_c} \times \sigma_{j_0}, && \widetilde{\sigma_{j_0}} \times \sigma_{j_c} \cong \sigma_{j_c} \times \widetilde{\sigma_{j_0}},\\
&\sigma_{j_1} \times \sigma_{j_c} \cong \sigma_{j_c} \times \sigma_{j_1}, && \widetilde{\sigma_{j_1}} \times \sigma_{j_c} \cong \sigma_{j_c} \times \widetilde{\sigma_{j_1}}.
\end{aligned}$$

By Theorem 3.1, we also have $\sigma_{j_0} \rtimes \tau_0 \cong \widetilde{\sigma_{j_0}} \rtimes \tau_0$, $\sigma_{j_1} \rtimes \tau_0 \cong \widetilde{\sigma_{j_1}} \rtimes \tau_0$. Write $Y_{i_0}^0 = \{a_1 < a_2 < \cdots < a_{l_0}\}$, $Y_{i_0}^1 = \{b_1 < b_2 < \cdots < b_{l_1}\}$, and $Y_{i_0}^c = \{d_1 < d_2 < \cdots < d_{l_c}\}$. Using the above isomorphisms, we get

$$\begin{aligned}
&\sigma_1 \times \cdots \times \sigma_l \rtimes \tau_0\\
&\quad\cong \sigma_{a_1} \times \cdots \times \sigma_{a_{l_0}} \times \sigma_{d_1} \times \cdots \times \sigma_{d_{l_c}} \times \sigma_{b_1} \times \cdots \times \sigma_{b_{l_1}} \rtimes \tau_0\\
&\quad\cong \sigma_{a_1} \times \cdots \times \sigma_{a_{l_0}} \times \sigma_{d_1} \times \cdots \times \sigma_{d_{l_c}} \times \sigma_{b_1} \times \cdots \times \sigma_{b_{l_1-1}} \times \widetilde{\sigma_{b_{l_1}}} \rtimes \tau_0\\
&\quad\cong \sigma_{a_1} \times \cdots \times \sigma_{a_{l_0}} \times \sigma_{d_1} \times \cdots \times \sigma_{d_{l_c}} \times \widetilde{\sigma_{b_{l_1}}} \times \sigma_{b_1} \times \cdots \times \sigma_{b_{l_1-1}} \rtimes \tau_0\\
&\quad\quad\vdots\\
&\quad\cong \sigma_{a_1} \times \cdots \times \sigma_{a_{l_0}} \times \widetilde{\sigma_{b_{l_1}}} \times \cdots \times \widetilde{\sigma_{b_1}} \times \sigma_{d_1} \times \cdots \times \sigma_{d_{l_c}} \rtimes \tau_0.
\end{aligned}$$

Similarly, we get

$$\sigma_1 \times \cdots \times \sigma_l \rtimes \tau_0 \cong \sigma_{b_1} \times \cdots \times \sigma_{b_{l_1}} \times \widetilde{\sigma_{a_{l_0}}} \times \cdots \times \widetilde{\sigma_{a_1}} \times \sigma_{d_1} \times \cdots \times \sigma_{d_{l_c}} \rtimes \tau_0.$$

It follows from the Frobenius reciprocity that the representations

$$\pi' = \sigma_{a_1} \otimes \cdots \otimes \sigma_{a_{l_0}} \otimes \widetilde{\sigma_{b_{l_1}}} \otimes \cdots \otimes \widetilde{\sigma_{b_1}} \otimes \sigma_{d_1} \otimes \cdots \otimes \sigma_{d_{l_c}} \otimes \tau_0,$$
$$\pi'' = \sigma_{b_1} \otimes \cdots \otimes \sigma_{b_{l_1}} \otimes \widetilde{\sigma_{a_{l_0}}} \otimes \cdots \otimes \widetilde{\sigma_{a_1}} \otimes \sigma_{d_1} \otimes \cdots \otimes \sigma_{d_{l_c}} \otimes \tau_0$$

are quotients of corresponding Jacquet modules. Note that $\sigma_{a_1} \times \cdots \times \sigma_{a_{l_0}} \times \sigma_{b_1} \times \cdots \times \sigma_{b_{l_1}}$ is a representation of $GL_u(F)$, $u \leq n$. Suppose $n' \geq 2$. Then

$$(e_*(\pi'), \xi_u) = -(e_*(\pi''), \xi_u)$$

which contradicts Proposition 3.2. Next we assume $n' = 0$. In this case, $M = GL_{n_1}(F) \times \cdots \times GL_{n_l}(F) \times \mathrm{GSpin}_0(F)$. If $u \neq n-1$, then

$$(e_*(\pi'), \xi_u) = -(e_*(\pi''), \xi_u) \tag{3.1}$$

and if $u = n-1$, then

$$(e_*(\pi'), \xi_{n-1}) + (e_*(\pi'), \xi_n) = -(e_*(\pi''), \xi_{n-1}) - (e_*(\pi''), \xi_n). \tag{3.2}$$

Both (3.1) and (3.2) contradict Proposition 3.2. We must have $\sigma_i^u \cong \widetilde{\sigma_i^u}$ for $i = 1, \ldots, l$. This completes the proof of (i) when $\sigma_1 \times \cdots \times \sigma_l \rtimes \tau_0$ contains a square integrable subrepresentation. If $\sigma_1 \times \cdots \times \sigma_l \rtimes \tau_0$ contains a square integrable subquotient, then $\widetilde{\sigma}_1 \times \cdots \times \widetilde{\sigma}_l \rtimes \tau_0$ contains a square integrable subrepresentation. Since $\widetilde{\sigma}_i = \nu^{-e(\sigma_i)}\widetilde{\sigma_i^u}$, its proof follows from the subrepresentation case. Finally, if $2e(\sigma_{i_0}) \notin \mathbb{Z}$ for some i_0, then we repeat a similar argument to complete (ii). ■

Remark 3.4. This type of argument for classical groups of type B_n or C_n is due to Tadić who also interpreted Casselman's square integrability criterion in the form of Proposition 3.2. Ban used a similar argument for SO_{2n} (of type D_n) in [3]. We refer the reader to [3, 26] for more details.

The following result is based on the *local-global* argument which was used in the proof of Lemma 3.6 of [23]. The result of Lemma 3.5 is stated implicitly in [24]. We note that Shahidi used spin groups to get the twisted symmetric or exterior L-functions in [24], while we get them from general spin groups.

Lemma 3.5 *Let σ be an irreducible unitary supercuspidal representation of $GL_k(F)$ and let χ be a unitary character of $GL_1(F) = F^\times$. Then*

$$L(s, \sigma \times (\sigma \otimes \chi^{-1})) = L(s, \sigma \otimes \chi, \textstyle\bigwedge^2 \rho_k \otimes \mu^{-1}) L(s, \sigma \otimes \chi, \mathrm{Sym}^2 \rho_k \otimes \mu^{-1}). \tag{3.3}$$

Proof Since $\mathrm{GSpin}_0(F) = \mathrm{GSpin}_1(F) = F^\times$, we may consider $\sigma \otimes \chi$ as a representation of $M = GL_k(F) \times \mathrm{GSpin}_0(F) \subset \mathrm{GSpin}_{2k}(F)$ or $M' = GL_k(F) \times \mathrm{GSpin}_1(F) \subset \mathrm{GSpin}_{2k+1}(F)$. The L-functions on the right hand side of (3.3) appear as the first L-functions of the representation $\sigma \otimes \chi$ of M and M', respectively. The L-function on the left hand side of (3.3) is the L-function for the Rankin-Selberg product of σ and $\sigma \otimes \chi^{-1}$.

