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TWO QUESTIONS ON SEMIGROUP LAWS

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B.H. Neumann recently proved some implications for semigroup laws in groups. This may help in the solution of a problem posed by G.M. Bergman in 1981.

INTRODUCTION

Let G be a group, and $S \subseteq G$ be a subsemigroup generating G. It is clear that if S is commutative, then G is commutative. The following question is equivalent to the one posed by Bergman [**2, 3**].

QUESTION 1. Let S generating G satisfy a law. Must G satisfy the same law?

For some laws the answer is positive ([**9, 5, 8, 1**]), however in general the question is open and in the opinion of Ivanov and Rips it has a negative answer. All semigroups we consider are cancellative.

QUESTION 2. Let a semigroup law $a = b$ imply a semigroup law $u = v$ in groups. Does the same implication hold in semigroups?

To show implication of laws in semigroups we can use only so-called positive endomorphisms, which map generators to positive words. It is shown in [**8**] (an example at the end of this paper), that all implications for positive laws of length $\leqslant 5$ which hold in groups, also are valid for semigroups. The fact that the law $x^2y^2x = yx^3y$ implies $xy = yx$ in semigroups (and hence in groups) is proved in [**5**, p.132].

We show the equivalence of the above Questions.

It is shown in [**10**], that the law $x^{s+t}y^2x^t = yx^{s+2t}y$, $\gcd(s,t) = 1$, implies $xy^2x = yx^2y$ in groups (which is equivalent to $[x, y, x] = 1$ [**12**]). So if there exists a semigroup satisfying $x^{s+t}y^2x^t = yx^{s+2t}y$, $\gcd(s,t) = 1$, but not $xy^2x = yx^2y$, the desired counterexample for Question 1 would be found.

Let $a = a(x_1, \dots, x_n)$, $b = b(x_1, \dots, x_n)$ be positive words. A semigroup law $a = b$ is called *balanced* if every x_i has the same exponent sum in a and b. The law is *trivial* if $ab^{-1} = 1$ in F. The law is called *cancelled* if the first (and the last) letters in a and b are different.

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Notation Let F be a free group and $\mathcal{F} \ni 1$ be a free semigroup, both generated by $x_1, x_2, x_3, \ldots$. Words in $\mathcal{F}$ are called positive. We denote:

End$^+$ – the set of positive endomorphisms which map x_i to positive words,

N_w – a normal End$^+$-invariant closure of a word w in F,

End – the set of all endomorphisms of the free group F,

V_w – a fully invariant subgroup generated by a word $w \in F$,

$(u, v)^\#$ – the smallest cancellative congruence in $\mathcal{F}$ implying the law $u = v$.

A relatively free cancellative semigroup, defined by the law $u = v$ is isomorphic to $\mathcal{F}/(u, v)^\#$ [**8**].

We note that if N_w contains a positive word, say x^2yz^4, then it contains x^7 and hence $x^{-1} \in x^6 N_w$ implies $F = \mathcal{F} \bmod N_w$.

Remark 1. Since each semigroup with a non-balanced law is a group, we have to consider only balanced non-trivial semigroup laws. Each such law implies a binary balanced and cancelled law $A(x, y) = B(x, y)$ ([**6**]).

Questions and Results

To formulate the above Questions in terms of normal subgroups we need

Lemma 1. *A semigroup law $u = v$ implies $a = b$ in semigroups if and only if $N_{ab^{-1}} \subseteq N_{uv^{-1}}$. The implication holds in groups if and only if $V_{ab^{-1}} \subseteq V_{uv^{-1}}$.*

Proof: The law $u = v$ implies $a = b$ in semigroups if and only if the corresponding smallest congruences satisfy $(a, b)^\# \subseteq (u, v)^\#$. If we map $F \to F/N$, then $\mathcal{F}$ is mapped onto $\mathcal{F}/N^\#$, where $N^\#$ is a cancellative congruence in $\mathcal{F}$ defined as: $N^\# = \{(s, t);\ st^{-1} \in N \cap \mathcal{F}\mathcal{F}^{-1}\}$. It is proved in [**7**, Theorem 2], that $N := N_{uv^{-1}}$ is the smallest normal subgroup such that $N^\# = (u, v)^\#$. So we have

$$(1) \qquad (u, v)^\# = \{(s, t);\ st^{-1} \in N_{uv^{-1}} \cap \mathcal{F}\mathcal{F}^{-1}\}.$$

