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ROOTS OF DEHN TWISTS ABOUT MULTICURVES

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Abstract. A *multicurve* $\mathcal{C}$ in a closed orientable surface S_g of genus g is defined to be a finite collection of disjoint non-isotopic essential simple closed curves. A *left-handed Dehn twist $t_{\mathcal{C}}$ about $\mathcal{C}$* is the product of left-handed Dehn twists about the individual curves in $\mathcal{C}$. In this paper, we derive necessary and sufficient conditions for the existence of a root of $t_{\mathcal{C}}$ in the mapping class group $\mathrm{Mod}(S_g)$. Using these conditions, we obtain combinatorial data that correspond to roots, and use it to determine upper bounds on the degree of a root. As an application of our theory, we classify all such roots up to conjugacy in $\mathrm{Mod}(S_4)$. Finally, we establish that no such root can lie in the level m congruence subgroup of $\mathrm{Mod}(S_g)$, for $m \geq 3$.

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1. Introduction. For $g \geq 0$, let S_g denote the closed, orientable surface of genus g, and let $\mathrm{Mod}(S_g)$ denote the mapping class group of S_g. By a *multicurve* $\mathcal{C}$ in S_g, we mean a finite collection of disjoint non-isotopic essential simple closed curves in S_g. Given a multicurve $\mathcal{C}$, we define the number $|\mathcal{C}|$ to be the *size* of $\mathcal{C}$. Let t_c denote the left-handed Dehn twist about an essential simple closed curve c on S_g. Since the Dehn twists about any two curves in $\mathcal{C}$ commute, we will define the *left-handed Dehn twist about* $\mathcal{C}$ to be $t_{\mathcal{C}} := \prod_{c \in \mathcal{C}} t_c$. A *root of $t_{\mathcal{C}}$ of degree n* is an element $h \in \mathrm{Mod}(S_g)$ such that $h^n = t_{\mathcal{C}}$.

When $\mathcal{C}$ comprises a single non-separating curve, Margalit and Schleimer [**5**] showed the existence of roots of $t_{\mathcal{C}}$ of degree $2g-1$ in $\mathrm{Mod}(S_g)$, for $g \geq 2$. This motivated [**6**], in which McCullough and the first author derived necessary and sufficient conditions for the existence of a root of degree n. As immediate applications of the main theorem in the paper, they showed that n must be odd and that $n \leq 2g-1$. These results were also independently derived by Monden [**7**]. When $\mathcal{C}$ consists of a single separating curve, the first author derived conditions [**9**] for the existence of a root of $t_{\mathcal{C}}$. Furthermore, a stable quadratic upper bound on the degree of the root and complete classifications of roots of Dehn twists about separating curves in $\mathrm{Mod}(S_2)$ and $\mathrm{Mod}(S_3)$ were obtained in [**9**] as corollaries to the main result. In this paper, we shall derive conditions for the existence of a root of $t_{\mathcal{C}}$ when $|\mathcal{C}| \geq 2$.

In general, a root h of $t_{\mathcal{C}}$ may permute some curves in $\mathcal{C}$, while preserving other curves (see Proposition 2.6). So, a root is said to be (r, k)-*permuting*, if it preserves r curves in $\mathcal{C}$, and induces k orbits on the remaining curves. The theory for

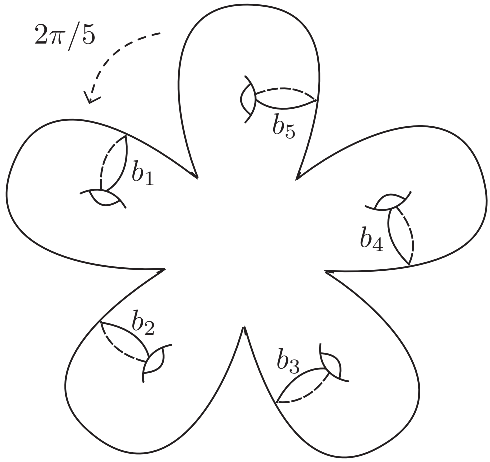

Figure 1. A non-separating multicurve of size 5 in S_5.

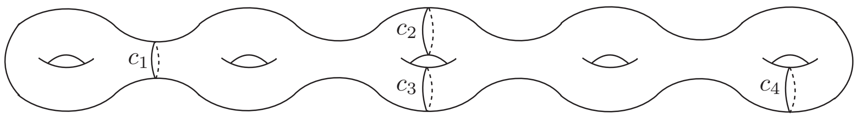

Figure 2. The surface S_5 with a separating multicurve.

$(r, 0)$-permuting roots, as we will see, can be obtained by generalizing the theories developed in **[6, 9]**, which involved the analysis of the fixed point data of finite cyclic actions.