Choose a globally generic cusp form $\Sigma = \bigotimes_v \Sigma_v$ of $GL_k(\mathbb{A}_F)$ such that $\Sigma_{v_0} = \sigma$ and Σ_v is unramified for any finite $v \neq v_0$, and a unitary grössen-character $X = \bigotimes_v X_v$ of $\mathbb{A}_F^\times / F^\times$ such that $X_{v_0} = \chi$ and X_v is unramified for any finite $v \neq v_0$. The equality

$$L(s, \Sigma_v \times (\Sigma_v \otimes X_v^{-1})) = L(s, \Sigma_v \otimes X_v, \bigwedge^2 \rho_k \otimes \mu^{-1}) L(s, \Sigma_v \otimes X_v, \mathrm{Sym}^2 \rho_k \otimes \mu^{-1})$$

holds for any finite $v \neq v_0$ by a direct computation and for archimedean places v by [19, Theorem 3.1]. The global L-functions

$$L(s, \Sigma \otimes X, \bigwedge^2 \rho_k \otimes \mu^{-1}), L(s, \Sigma \otimes X, \mathrm{Sym}^2 \rho_k \otimes \mu^{-1}), \text{ and } L(s, \Sigma \times (\Sigma \otimes X^{-1}))$$

satisfy the required functional equations [17, 22, 24]. The uniqueness of γ-factors implies (3.3). ■

Using multiplicativity of γ-factors (see Theorem 1.2 or [21]) together with Lemma 3.5, we obtain the following proposition.

Proposition 3.6 *Let σ be an irreducible square integrable representation of $GL_k(F)$ and χ a unitary character of $GL_1(F)$. Choose an irreducible unitary supercuspidal representation σ_0 of $GL_a(F)$ with $k = ab$ such that σ is the unique subrepresentation of $\sigma_1 \times \cdots \times \sigma_b, \sigma_i = \nu^{(b+1)/2-i}\sigma_0, 1 \leq i \leq b$. For the sake of simplicity, let $r_{2,k}^e = \bigwedge^2 \rho_k \otimes \mu^{-1}$ and $r_{2,k}^s = \mathrm{Sym}^2 \rho_k \otimes \mu^{-1}$. If k is even, then*

$$L(s, \sigma \otimes \chi, r_{2,k}^e) = \prod_{i=1}^{b/2} L(s, \sigma_i \otimes \chi, r_{2,a}^e) L(s, \nu^{-1/2}\sigma_i \otimes \chi, r_{2,a}^s)$$
$$L(s, \sigma \otimes \chi, r_{2,k}^s) = \prod_{i=1}^{b/2} L(s, \sigma_i \otimes \chi, r_{2,a}^s) L(s, \nu^{-1/2}\sigma_i \otimes \chi, r_{2,a}^e).$$

If k is odd, then

$$L(s, \sigma \otimes \chi, r_{2,k}^e) = \prod_{i=1}^{(b+1)/2} L(s, \sigma_i \otimes \chi, r_{2,a}^e) \prod_{i=1}^{(b-1)/2} L(s, \nu^{-1/2}\sigma_i \otimes \chi, r_{2,a}^s)$$
$$L(s, \sigma \otimes \chi, r_{2,k}^s) = \prod_{i=1}^{(b+1)/2} L(s, \sigma_i \otimes \chi, r_{2,a}^s) \prod_{i=1}^{(b-1)/2} L(s, \nu^{-1/2}\sigma_i \otimes \chi, r_{2,a}^e).$$

Remark 3.7. Suppose that $\sigma \otimes \tau$ is a generic square integrable representation of $M = GL_k(F) \times \mathrm{GSpin}_m(F)$. Let χ be the central character of τ. Since the second L-function $L(s, \sigma \otimes \tau, r_2)$ depends only on σ and χ, we have

$$L(s, \sigma \otimes \tau, r_2) = L(s, \sigma \otimes \chi, r_2)$$

where $L(s, \sigma \otimes \chi, r_2)$ is the first L-function attached to the representation $\sigma \otimes \chi$ of $M' = GL_k(F) \times \mathrm{GSpin}_0(F)$.

Corollary 3.8 *Let $\sigma \otimes \tau$ be an irreducible generic representation of $M = GL_k(F) \times \mathrm{GSpin}_m(F)$. If $\sigma \otimes \tau$ is square integrable modulo center, then the second L-function $L(s, \sigma \otimes \tau, r_2)$ is holomorphic for $Re(s) > 0$.*

Proof By Remark 3.7, we may replace τ by its central character χ. Then χ is unitary by Theorem 3.3. Choose an irreducible unitary supercuspidal representation σ_0 of $GL_a(F)$ with $k = ab$ such that σ is the unique square integrable subrepresentation of $\sigma_1 \times \cdots \times \sigma_b, \sigma_i = \nu^{(b+1)/2-i}\sigma_0, i = 1, \dots, b$. Assume that $r_2 = \bigwedge^2 \rho_k \otimes \mu^{-1}$ and k is even. By Proposition 3.6,

$$L(s, \sigma \otimes \chi, r_{2,k}^e) = \prod_{i=1}^{b/2} L(s + b + 1 - 2i, \sigma_0 \otimes \chi, r_{2,a}^e) L(s + b - 2i, \sigma_0 \otimes \chi, r_{2,a}^s),$$

where $r_{2,k}^e = \bigwedge^2 \rho_k \otimes \mu^{-1}$ and $r_{2,k}^s = \mathrm{Sym}^2 \rho_k \otimes \mu^{-1}$. For any $i, 1 \le i \le b/2$,

$$Re(s + b + 1 - 2i) > Re(s + b - 2i) \ge Re(s) > 0.$$

Now the holomorphy of the second L-function follows from [22, Proposition 7.3]. The other cases can be proved similarly. ■

Corollary 3.9 *Let σ be a square integrable representation attached to a segment $\Delta = [\nu^{-n}\sigma_0, \nu^n\sigma_0]$, where σ_0 is an irreducible unitary supercuspidal representation of $GL_a(F)$, and let τ be a square integrable representation of $\mathrm{GSpin}_m(F)$. Then $L(s, \sigma \otimes \tau, r_2)$ is holomorphic for $Re(s) > 0$ and it has a pole at $s = 1 - 2c, c \in \mathbb{R}$, if and only if one of the following holds:*

(i) $L(0, \sigma_0 \otimes \tau, r_2) = \infty, \quad n + c \in 1/2 + \mathbb{Z},\ 0 < c \le n + 1/2.$
(ii) $L(0, \sigma_0 \otimes \tau, r_2) \ne \infty, \quad n + c \in \mathbb{Z},\ 0 < c \le n.$

Proof The holomorphy of L-functions is proved in Corollary 3.8. After replacing τ by its central character χ, it follows immediately from Lemma 3.3 and Proposition 3.6. ■

We are ready to prove the holomorphy of L-functions attached to a generic tempered representation of $M = GL_k(F) \times \mathrm{GSpin}_m(F)$. By multiplicativity of L-functions it is enough to prove the holomorphy of L-functions for generic square integrable representations.