Since $\mathcal{F}/(u, v)^\#$ is embeddable into a group $F/N_{uv^{-1}}$, and $N_{uv^{-1}}$ is the smallest normal subgroup with this property, it follows by [**4**, 12.3], that

$$(2) \qquad N_{uv^{-1}} = gpn(st^{-1};\ (s, t) \in (u, v)^\#).$$

Hence by (1), (2): $(a, b)^\# \subseteq (u, v)^\#$ if and only if $N_{ab^{-1}} \subseteq N_{uv^{-1}}$, which gives the first statement of the Lemma. The second statement is known [**11**]. ∎

In terms of normal subgroups the above Questions are:

Question 1′. Does $N_{ab^{-1}} = V_{ab^{-1}}$ hold for each semigroup law $a = b$?

Question 2′. Does $V_{ab^{-1}} \subseteq V_{uv^{-1}}$ imply $N_{ab^{-1}} \subseteq N_{uv^{-1}}$ for semigroup laws $a = b$ and $u = v$?

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We shall prove that for each semigroup law $a = b$ there exists a semigroup law $u = v$ such that the fully invariant closure of ab^{-1} coincides with the End$^+$-invariant normal closure of uv^{-1}. This will imply the equivalence of the Questions.

THEOREM. *For every n-variable semigroup law $a = b$ there exists an $n+1$-variable semigroup law $u = v$ such that the equality $V_{ab^{-1}} = N_{uv^{-1}}$ holds.*

COROLLARY. *Questions 1 and 2 are equivalent.*

PROOF: We have to show that for each semigroup law $a = b$ the equality holds: $N_{ab^{-1}} = V_{ab^{-1}}$. Take $u = v$ as in the Theorem, then $V_{ab^{-1}} \stackrel{\mathrm{T}}{=} N_{uv^{-1}}$. By taking the fully invariant closure we get $V_{ab^{-1}} = V_{uv^{-1}}$. If Question 2 has a positive answer then we have $N_{ab^{-1}} = N_{uv^{-1}} \stackrel{\mathrm{T}}{=} V_{ab^{-1}}$, as required. ∎

LEMMAS AND PROOF OF THE THEOREM

LEMMA 2. *Let $A(x,y) = B(x,y)$ be a balanced and cancelled semigroup law such that the first letter in $A(x,y)$ is x. Then there exist $a_i = a_i(x,y)$, $b_i = b_i(x,y) \in \mathcal{F}$, $i = 1,2$, such that*

(i) $x^{-1}y = a_1 b_1^{-1} \cdot (A^{-1}B)^{b_1^{-1}}$,

(ii) $xy^{-1} = a_2^{-1} b_2 \cdot (AB^{-1})^{\varepsilon b_2}$, $\varepsilon = \pm 1$,

(iii) $F = \mathcal{F}\mathcal{F}^{-1} N_{AB^{-1}} = \mathcal{F}^{-1}\mathcal{F} N_{AB^{-1}}$.

PROOF: Since the law $A = B$ is cancelled, it can be written as $x \cdot a_1 = y \cdot b_1$, which gives $A^{-1}B = a_1^{-1}x^{-1}yb_1$ and hence (i). The law $A = B$, (or $B = A$) can be written as $a_2 \cdot x = b_2 \cdot y$. In the first case $AB^{-1} = a_2 xy^{-1} b_2$ gives $xy^{-1} = a_2^{-1}b_2 \cdot (AB^{-1})^{b_2}$. If $B = a_2 \cdot x$, $A = b_2 \cdot y$, then $xy^{-1} = a_2^{-1}b_2 \cdot (AB^{-1})^{-b_2}$, which gives (ii).

Since $A^{-1}B = (AB^{-1})^{-B} \in N_{AB^{-1}}$, we get from (i), that $x^{-1}y \in \mathcal{F}\mathcal{F}^{-1} \bmod N_{AB^{-1}}$, which holds under every substitution of elements from $\mathcal{F}$ for x and y. Since every word in F is a product of words in $\mathcal{F} \cup \mathcal{F}^{-1}$, we get $F = \mathcal{F}\mathcal{F}^{-1}N_{AB^{-1}}$. Similarly, from (ii) we get $F = \mathcal{F}^{-1}\mathcal{F}N_{AB^{-1}}$. ∎

The following Lemma is well known in terms of a group of fractions and Ore conditions.