The theory that we intend to develop for (r, k)-permuting roots when $k > 0$ can be motivated by the following example. Consider the multicurve $\mathcal{C} = \{b_1, b_2, b_3, b_4, b_5\}$ in S_5 shown in Figure 1.
It is apparent that the rotation of S_5 by $2\pi/5$ composed with t_{b_i} for some fixed $b_i \in \mathcal{C}$ is a 5th root of $t_{\mathcal{C}}$ in $\mathrm{Mod}(S_5)$. This is a simple example of a $(0, 1)$-permuting root, which is obtained by removing invariant disks around pairs of points in two distinct orbits of a $2\pi/5$ rotation of S_0, and then attaching five 1-handles with full twists. This example indicates that a classification of roots would require the examination of the orbit information of finite cyclic actions, in addition to their fixed point data. This is a significant departure from the existing theories that have been developed in **[6, 9]**. A multicurve $\mathcal{C}$ in S_g is said to be *non-separating* if $S_g \setminus \mathcal{C}$ is connected, and is called a *separating multicurve* otherwise. In Figure 2, the collection of curves $\{c_1, c_2, c_3, c_4\}$, and its subcollections $\{c_2, c_3\}$, $\{c_1, c_2, c_3\}$ are separating multicurves, while the subcollection $\{c_2, c_4\}$ is a non-separating multicurve.

We start by generalizing the notion of a nestled (n, ℓ)-action from **[9]** to a *permuting* (n, r, k)*-action*. These are C_n-actions on S_g that have r distinguished fixed points, and k distinguished non-trivial orbits. In Section 3, we introduce the notion of a *permuting* (n, r, k)*-data set*, which is a generalization of a data set from **[9]**. We use Thurston's orbifold theory **[11**, Chapter 13**]** in Theorem 3.9 to establish a correspondence between permuting (n, r, k)-actions on S_g and permuting (n, r, k)-data sets of genus g. In other words, permuting data sets algebraically encode these permuting actions and contain all the relevant orbit and fixed-point information required to classify the roots that will be constructed from these actions.

Let $S_g(\mathcal{C})$ denote the surface obtained from S_g by removing a closed annular neighbourhood N of $\mathcal{C}$ and then capping $\overline{S_g \setminus N}$. In Section 4, we prove that conjugacy classes of roots of Dehn twists about non-separating multicurves correspond to a special subclass of permuting actions on the connected surface $S_g(\mathcal{C})$. We use this to obtain the following bounds for the degree of such a root.

Corollary . *Let $\mathcal{C}$ be a non-separating multicurve in S_g of size m, and let h be an (r, k)-permuting root of $t_{\mathcal{C}}$ of degree n.*

(*i*) *If $r \geq 0$, then*

$$n \leq \begin{cases} 4(g-m)+2 & : g-m \geq 1 \\ g & : g = m. \end{cases}$$

Furthermore, if $g = m$, then this upper bound is realizable.

(*ii*) *If $r = 1$, then $n \leq 2(g-m)+1$.*

(*iii*) *If $r \geq 2$, then $n \leq \dfrac{g-m+r-1}{r-1}$.*

Note that the bound obtained in (i) is not, in general, realizable as we will show in Section 6.

When $\mathcal{C}$ is a separating, the action induced on the components of $S_g(\mathcal{C})$ by a root can have orbits that we call *surface orbits*. When a surface orbit is trivial, it is homeomorphic to $S_{g'}$, for some $g' \leq g$. In Section 5, we show that the root induces a permuting (n', r', k')-action on $S_{g'}$ for some $n' \mid n$. Moreover, when a surface orbit is non-trivial, it is homeomorphic to a disjoint union of $\widetilde{m}$ copies of $S_{\widetilde{g}}$ (for some $\widetilde{g} \leq g$ and $\widetilde{m} \mid n$,) that we will denote by $\mathbb{S}_{\widetilde{g}}(\widetilde{m})$. As the induced action cyclically permutes the $\widetilde{m}$ components of $\mathbb{S}_{\widetilde{g}}(\widetilde{m})$, we show that it is of the form $\sigma_{\widetilde{m}} \circ \widetilde{t}$, where $\sigma_{\widetilde{m}}$ can be viewed as an $\widetilde{m}$-cycle and $\widetilde{t}$ is a permuting $(\widetilde{n}/\widetilde{m}, \widetilde{r}, \widetilde{k})$-action on one of its components where $\widetilde{n} \mid n$. So, in general, a root induces a non-trivial partition $S_g(\mathcal{C}) = \sqcup_{i=1}^{s} \mathbb{S}_{g_i}(m_i)$, and an action of the form $\sigma_{m_i} \circ t_i$ on each $\mathbb{S}_{g_i}(m_i)$, where t_i is a permuting $(n_i/m_i, r_i, k_i)$-action on a surface homeomorphic to S_{g_i} where $n_i \mid n$. Conversely, given a partition $S_g(\mathcal{C}) = \sqcup_{i=1}^{s} \mathbb{S}_{g_i}(m_i)$ and a collection of actions t_i on the $\mathbb{S}_{g_i}(m_i)$, for $1 \leq i \leq s$, they can be extended to a root of $t_{\mathcal{C}}$, provided these actions satisfy certain compatibility criteria involving their distinguished orbits and fixed points. Using this theory, we shall obtain bounds for the degree of the root.