Theorem 3.10 *Let $\sigma \otimes \tau$ be an irreducible generic representation of $GL_k(F) \times \mathrm{GSpin}_m(F)$. If $\sigma \otimes \tau$ is square integrable modulo center, then each L-function $L(s, \sigma \otimes \tau, r_i)$ is holomorphic for $Re(s) > 0$ ($i = 1, 2$).*

Proof Since this theorem is proved for split generalized spin groups by Asgari in [1], we only consider quasisplit $\mathbf{G} = \mathrm{GSpin}_{2n}(m = 2n)$. According to [20], there are three cases to be considered. In the case of ${}^2D_n - 2$ or ${}^2D_n - 3$, the holomorphy follows immediately from multiplicativity of γ-factors and Proposition 7.3 of [22]. The holomorphy of the second L-function is proved in Corollary 3.8.

Let $L(s, \sigma \otimes \tau, r_1)$ be the first L-function attached to a square integrable representation $\sigma \otimes \tau$ of $M = GL_k(F) \times \mathrm{GSpin}_{2n}(F)$ in the case of ${}^2D_n - 1$. After replacing τ by its contragredient, we denote the given L-function by $L(s, \sigma \times \tau)$. Choose an irreducible unitary supercuspidal representation σ_0 of $GL_a(F)$ with $k = ab$ such that σ is the unique subrepresentation of $\sigma_1 \times \cdots \times \sigma_b$ where $\sigma_i = \nu^{(b+1)/2-i}\sigma_0$, $1 \le i \le b$ ($\nu(g) = |\det(g)|_F$). Assume that τ is a subrepresentation of $\rho_1 \times \cdots \times \rho_c \rtimes \tau_0$ where ρ_j^u, $1 \le j \le c$, are supercuspidal representations of $GL_{n_j}(F)$ and τ_0 is a unitary supercuspidal representation of $\mathrm{GSpin}_{2n'}(F)$. We write $\rho_j = \nu^{e_j}\rho_j^u$, $e_j \in \mathbb{R}$, $1 \le j \le c$. By Proposition 3.3, we have $e_j \in \frac{1}{2}\mathbb{Z}$ and $\rho_j^u \cong \widetilde{\rho_j^u}$ for $1 \le j \le c$. Set $s_i = s + \frac{b+1}{2} - i$, $1 \le i \le b$. For the simplicity of notation, we write $\gamma(s, \pi)$ for $\gamma(s, \pi, \psi_F)$. By multiplicativity of γ-factors (Theorem 1.2 or [21]),

$$\begin{aligned}\gamma(s, \sigma \times \tau) &= \prod_{i=1}^{b} \gamma(s, \sigma_i \times \tau) = \prod_{i=1}^{b} \gamma(s_i, \sigma_0 \times \tau)\\ &= \prod_{i=1}^{b} \Big[\gamma(s_i, \sigma_0 \times \tau_0) \prod_{j=1}^{c} \gamma(s_i + e_j, \sigma_0 \times \rho_j^u)\gamma(s_i - e_j, \sigma_0 \times \widetilde{\rho_j^u})\Big].\end{aligned}$$

The L-functions in the product

$$\prod_{j=1}^{c} L(s_i + e_j, \sigma_0 \times \rho_j^u) L(s_i - e_j, \sigma_0 \times \rho_j^u)$$

are non-trivial only if $\rho_j^u \cong \nu^{d_j}\widetilde{\sigma}_0$ for some $d_j \in \sqrt{-1} \cdot \mathbb{R}$. Since we are interested in the holomorphy of L-functions, we may assume that $\rho_j^u = \sigma_0$ after shifting s by $d_j \in \sqrt{-1} \cdot \mathbb{R}$. From this observation it suffices to prove that

$$\prod_{i=1}^{b} \Big[\gamma(s_i, \sigma_0 \times \tau_0) \prod_{j=1}^{c} \gamma(s_i + e_j, \sigma_0 \times \sigma_0)\gamma(s_i - e_j, \sigma_0 \times \sigma_0)\Big]$$

is non-zero for $Re(s) > 0$. First, we prove that

$$\prod_{i=1}^{b} \gamma(s_i, \sigma_0 \times \tau_0) \neq 0 \quad \text{for } Re(s) > 0 \quad (s_i = s + (b+1)/2 - i). \tag{3.4}$$

We recall

$$L(s, \widetilde{\pi}, r_i) = L(-s, \pi, r_i) \tag{3.5}$$

for any unitary supercuspidal representation π (see the proof of [21, Theorem 5.5] for details). The product $\prod_{i=1}^{b} \gamma(s_i, \sigma_0 \times \tau_0)$ of γ-functions is equal to

$$\prod_{i=1}^{b} \frac{L(s + (b+1)/2 - i - 1, \sigma_0 \times \tau_0)}{L(s + (b+1)/2 - i, \sigma_0 \times \tau_0)} = \frac{L(s - (b+1)/2, \sigma_0 \times \tau_0)}{L(s + (b-1)/2, \sigma_0 \times \tau_0)}$$

https://doi.org/10.4153/CJM-2009-033-3 Published online by Cambridge University Press

up to a monomial in q^{-s} by (3.5). Now (3.4) follows from [22, Proposition 7.3].

To continue, we use the technique of Casselman and Shahidi in [8, Section 3]. We define a σ_0-chain to be a sequence of representations $\nu^{e_1}\sigma_0, \dots, \nu^{e_l}\sigma_0$ such that $e_{j+1} - e_j = 1, 1 \le j \le l-1$. Since σ_0 is fixed, we also write $\{e_j\}_{j=1}^l$ for $\{\nu^{e_j}\sigma_0\}_{j=1}^l$. For a σ_0-chain $\{e_j\}_{j=1}^l$, we define

$$\gamma(s; e_1, \dots, e_l) = \prod_{j=1}^{l} \gamma(s + e_j, \sigma_0 \times \sigma_0)\gamma(s - e_j, \sigma_0 \times \sigma_0). \tag{3.6}$$

In terms of L-functions, by (3.5), (3.6) reads

$$\begin{aligned}\gamma(s; e_1, \dots, e_l) &= \prod_{j=1}^{l} \frac{L(s + e_j - 1, \sigma_0 \times \sigma_0)L(s - e_j - 1, \sigma_0 \times \sigma_0)}{L(s + e_j, \sigma_0 \times \sigma_0)L(s - e_j, \sigma_0 \times \sigma_0)} \\ &= \frac{L(s + e_1 - 1, \sigma_0 \times \sigma_0)L(s - e_l - 1, \sigma_0 \times \sigma_0)}{L(s - e_1, \sigma_0 \times \sigma_0)L(s + e_l, \sigma_0 \times \sigma_0)}.\end{aligned}$$

Now we define several concepts related to σ_0-chains.