LEMMA 3. *Let $a = b$ be a nontrivial semigroup law, and $g_1, g_2, \dots, g_n$ be elements in F. Then there exist elements $s_1, s_2, \dots, s_n$ and d in $\mathcal{F}$ such that $g_i = s_i d^{-1} \bmod N_{ab^{-1}}$.*

PROOF: By [6], the law $a = b$ implies a balanced and cancelled binary law $A = B$. Since $N_{AB^{-1}} \subseteq N_{ab^{-1}}$, the inclusions in Lemma 2 are valid $\bmod N_{ab^{-1}}$. Then by (iii) we have *modulo* $N_{ab^{-1}}$: $g_i = a_i b_i^{-1}$ for some $a_i, b_i \in \mathcal{F}$. For $n = 2$, $g_1 = a_1 b_1^{-1}$, $g_2 = a_2 b_2^{-1}$. Also by (iii), there exist c, d such that $b_2^{-1}b_1 = cd^{-1}$. We introduce $r := b_1 d = b_2 c$, then $g_1 = a_1 b_1^{-1} = a_1 dd^{-1} b_1^{-1} = a_1 d r^{-1} =: sr^{-1}$, $g_2 = a_2 b_2^{-1} = a_2 cc^{-1} b_2^{-1} = a_2 c r^{-1} =: tr^{-1}$, $s, t, r \in \mathcal{F}$. So, by repeating this step we can write $g_1, \dots, g_n$ with a "common denominator" $\bmod N_{ab^{-1}}$ as required. ∎

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To compare End$^+$-invariant and End-invariant closures of words we make an observation that by positive endomorphisms we can map xy^{-1} into any word $g \in F$ mod $N_{ab^{-1}}$ if we write $g = st^{-1}$ and map x to s, and y to t.

LEMMA 4. *There exists an automorphism $\alpha \in \operatorname{Aut} F$ such that for any $w \in F$, N_{w^α} is fully invariant mod $N_{ab^{-1}}$, for any nontrivial $ab^{-1} \in \mathcal{F}\mathcal{F}^{-1}$. That is $V_w \subseteq N_{w^\alpha} N_{ab^{-1}}$.*

PROOF: Let $w = w(x_1, \dots, x_n)$. We take $\alpha \in \operatorname{Aut} F$ which maps $x_i \to x_i x_{n+1}^{-1}$, $i = 1, \dots, n$ and leaves x_i, $i > n$, fixed. It is enough to show that for any $g_1, \dots, g_n$ in F, $w(g_1, \dots, g_n) \in N_{w^\alpha} N_{ab^{-1}}$. By Lemma 3, we write $g_i = s_i d^{-1}$ mod $N_{ab^{-1}}$ and define $\nu \in \mathrm{End}^+$ by $x_i^\nu = s_i$, $i \leqslant n$, and $x_{n+1}^\nu = d$. Then *modulo* $N_{ab^{-1}}$ we have $(x_i x_{n+1}^{-1})^\nu = g_i$ and $w(g_1, \dots, g_n) = w(x_1 x_{n+1}^{-1}, \dots, x_n x_{n+1}^{-1})^\nu = \left(w(x_1, \dots, x_n)^\alpha\right)^\nu \in N_{w^\alpha}^\nu \subseteq N_{w^\alpha}$, as required. ∎

COROLLARY 1. *For a nontrivial semigroup law $a = b$ the following equality holds:*

$$V_{ab^{-1}} = N_{(ab^{-1})^\alpha}.$$

PROOF: We have $ab^{-1} \in N_{(ab^{-1})^\alpha}^{\alpha^{-1}}$. Since α^{-1} is in End$^+$, then $N_{(ab^{-1})^\alpha}^{\alpha^{-1}} \subseteq N_{(ab^{-1})^\alpha}$ and hence $ab^{-1} \in N_{(ab^{-1})^\alpha}$, which gives

$$N_{ab^{-1}} \subseteq N_{(ab^{-1})^\alpha}. \tag{3}$$

By Lemma 4 for $w := ab^{-1}$, by (3), and since End$^+ \subseteq$ End, we have:

$$V_{ab^{-1}} \subseteq N_{(ab^{-1})^\alpha} N_{ab^{-1}} = N_{(ab^{-1})^\alpha} \subseteq V_{ab^{-1}},$$

which implies $V_{ab^{-1}} = N_{(ab^{-1})^\alpha}$. ∎

We denote by δ the endomorphism which maps $x_{n+1} \to 1$ and leaves other generators fixed, then $\delta \in \mathrm{End}^+$. As above, $\alpha \in \operatorname{Aut} F$ maps $x_i \to x_i x_{n+1}^{-1}$, $i = 1, \dots, n$ and leaves x_i, $i > n$, fixed.