Definition 3.1 Two σ_0-chains are equivalent if they have the same γ-functions. A nonnegative σ_0-chain $\{e_j\}_{j=1}^l$ is called *regular* if either $e_1 = 0, 1/2$ or $e_1 = 1$ and $L(s, \sigma_0 \times \sigma_0)^{-1}$ divides $L(1 - s, \sigma_0 \times \tau_0)^{-1}$. A pair of σ_0-chains $\{e_j\}_{j=1}^l$ and $\{e_{j'}\}_{j'=1}^{l'}$ with $e_1 \ne 1/2, e_1' \ne 1/2$ is called a *singular pair* if $e_1 + e_1' = 1$.

The meaning of a σ_0-chain being completed to a larger σ_0-chain is understood in the obvious manner. Then every σ_0 chain is γ-equivalent to a σ_0-chain that can be completed to a chain which is γ-equivalent to either a regular chain, a pair of regular chains, or a singular pair [8, Proposition 4.16]. The same method used in the proofs of Lemmas 18, 19, and 20 of [8, Theorem 4.1] can be adapted to prove the following lemma. (The proofs in [8] do not use any special properties of a classical group $\mathbf{G}$ except the property that maximal Levi subgroups are of the form $GL_k \times \mathbf{G}'$, which is true for GSpin_{2n}.)

Lemma 3.11 *We keep the notation $s_i = s + (b+1)/2 - i, 1 \le i \le b$.*

(a) *Let $\{e_j\}_{j=1}^l$ be a regular chain with $e_1 = 0$ or $1/2$. Then*

$$\prod_{i=1}^{b} \gamma(s_i; e_1, \dots, e_l) \ne 0 \quad \text{for } Re(s) > 0.$$

(b) *Let $\{e_j\}_{j=1}^l$ and $\{e'_{j'}\}_{j'=1}^{l'}$ be a singular pair. Then*

$$\prod_{i=1}^{b} \gamma(s_i; e_1, \dots, e_l)\gamma(s_i; e_1', \dots, e_{l'}') \ne 0 \quad \text{for } Re(s) > 0.$$

(c) *Let* $\cup_{k=1}^{h}\{e_j^k\}_{j=1}^{l_k}$ *be the union of* σ_0*-chains with* $e_1^k = 1, 1 \leq k \leq h$. *Then*

$$\prod_{i=1}^{b} \gamma(s_i, \sigma_0 \times \tau_0) \prod_{i=1}^{b} \prod_{k=1}^{h} \gamma(s_i; e_1^k, \dots, e_{l_k}^k) \neq 0 \quad \textit{for } Re(s) > 0.$$

Now we apply Lemma 3.11 to each σ_0-chain appearing in $\rho_1 \times \cdots \times \rho_c$ to prove $\gamma(s, \sigma \times \tau) \neq 0$ for $Re(s) > 0$. ∎

4 A Square Integrability Criterion in Terms of γ-Factors

We now generalize a result on classical groups proved by Muić to our case (Theorem 4.4). It is not only important for the proof of the standard module conjecture but also interesting on its own. To begin with, we state a well-known result concerning poles of Rankin-Selberg L-functions attached to a pair of representations of GL_n. The proof follows immediately from multiplicativity of L-functions and the knowledge of L-functions of unitary supercuspidal representations. In [14] Muić reformulated the result as follows.

Lemma 4.1 *Let* $\Delta = [\nu^{-n}\sigma_0, \nu^{n}\sigma_0]$ *and* $\Delta' = [\nu^{-n'}\sigma_0', \nu^{n'}\sigma_0']$ *be segments where* σ_0, σ_0' *are irreducible unitary supercuspidal representations. Denote by* $\sigma = \sigma(\Delta)$ *and* $\sigma' = \sigma'(\Delta')$ *the square integrable representations attached to the segments* Δ *and* Δ', *respectively (see* [29] *for the definition of segments and the related theory of representations of general linear groups). Let c be a nonnegative real number. Then* $L(s, \rho \times \rho')$ *is holomorphic for* $Re(s) > 0$, *and it has a pole at* $s = -c$ *if and only if*

$$\sigma_0' \cong \widetilde{\sigma}_0,\ n' + c \geq n \geq |c - n'| \textit{ and } n + n' - c \in \mathbb{Z}.$$

The method used in the proof of [14, Theorem 5.1] can be employed to prove the following lemma.

Lemma 4.2 *Let* τ *be a generic square integrable representation of* $\mathrm{GSpin}_m(F)$, *and let* σ *be an irreducible square integrable representation of* $GL_k(F)$. *Assume that* $L(s, \sigma \times \tau)$ *has a pole at* $s = 0$. *Then, for any square integrable representation* σ' *of* $GL_n(F)$ *and for any* $s_0 \in \mathbb{R}$,

$$ord_{s_0} L(s, \sigma' \times \tau) \geq ord_{s_0} L(s, \sigma' \times \widetilde{\sigma})$$

where ord_{s_0} *denotes the order of the pole at* s_0.

Remark 4.3. Let $\sigma \otimes \tau$ be an irreducible ψ_F-generic representation of $GL_k(F) \times \mathrm{GSpin}_m(F)$. Then we define $L(s, \sigma \otimes \tau, r_1)$ via the Langlands–Shahidi method. On the other hand, we have $L(s, \sigma \times \tau)$ defined as the Rankin–Selberg L-function attached to (σ, τ). Note that $L(s, \sigma \times \tau) = L(s, \sigma \otimes \widetilde{\tau}, r_1)$. There is no confusion if we only consider the first L-functions as in the proof of Theorem 3.10, but it may cause some notational confusion when we consider first and second L-functions at the same time. Since we want to keep Rankin–Selberg L-function notation for $L(s, \sigma \otimes \tau, r_1)$, we replace τ by its contragredient. We use the same convention for γ-factors, *i.e.,* $\gamma(s, \sigma \times \tau) = \gamma(s, \sigma \otimes \widetilde{\tau}, r_1)$. For example, the local coefficient reads

$$C(s, \sigma \otimes \widetilde{\tau}, w_0, \psi_F) = \gamma(s, \sigma \times \tau, \psi_F)\gamma(s, \sigma \otimes \widetilde{\tau}, r_2, \psi_F).$$

We sometimes drop ψ_F from γ-factors when their dependence on the generic character ψ_F is clearly understood.

Let τ be an irreducible ψ_F-generic representation of $\mathbf{G}(F)$, where $\mathbf{G}$ is either GSpin_{2n+1} or GSpin_{2n}. There exists a standard parabolic subgroup $\mathbf{P} = \mathbf{MN}$ with

$$\mathbf{M} \cong GL_{n_1} \times \cdots \times GL_{n_l} \times \mathbf{G}_0,$$

where $\mathbf{G}_0$ is of the same type as $\mathbf{G}$ with a smaller rank, and a ψ_F-generic supercuspidal representation $\sigma_1 \otimes \cdots \otimes \sigma_l \otimes \tau_0$ of $\mathbf{M}(F)$ such that τ is a subquotient of $\sigma_1 \times \cdots \times \sigma_l \rtimes \tau_0$. Write $\sigma_i = \nu^{e(\sigma_i)} \sigma_i^u$ where σ_i^u is unitarizable and $e(\sigma_i) \in \mathbb{R}, 1, \leq i \leq l$.

Theorem 4.4 *An irreducible ψ_F-generic representation τ is square integrable modulo center if and only if the central character of τ is unitary and τ satisfies the following two conditions:*

(i) $\sigma_i^u \cong \widetilde{\sigma_i^u}$ *and* $2e(\sigma_i) \in \mathbb{Z}$.
(ii) *The function* $\gamma(s, \sigma \otimes \tau, r_1, \psi_F)\gamma(2s, \sigma \otimes \tau, r_2, \psi_F)$ *is holomorphic along the imaginary axis* $Re(s) = 0$ *and it has at most a simple zero at* $s = 0$ *for any irreducible square integrable representation* σ *of* GL_*.

Proof Assume that τ is square integrable modulo center. Then the central character of τ is unitary and (i) is true by Theorem 3.3. By Theorem 1.4, we have

$$\mu(s, \sigma \otimes \tau) = |\gamma(s, \sigma \otimes \tau, r_1, \psi_F)\gamma(2s, \sigma \otimes \tau, r_2, \psi_F)|^2, \quad Re(s) = 0,$$

for any ψ_F-generic representation $\sigma \otimes \tau$. Since $\sigma \otimes \tau$ is square integrable, we may apply Theorem 1.3 to obtain (ii).

For the converse, assume that the central character of τ is unitary and τ satisfies (i) and (ii). If τ is not square integrable, then there exists a standard parabolic subgroup $\mathbf{P} = \mathbf{MN}$ with $\mathbf{M} \cong GL_{n_1} \times \cdots \times GL_{n_k} \times \mathbf{G}_1$, where $\mathbf{G}_1$ is of the same type as $\mathbf{G}$ with a smaller rank, and a representation $\sigma_1 \otimes \cdots \otimes \sigma_k \otimes \tau_1$ of $\mathbf{M}(F)$ such that τ is a unique irreducible subquotient of $\sigma_1 \times \cdots \times \sigma_k \rtimes \tau_1$, where σ_i's are essentially square integrable representations of $GL_{n_i}(F), 1 \leq i \leq k$, and τ_1 is a generic tempered representation of $\mathbf{G}_1(F)$ (the Langlands quotient theorem, [6, Theorem 2.11]). Each essentially square integrable representation σ_i can be written as the representation attached to a segment $\Delta_i = [\nu^{-m_i+e_i}\sigma_{i,0}, \nu^{m_i+e_i}\sigma_{i,0}]$ where $2m_i \in \mathbb{Z}, m_i \geq 0$, and $\sigma_{i,0}$ is a unitary supercuspidal representation for $i = 1, \ldots, k$. The Langlands data imply that

$$e_1 \geq \cdots \geq e_k > 0 \tag{4.1}$$

unless $\mathbf{G} = \mathrm{GSpin}_{2n}$ splits and $\mathbf{M} \cong GL_{n_1} \times \cdots \times GL_{n_{k-1}} \times GL_1 \times \mathrm{GSpin}_0$ $(n_k = 1)$. In that case,

$$e_1 \geq \cdots \geq e_{k-1} > |e_k|. \tag{4.2}$$

Assumption (i) implies

$$\sigma_i^u \cong \widetilde{\sigma_i^u} \quad \text{and} \quad 2e_i \in \mathbb{Z}, \quad 1 \leq i \leq k.$$

Since τ is generic, by Proposition 5.4 of [8], we have

$$L(1,\widetilde{\sigma}_i\times\sigma_j)^{-1}\neq 0,\ L(1,\widetilde{\sigma}_i\times\widetilde{\sigma}_j)^{-1}\neq 0,\ 1\le i,j\le k, \tag{4.3}$$
$$L(1,\widetilde{\sigma}_i\times\tau_1)^{-1}\neq 0,\ L(1,\widetilde{\sigma}_i\otimes\widetilde{\tau}_1,r_2)^{-1}\neq 0, 1\le i\le k.$$

To handle (4.2), we need the following lemma.

Lemma 4.5 *If* $\mathbf{G}=\mathrm{GSpin}_{2n}$ *splits over* F *and*

$$\mathbf{M}\cong GL_{n_1}\times\cdots\times GL_{n_{k-1}}\times GL_1\times\mathrm{GSpin}_0\quad(n_k=1),$$

then $e_k\neq 0$.

Proof Suppose $e_k=0$. Then σ_k is a self-dual unitary character of $GL_1(F)=F^\times$. Since $\mathbf{G}_1=\mathrm{GSpin}_0=GL_1$, τ_1 is just a unitary character of $GL_1(F)$. Note that $\gamma(2s,\sigma_k\otimes\tau,r_2)$ is trivial. By multiplicativity of γ-factors (Theorem 1.2 or [21]), we have

$$\gamma(s,\sigma_k\times\widetilde{\tau})=\gamma(s,\sigma_k\times\widetilde{\tau}_1)\gamma(s,\sigma_k\times\sigma_k)^2\prod_{i=1}^{k-1}\gamma(s,\sigma_k\times\sigma_i)\gamma(s,\sigma_k\times\widetilde{\sigma}_i).$$

It follows from (4.3) that $\gamma(0,\sigma_k\times\widetilde{\tau}_1)\neq\infty$ and $\gamma(0,\sigma_k\times\widetilde{\sigma}_i)\neq\infty$ for all $i=1,\dots,k-1$. Since $\gamma(0,\sigma_k\times\sigma_i)\neq\infty$, $1\le i\le k-1$, and $\gamma(0,\sigma_k\times\sigma_k)=0$ by (4.3) and Lemma 4.1, $\gamma(s,\sigma_k\times\widetilde{\tau})$ has a double zero. It contradicts assumption (ii) (with $\sigma=\sigma_k$). ■

If $e_k<0$, then we may apply an outer automorphism $\widetilde{s}$ of GSpin_{2n} (see Section 2) so that we can assume $e_k>0$. From now on, we shall assume that the Langlands data is of the form (4.1).

Now we consider assumption (ii). Let σ be an arbitrary square integrable representation of GL_* such that $\sigma\cong\widetilde{\sigma}$, and let $\Delta=[\nu^{-m}\sigma_0,\nu^m\sigma_0]$ be the corresponding segment of σ where σ_0 is a self-dual unitary supercuspidal representation. By multiplicativity of γ-factors,

$$\gamma(s,\sigma\times\widetilde{\tau})=\gamma(s,\sigma\times\widetilde{\tau}_1)\prod_{i=1}^{k}\gamma(s,\sigma\times\sigma_i)\gamma(s,\sigma\times\widetilde{\sigma}_i).$$

Since the L-functions attached to tempered representations are holomorphic for $Re(s)>0$ by Theorem 3.10, assumption (ii) implies that

$$L(s,\sigma\times\widetilde{\tau}_1)L(2s,\sigma\otimes\tau_1,r_2)\prod_{i=1}^{k}\frac{L(s-e_i,\sigma\times\widetilde{\sigma_i^u})}{L(1-s-e_i,\sigma\times\widetilde{\sigma_i^u})}$$

is nonzero at $s=0$ and it has at most simple pole at $s=0$ (note that $L(2s,\sigma\otimes\tau,r_2)=L(2s,\sigma\otimes\tau_1,r_2)$ follows from Remark 3.7). Let

$$H(s,\sigma)=\prod_{i=1}^{k}\frac{L(s-e_i,\sigma\times\widetilde{\sigma_i^u})}{L(1-s-e_i,\sigma\times\widetilde{\sigma_i^u})}.$$

By multiplicativity of L-functions, we have

$$(4.4)\qquad H(s,\sigma)=\prod_{i=1}^{k}\prod_{j=1}^{\min(2m+1,2m_i+1)}\frac{L(s-e_i+m+m_i-j,\sigma_0\times\sigma_{0,i})}{L(1-s-e_i+m+m_i-j,\sigma_0\times\sigma_{0,i})}$$

for any square integrable representation $\sigma=\sigma(\Delta),\Delta=[\nu^{-m}\sigma_0,\nu^{m}\sigma_0]$. The representation τ_1 is a priori tempered but we shall show that it is square integrable.