LEMMA 5. *Let $a = b$ be a nontrivial semigroup law, and $\mathcal{F}_{n+1}$ be a free subsemigroup generated by $x_1, \dots, x_{n+1}$. Then for any positive word $p(x_1, \dots, x_n)$, there exist positive words $u_i = u_i(x_1, \dots, x_{n+1})$, $v_i = v_i(x_1, \dots, x_{n+1})$, $i = 1, 2$, such that $p^\alpha = u_1 v_1^{-1} = u_2^{-1} v_2$ mod $(N_{ab^{-1}} \cap \operatorname{Ker} \delta)$.*

PROOF: We show first that for any words $c, q \in \mathcal{F}_{n+1}$ the following inclusions hold:

$$cx_{n+1}^{-1} \in \mathcal{F}_{n+1}^{-1} \mathcal{F}_{n+1} \bmod (N_{ab^{-1}} \cap \operatorname{Ker} \delta), \tag{$*$}$$

$$x_{n+1}^{-1} q \in \mathcal{F}_{n+1} \mathcal{F}_{n+1}^{-1} \bmod (N_{ab^{-1}} \cap \operatorname{Ker} \delta). \tag{$**$}$$

The law $a = b$ implies the balanced and cancelled binary law $A = B$, so it is enough to prove the inclusions for the law $A(x, y) = B(x, y)$.

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If we apply δ to the balanced equality $A(c, x_{n+1}) = B(c, x_{n+1})$, it becomes trivial, and hence the word $AB^{-1}(c, x_{n+1})$ is in $\operatorname{Ker}\delta$. Similarly we get $A^{-1}B(x_{n+1}, q) \in \operatorname{Ker}\delta$. We put now c, x_{n+1}, for x, y, in (ii) (Lemma 2) to get $(*)$, and then put x_{n+1}, q, in (i) (Lemma 2) to get $(**)$.

We continue the proof *modulo* $(N_{ab^{-1}} \cap \operatorname{Ker}\delta)$. To show that:

$$p(x_1x_{n+1}^{-1}, \dots, x_nx_{n+1}^{-1}) \in \mathcal{F}_{n+1}\mathcal{F}_{n+1}^{-1},$$

and

$$p(x_1x_{n+1}^{-1}, \dots, x_nx_{n+1}^{-1}) \in \mathcal{F}_{n+1}^{-1}\mathcal{F}_{n+1},$$

we use induction on the length $|p| = m$. Let $p(x_1, \dots, x_n) = c_mc_{m-1}\dots c_2c_1$, $c_i \in \{x_1, \dots, x_n\}$, then $p^\alpha = c_mx_{n+1}^{-1}c_{m-1}x_{n+1}^{-1}\dots c_2x_{n+1}^{-1}c_1x_{n+1}^{-1}$. For $m = 1$, $p^\alpha = cx_{n+1}^{-1} \in \mathcal{F}_{n+1}\mathcal{F}_{n+1}^{-1}$ and by $(*)$, $p^\alpha = cx_{n+1}^{-1} \in \mathcal{F}_{n+1}^{-1}\mathcal{F}_{n+1}$.

Let $|p| = m$, then $p = c_mc_{m-1}\dots c_2c_1$ and by inductive assumption $p^\alpha = c_mx_{n+1}^{-1} \cdot qr^{-1}$. Then by $(**)$, there exist $s, t \in \mathcal{F}_{n+1}$, such that $x_{n+1}^{-1}q = st^{-1}$ and hence $p^\alpha = c_m(x_{n+1}^{-1}q)r^{-1} = c_m(st^{-1})r^{-1} = (c_ms)(rt)^{-1} \in \mathcal{F}_{n+1}\mathcal{F}_{n+1}^{-1}$.

Again for $|p| = m$, we get by assumption $p^\alpha = r^{-1}s \cdot c_1x_{n+1}^{-1} = r^{-1}(sc_1)x_{n+1}^{-1}$. By $(*)$ for sc_1 instead of c, there exist $t, u \in \mathcal{F}_{n+1}$, such that $sc_1x_{n+1}^{-1} = t^{-1}u$. Then $p^\alpha = r^{-1}(sc_1)x_{n+1}^{-1} = r^{-1}t^{-1}u = (tr)^{-1}u \in \mathcal{F}_{n+1}^{-1}\mathcal{F}_{n+1}$ as required. ∎

Proof of the Theorem

We have to show that for every nontrivial n-variable semigroup law $a = b$ there exists an $n+1$-variable semigroup law $u = v$ such that $V_{ab^{-1}} = N_{uv^{-1}}$.