Lemma 4.6 *τ_1 is square integrable.*

Proof Suppose τ_1 is a subquotient of $\rho_1\times\cdots\times\rho_l\rtimes\tau_2$ where $\rho_1\otimes\cdots\otimes\rho_l\otimes\tau_2$ is a square integrable representation. By assumption (i) of Theorem 3.3, the ρ_i's are self-dual. By multiplicativity of L-functions,

$$L(s,\rho_1\times\widetilde{\tau}_1)=L(s,\rho_1\times\widetilde{\tau}_2)\prod_{a=1}^{l}L(s,\rho_1\times\rho_a)L(s,\rho_1\times\widetilde{\rho}_a).$$

Since ρ_1 is self-dual by assumption (i), $L(s,\rho_1\times\widetilde{\tau}_1)$ has at least a double pole at $s=0$. For each i, $1\le i\le k$, we apply multiplicativity of L-functions,

$$L(s-e_i,\widetilde{\sigma_i^u}\times\tau_1)=L(s-e_i,\widetilde{\sigma_i^u}\times\tau_2)\prod_{a=1}^{l}L(s-e_i,\widetilde{\sigma_i^u}\times\rho_a)L(s-e_i,\widetilde{\sigma_i^u}\times\widetilde{\rho}_a).$$

Since $L(1-e_i,\widetilde{\sigma_i^u}\times\tau_1)^{-1}=L(1,\widetilde{\sigma}_i\times\tau_1)^{-1}\neq0$ by (4.3), we must have

$$L(1,\rho_1\times\widetilde{\sigma}_i)^{-1}=L(1-e_i,\rho_1\times\widetilde{\sigma_i^u})^{-1}\neq0\qquad\text{for all } i,\ 1\le i\le k.$$

It follows that $H(0,\rho_1)\neq0$. Now the function $L(s,\rho_1\times\widetilde{\tau}_1)L(s,\rho_1\otimes\tau_1,r_2)H(s,\rho_1)$ has at least a double pole at $s=0$, which contradicts assumption (ii). ■

Now we fix $d\in\{0,1/2\}$ and an irreducible self-dual supercuspidal representation σ_0 of GL_*. Then consider a set S such that

$$S=\left\{i\in\{1,\dots,k\}\ :\ \sigma_{i,0}\cong\sigma_0\text{ and }m_i+e_i\in d+\mathbb{Z}\right\}.$$

(Recall that τ is a subquotient of $\sigma_1\times\cdots\times\sigma_k\rtimes\tau_1$ and $\sigma_i=\sigma_i(\Delta_i)$ where $\Delta_i=[\nu^{-m_i+e_i}\sigma_{i,0},\nu^{m_i+e_i}\sigma_{i,0}]$.) Note that the set S depends only on d and σ_0. We may assume S is not empty by the assumption (i). Choose i_0 such that $e_{i_0}+m_{i_0}$ is maximal among e_i+m_i, $i\in S$. Let σ be the square integrable representation attached to the segment $\Delta=[\nu^{-e_{i_0}-m_{i_0}}\sigma_0,\nu^{e_{i_0}+m_{i_0}}\sigma_0]$. With this choice of σ, it is easy to see that $H(s,\sigma)$ has a pole at $s=0$. If there is $i\neq i_0$ such that $e_{i_0}+m_{i_0}=e_i+m_i$, then $H(s,\sigma)$ will have at least a double pole at $s=0$, which contradicts assumption (ii). It means that i_0 is unique. Since $L(s,\sigma\times\widetilde{\tau}_1)L(2s,\sigma\otimes\tau_1,r_2)H(s,\sigma)$ can have at most a simple pole at $s=0$ by assumption (ii), it follows that $L(2s,\sigma\otimes\tau_1,r_2)$ cannot have a pole at $s=0$. This fact, in turn, implies the following (by Corollary 3.9):

$$(4.5)\qquad \begin{cases}L(0,\sigma_0\otimes\tau_1,r_2)\neq\infty\text{ if } d=0,\ \text{or}\\ L(0,\sigma_0\otimes\tau_1,r_2)=\infty\text{ if } d=1/2.\end{cases}$$

Finally, if $e_i - m_i < 1$ for some $i \in S$, then Corollary 3.9 and (4.5) imply $L(1 - 2e_i, \sigma_i^u \otimes \tau_1, r_2) = \infty$. It follows from [22, Proposition 7.8] that

$$L(1 - 2e_i, \widetilde{\sigma_i^u} \otimes \widetilde{\tau}_1, r_2) = L(1, \widetilde{\sigma}_i \otimes \widetilde{\tau}_1, r_2) = \infty$$

which is against (4.3). We must have $e_i - m_i \geq 1$ for all $i \in S$.

Let $m = e_{i_0} - m_{i_0} - 1$. Then m is nonnegative by what we have observed from above. We define another square integrable representation σ attached to $\Delta = [\nu^{-m}\sigma_0, \nu^m\sigma_0]$ with m we just defined. Using the fact that Δ_i and Δ_{i_0} is not linked for any $i \neq i_0$, we deduce from (4.4) that $H(0, \sigma) = 0$. Since $L(2s, \sigma \otimes \tau_1, r_2)$ cannot have a pole at $s = 0$ by Corollary 3.9 and (4.5), it follows from assumption (ii) that $L(s, \sigma \times \widetilde{\tau}_1)$ has a pole at $s = 0$. Note that $L(1 - e_{i_0}, \sigma_{i_0}^u \times \sigma) = \infty$ by Lemma 4.1. Now apply Lemma 4.2: $\operatorname{ord}_{s_0} L(s, \sigma' \times \widetilde{\tau}_1) \geq \operatorname{ord}_{s_0} L(s, \sigma' \times \widetilde{\sigma}) > 0$ with $\sigma' = \sigma_{i_0}^u$ and $s_0 = 1 - e_{i_0}$. It follows that $L(1 - e_{i_0}, \sigma_{i_0}^u \times \widetilde{\tau}_1) = \infty$. By [22, Proposition 7.8], we have $L(1 - e_{i_0}, \widetilde{\sigma_{i_0}^u} \times \tau_1) = L(1, \widetilde{\sigma_{i_0}} \times \tau_1) = \infty$, which contradicts (4.3). ■

5 The Standard Module Conjecture and its Application to Functoriality

In this section we prove the standard module conjecture for GSpin_m. First we give the definition of standard modules in the general setting. Let $\mathbf{G}$ be a connected, reductive, quasisplit algebraic group, not necessarily a general spin group, defined over F. Suppose π is an irreducible tempered representation of $\mathbf{M}(F)$ where $\mathbf{M}$ is a Levi subgroup of a standard parabolic subgroup $\mathbf{P} = \mathbf{MN}$ of $\mathbf{G}$. A standard module is an induced representation

$$I(\nu, \pi) = \operatorname{Ind}_P^G(q^{\langle \nu, H_M(\cdot) \rangle} \otimes \pi) = \operatorname{Ind}_P^G(\pi_\nu), \quad \pi_\nu = q^{\langle \nu, H_M(\cdot) \rangle} \otimes \pi$$

with ν in the positive Weyl chamber of the complex dual of the Lie algebra of the split component $\mathbf{A}$ of $\mathbf{M}$. This induced representation has the Langlands quotient $J(\nu, \pi)$. Now we state the standard module conjecture.

Conjecture 5.1 Let π be a generic tempered representation of $\mathbf{M}(F)$. A standard module $I(\nu, \pi)$ is irreducible if and only if the Langlands quotient $J(\nu, \pi)$ is generic.

From now on, we assume $\mathbf{G}$ is a general spin group. We start with a consequence of Theorem 4.4.

Proposition 5.2 *Let $\sigma_1 \otimes \cdots \otimes \sigma_l \otimes \tau_0$ be an irreducible ψ_F-generic supercuspidal representation of $\mathbf{M}(F)$ where $\mathbf{M} \cong GL_{n_1} \times \cdots \times GL_{n_l} \times \mathrm{GSpin}_{m_0}$. If $\sigma_1 \times \cdots \times \sigma_l \rtimes \tau_0$ contains a square integrable representation, then the unique ψ_F-generic irreducible subquotient is square integrable.*

Proof There is a unique ψ_F-generic subquotient of $\sigma_1 \times \cdots \times \sigma_l \rtimes \tau_0$ by Rodier's Theorem [15]. Denote by τ the unique ψ_F-generic subquotient. Let τ' be a square integrable subquotient of $\sigma_1 \times \cdots \times \sigma_l \rtimes \tau_0$. We shall use Theorem 4.4 to prove that τ is square integrable. The necessary condition (i) of Theorem 4.4 is satisfied

https://doi.org/10.4153/CJM-2009-033-3 Published online by Cambridge University Press

by Theorem 3.3 applied to the assumption of this proposition that τ' is a square integrable subquotient of $\sigma_1 \times \cdots \times \sigma_l \rtimes \tau_0$. To check another necessary condition (ii) of Theorem 4.4, let σ be any square integrable representation of GL_*. Multiplicativity of γ-factors (Theorem 1.2) implies that

$$\gamma(s, \sigma \otimes \tau, r_1) = \gamma(s, \sigma \otimes \tau_0, r_1) \prod_{i=1}^{l} \gamma(s, \sigma \otimes \tilde{\sigma}_i, r_1)\gamma(s, \sigma \otimes \sigma_i, r_1).$$

On the other hand, by multiplicativity of Plancherel measures [25], we have

$$\mu(s, \sigma \otimes \tau') = \mu(s, \sigma \otimes \tau_0) \prod_{i=1}^{l} \mu(s, \sigma \otimes \sigma_i)\mu(s, \sigma \otimes \tilde{\sigma}_i).$$

By [18, Theorem 6.1],

$$\mu(s, \sigma \otimes \sigma_i) = \gamma(s, \sigma \otimes \sigma_i)\gamma(-s, \widetilde{\sigma} \otimes \widetilde{\sigma_i}), \quad i = 1, \ldots, l. \tag{5.1}$$

Using (5.1), Remark 3.7, and Theorem 1.4, we obtain the following equality

$$|\gamma(s, \sigma \otimes \tau, \psi_F)\gamma(2s, \sigma \otimes \tau, r_2, \psi_F)|^2 = \mu(s, \sigma \otimes \tau').$$

Since τ' is square integrable modulo center, $\mu(s, \sigma \otimes \tau')$ is holomorphic and can have at most double zero at $s = 0$ by Theorem 1.3. This completes the proof. ∎

The following corollary of Theorem 5.1 concerns tempered subquotients.

Corollary 5.3 *Let $\sigma_1 \otimes \cdots \otimes \sigma_l \otimes \tau_0$ be an irreducible ψ_F-generic supercuspidal representation of $\mathbf{M}(F)$ where $\mathbf{M} \cong GL_{n_1} \times \cdots \times GL_{n_l} \times \mathrm{GSpin}_{m_0}$. If $\sigma_1 \times \cdots \times \sigma_l \rtimes \tau_0$ contains an irreducible tempered representation, then the unique ψ_F-generic irreducible subquotient is tempered.*

Proof Let τ be a tempered subquotient of $\sigma_1 \times \cdots \times \sigma_l \rtimes \tau_0$. Since τ is tempered, τ can be realized as an irreducible direct summand of $\rho_1 \times \cdots \times \rho_k \rtimes \tau_1$ where $\rho_1 \otimes \cdots \otimes \rho_k \otimes \tau_1$ is a square integrable representation of $\mathbf{M}_1(F)$. We may assume that τ_1 is an irreducible subquotient of $\sigma_{l'} \times \cdots \times \sigma_l \rtimes \tau_0$ for some l'. Let τ' be the unique ψ_F-generic irreducible subquotient of $\sigma_{l'} \times \cdots \times \sigma_l \rtimes \tau_0$. By Proposition 5.2, τ' is square integrable. Any irreducible subquotient of $\rho_1 \times \cdots \times \rho_k \rtimes \tau'$ is a subquotient of $\sigma_1 \times \cdots \times \sigma_l \rtimes \tau_0$, in particular, the unique ψ_F-generic subquotient of $\rho_1 \times \cdots \times \rho_k \rtimes \tau'$ is a subquotient of $\sigma_1 \times \cdots \times \sigma_l \rtimes \tau_0$. It follows that the unique ψ_F-generic irreducible subquotient of $\sigma_1 \times \cdots \times \sigma_l \rtimes \tau$ is tempered. ∎

Now we provide a proof of the standard module conjecture for GSpin_m.

Theorem 5.4 *Suppose $\rho_1, \ldots, \rho_k$ are essentially tempered representations of $GL_{n_i}(F)$ with $e(\rho_1) > \cdots > e(\rho_k) > 0$, where $\rho_i = \nu^{e(\rho_i)}\rho_i^u$, $e(\rho_i) \in \mathbb{R}$, and τ is a ψ_F-generic tempered representation of $\mathrm{GSpin}_m(F)$. The standard module $\rho_1 \times \cdots \times \rho_k \rtimes \tau$ reduces if and only if the Langlands quotient $J(\rho_1 \otimes \cdots \otimes \rho_k \otimes \tau)$ is not ψ_F-generic.*

Proof Denote by π the Langlands quotient $J(\rho_1 \otimes \cdots \otimes \rho_k \otimes \tau)$. Let $\mathbf{T}$ be the (fixed) torus of $\mathbf{G}$, and let $\mathbf{Z}$ the center of $\mathbf{G}$. Define $\mathbf{T}^s$ be the quotient of $\mathbf{T}$ by the connected component of $\mathbf{Z}$. We associate to π its real parameter

$$\nu_\pi = (\underbrace{e(\rho_1), \dots, e(\rho_1)}_{n_1 \text{ times}}, \dots, \underbrace{e(\rho_k), \dots, e(\rho_k)}_{n_k \text{ times}}, \underbrace{0, \dots, 0}_{m \text{ times}}) \in X(\mathbf{T}^s)_F \otimes_{\mathbb{Z}} \mathbb{R}.$$