By Lemma 5 for the words $a = a(x_1, \dots, x_n)$ and $b = b(x_1, \dots, x_n)$ we get respectively:

$$a^\alpha = u_1v_1^{-1} \bmod (N_{ab^{-1}} \cap \operatorname{Ker}\delta),$$

and

$$b^\alpha = u_2^{-1}v_2 \bmod (N_{ab^{-1}} \cap \operatorname{Ker}\delta).$$

Then

$$(ab^{-1})^\alpha = u_1v_1^{-1}v_2^{-1}u_2 = u_2^{-1}(u_2u_1)(v_2v_1)^{-1}u_2 \bmod (N_{ab^{-1}} \cap \operatorname{Ker}\delta).$$

We denote $u := u_2u_1$, $v := v_2v_1$, then

(4) $$(ab^{-1})^\alpha = (uv^{-1})^{u_2} \bmod (N_{ab^{-1}} \cap \operatorname{Ker}\delta)$$

This implies:

(5) $$N_{(ab^{-1})^\alpha} \subseteq N_{uv^{-1}}N_{ab^{-1}}$$

and

(6) $$N_{uv^{-1}} \subseteq N_{(ab^{-1})^\alpha}N_{ab^{-1}}.$$

To prove the equality

$$N_{(ab^{-1})^\alpha} = N_{uv^{-1}}, \tag{7}$$

we apply δ to (4). Since $\alpha\delta$ is the identity map on x_i, $i \leqslant n$, and δ is in End^+, we have that $ab^{-1} = (ab^{-1})^{\alpha\delta}$ is conjugate to $(uv^{-1})^\delta \in N^\delta_{uv^{-1}} \subseteq N_{uv^{-1}}$. This implies $N_{ab^{-1}} \subseteq N_{uv^{-1}}$ which, together with (5) gives $N_{(ab^{-1})^\alpha} \subseteq N_{uv^{-1}}$. Since by (3), $N_{ab^{-1}} \subseteq N_{(ab^{-1})^\alpha}$, it follows from (6), that $N_{uv^{-1}} \subseteq N_{(ab^{-1})^\alpha}$, and hence (7) holds.

Now, since by Corollary 1, $V_{ab^{-1}} = N_{(ab^{-1})^\alpha}$, we have by (7), the required equality $V_{ab^{-1}} = N_{uv^{-1}}$. ∎

Example of implications in semigroups

[8] The law $(xy)^2 = (yx)^2$ implies $xy^2 = y^2x$ for groups because we can apply the automorphism $\alpha : x \to x,\ y \to x^{-1}y$. For semigroups we can not use this automorphism. To prove that $(xy)^2 = (yx)^2$ implies $xy^2 = y^2x$ for semigroups we show first that $(xy)^2 = (yx)^2$ implies:

(i) $(yx)^2y = y(yx)^2$, (use the word $y(xy)^2$),

(ii) $x\big((yx)^2y\big)^2 = \big((yx)^2y\big)^2x$, (use $(i)^\alpha$, $x^\alpha = xyx^2$, $y^\alpha = y$; and $x \leftrightarrow y$)

(iii) $\big((yx)^2y\big)^2 = (yx)^4y^2$, (use $\big((yx)^2y\big)\big((xy)^2y\big)$),

(iv) $(yx)^4 = (xy)^4$.

Then for some word p we start with $p \cdot xy^2$ and by using (i)—(iv) obtain $p \cdot y^2x$, which by cancellation, implies required $xy^2 = y^2x$. Namely, for $p = (xy)^4$ we have

$$\begin{aligned} pxy^2 = (xy)^4xy^2 = x(yx)^2\underline{(yx)^2yy} &\overset{(i)}{=} x(yx)^2\underline{y(yx)^2}y \\ &= x\big((yx)^2y\big)^2 \overset{(ii)}{=} \big((yx)^2y\big)^2x \overset{(iii)}{=} (yx)^4y^2x \overset{(iv)}{=} (xy)^4y^2x = py^2x, \end{aligned}$$

which gives $pxy^2 = py^2x$ and hence $xy^2 = y^2x$ as required.

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