We shall use the partial ordering on $X(\mathbf{T}^s) \otimes_{\mathbb{Z}} \mathbb{R}$ defined by Borel and Wallach [6]. Suppose $\rho_1 \times \cdots \times \rho_k \rtimes \tau$ reduces. We need to show that π canot be ψ_F-generic. Let π' be an irreducible subquotient of $\rho_1 \times \cdots \times \rho_k \rtimes \tau$ which is not isomorphic to π. Then π' can be written as the unique Langlands quotient $J(\rho_1' \otimes \cdots \otimes \rho_{k'}' \otimes \tau')$, where $\rho_1', \dots, \rho_{k'}'$ are essentially tempered representations of $GL_{n_i'}(F)$, with $e(\rho_1') > \cdots > e(\rho_{k'}') > 0$, and τ' is a tempered representation of $\mathrm{GSpin}_{m'}(F)$. By [6, Lemma 2.13], we have $\nu_{\pi'} < \nu_\pi$. We observe that π and π' have the same supercuspidal support. To be precise, suppose that $\rho_1' \times \cdots \times \rho_{k'}'$ is a subquotient of $\sigma_1 \times \cdots \times \sigma_{l'}$ where $\sigma_1, \dots, \sigma_{l'}$ are supercuspidal representations of $GL_*(F)$, and τ' is a subquotient of $\sigma_{l'+1} \times \cdots \times \sigma_l \rtimes \tau_0$ where $\sigma_{l'+1}, \dots, \sigma_l$ and τ_0 are supercuspidal representations of $GL_*(F)$ and $\mathrm{GSpin}_{m_0}(F)$, respectively. Then

$$\sigma_1 \times \cdots \times \sigma_{l'} \times \sigma_{l'+1} \times \cdots \times \sigma_l \rtimes \tau_0$$

contains all the irreducible subquotients of $\rho_1 \times \cdots \times \rho_k \rtimes \tau$ and $\rho_1' \times \cdots \times \rho_{k'}' \rtimes \tau'$. Note that τ_0 is ψ_F-generic. Let τ'' be the unique ψ_F-generic irreducible subquotient of $\sigma_{l'+1} \times \cdots \times \sigma_l \rtimes \tau_0$. Then τ'' is tempered by Corollary 5.3. Let π'' be the Langlands quotient of $\rho_1' \times \cdots \times \rho_{k'}' \rtimes \tau''$. Now we have $\nu_{\pi''} = \nu_{\pi'} < \nu_\pi$. It follows that π cannot be an irreducible subquotient of $\rho_1' \times \cdots \times \rho_{k'}' \rtimes \tau''$ by [6, Lemma 2.13]. On the other hand, the unique irreducible ψ_F-generic subquotient of $\sigma_1 \times \cdots \times \sigma_{l'} \times \sigma_{l'+1} \times \cdots \times \sigma_l \rtimes \sigma$ is a subquotient of $\rho_1' \times \cdots \times \rho_{k'}' \rtimes \tau''$. We conclude that π cannot be ψ_F-generic. ■

Remark 5.5. In the statement of Theorem 5.4, $e(\rho_1) > \cdots > e(\rho_k) > 0$ is not a restrictive assumption. There are two types of Langlands situation, either $e(\rho_1) > \cdots > e(\rho_k) > 0$ or $e(\rho_1) > \cdots > |e(\rho_k)|$. The latter can happen if $\mathbf{G} = \mathrm{GSpin}_{2n}$ splits over F and

$$\mathbf{M} \cong GL_{n_1} \times \cdots \times GL_{n_{k-1}} \times GL_1 \times \mathrm{GSpin}_0 \quad (n_k = 1).$$

As in the proof of Theorem 4.4, we may apply an outer automorphism of GSpin_{2n} if $e(\rho_k)$ is negative. So we may assume $e(\rho_1) > \cdots > e(\rho_k) > 0$ without loss of generality.

Theorem 5.4 also implies the standard module conjecture for spin groups. This fact follows from a more general argument. More precisely, we have the following proposition by Asgari and Shahidi [2].

Proposition 5.6 *Let $\mathbf{G}$ and $\mathbf{G}'$ be reductive quasisplit algebraic groups such that $\mathbf{G} \subset \mathbf{G}'$. Assume that $\mathbf{G}$ and $\mathbf{G}'$ have the same derived group. Let $\mathbf{P}' = \mathbf{M}'\mathbf{N}$ be a maximal standard parabolic subgroup of $\mathbf{G}'$ and $\mathbf{P} = \mathbf{MN}$ the corresponding one of*

https://doi.org/10.4153/CJM-2009-033-3 Published online by Cambridge University Press

G *with* $\mathbf{M} = \mathbf{M}' \cap \mathbf{G}$*. Suppose* π' *is an irreducible quasi-tempered representation of* $\mathbf{M}'(F)$*. Let* π *be the restriction of* π' *to* $\mathbf{M}(F)$*. Denote by* π_i *the irreducible components of* $\pi = \bigoplus \pi_i$*. Then* $Ind_{P'}^{G}(\pi')$ *is standard and irreducible if and only if each* $Ind_{P}^{G}(\pi_i)$ *is standard and irreducible.*

Finally we consider an application of the standard module conjecture. Let **G** be a reductive quasisplit algebraic group, not necessarily a general spin group. Suppose that $\mathbf{P} = \mathbf{MN}$ is a maximal parabolic subgroup of **G**. Let $\pi = \bigotimes_v \pi_v$ be a generic cuspidal representation of $\mathbf{M}(\mathbb{A}_F)$. Denote by $A(s, \pi_v, w_0)$ the local intertwining operator for $I(s, \pi_v)$. The normalized intertwining operator is defined by

$$N(s, \pi_v, w_0) = \prod_{i=1}^{m} \frac{L(1 + is, \pi_v, r_i)\epsilon(is, \pi_v, r_i, \psi)}{L(is, \pi_v, r_i)} A(s, \pi_v, w_0).$$

The following conjecture concerning holomorphy of normalized intertwining operators has a profound implication in Langlands functorial lifting problems.

Conjecture 5.7 $N(s, \pi_v, w_0)$ is holomorphic and non-vanishing for $Re(s) \geq 1/2$.

This conjecture is true, by the result of [11, 12], if the standard module conjecture is proved and if L-functions attached to tempered representations are holomorphic for $Re(s) > 0$ (Conjecture 1.1). Since the holomorphy of L-functions and the standard module conjecture are proved for general spin groups in this paper, Conjecture 5.7 is a theorem for general spin groups. This fact is a necessary ingredient in the proof of functoriality of a generic transfer from general spin groups to general linear groups [2].

Acknowledgments I would like thank my thesis advisor Prof. F. Shahidi for his encouragement and constant help during the preparation of this paper. I also benefited from useful discussions with Prof. D. Goldberg.

I am grateful to the anonymous referee for his (her) constructive criticism which helped correct some mistakes and improve the quality of the paper.

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