In Section 6, we use this theory to obtain a complete classification of roots of $t_{\mathcal{C}}$ in $\mathrm{Mod}(S_4)$. Finally, in Section 7, we conclude by proving that a root of $t_{\mathcal{C}}$ cannot lie in the level d congruence subgroup of $\mathrm{Mod}(S_g)$ for any integer $d \geq 3$. In particular, this implies that a root of $t_{\mathcal{C}}$ cannot lie in the Torelli group, which is all the more interesting, as Dehn twists about separating curves do lie in the Torelli group. We end this paper by indicating how our results could be extended to classify roots of finite products of powers of commuting Dehn twists.

2. Roots and their induced partitions. In this section, we shall introduce some preliminary notions, which will be used in later sections.

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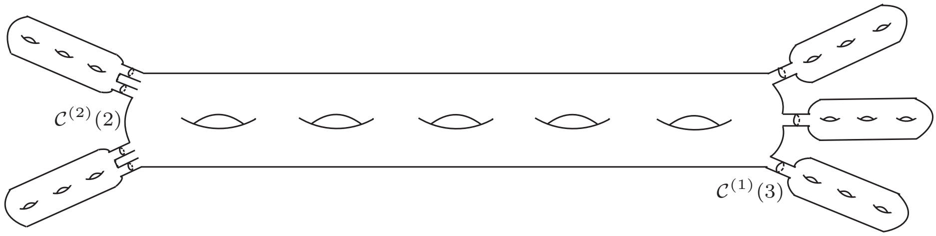

Figure 3. The surface $S_{22} = \mathbb{S}_3(2)\#_{\mathcal{C}_1} S_5 \#_{\mathcal{C}_2} \mathbb{S}_3(3)$, where $\mathcal{C}_1 = \mathcal{C}^{(2)}(2)$ and $\mathcal{C}_2 = \mathcal{C}^{(1)}(3)$.

NOTATION 2.1. Let $\mathcal{C}$ be a multicurve in S_g, and let N be a closed annular neighbourhood of $\mathcal{C}$.

(i) We denote the surface $\overline{S_g \setminus N}$ by $\widehat{S_g(\mathcal{C})}$.
(ii) The closed orientable surface obtained from $\widehat{S_g(\mathcal{C})}$ by capping off its boundary components is denoted by $S_g(\mathcal{C})$.

DEFINITION 2.2. Let $\mathcal{C}$ be a multicurve in S_g. The multicurve $\mathcal{C}$ is said to be *bounding* if $\mathcal{C}$ separates S_g, but no proper submulticurve of $\mathcal{C}$ separates S_g. In other words, $\mathcal{C}$ cobounds two subsurfaces of S_g.

Note that we allow for the possibility that $|\mathcal{C}| = 1$, in which case $\mathcal{C}$ consists of a single separating curve. When $|\mathcal{C}| = 2$, $\mathcal{C}$ is simply a bounding pair in the usual sense.

NOTATION 2.3.

(i) We will denote a bounding multicurve of size k by $\mathcal{C}^{(k)}$.
(ii) A disjoint union of m copies of $\mathcal{C}^{(k)}$ is denoted by $\mathcal{C}^{(k)}(m)$, as illustrated in Figure 3.
(iii) For integers $g \geq 0$ and $m \geq 1$, we define $\mathbb{S}_g(m)$ to be the disjoint union of m copies $\{S_g^1, S_g^2, \ldots, S_g^m\}$ of S_g isometrically imbedded in $\mathbb{R}^3$. In particular, $\mathbb{S}_g(1) \approx S_g$, and hence we shall write S_g for $\mathbb{S}_g(1)$.
(iv) Given two surfaces S_{g_1} and $\mathbb{S}_{g_2}(m)$ and a fixed $k \in \mathbb{N}$, we construct a new surface S_g with $g = (g_1 + mg_2 + (k-1)m)$ containing a multicurve of type $\mathcal{C}^{(k)}(m)$, in the following manner. We remove km disks $\{D_{i,j}^1 : 1 \leq j \leq k, 1 \leq i \leq m\}$ on S_{g_1} and k disks $\{D_{i,j}^2 : 1 \leq j \leq k\}$ on each $S_{g_2}^i$. Now, connect $\partial D_{i,j}^1$ to $\partial D_{i,j}^2$ along a 1-handle $A_{i,j}$, and choose the unique curve (up to isotopy) $c_{i,j}$ on each $A_{i,j}$. Let $\mathcal{C} = \{c_{i,j}\}$, then note that $\mathcal{C} = \mathcal{C}^{(k)}(m)$, so we write $S_{g_1}\#_{\mathcal{C}} \mathbb{S}_{g_2}(m)$ for the new surface S_g.
(v) Similarly, given surfaces $\{S_{g_1}, \mathbb{S}_{g_{2,1}}(m_1), \ldots, \mathbb{S}_{g_{2,s}}(m_s)\}$ and non-negative integers $\{k_1, k_2, \ldots, k_s\}$, we construct a new surface S_g with $g = g_1 + \sum_{i=1}^s m_i(g_{2,i} + k_i - 1)$, containing a multicurve of type $\mathcal{C} = \sqcup_{i=1}^s \mathcal{C}^{(k_i)}(m_i)$ in the following manner. Let $S_{g_i'} := S_{g_1}\#_{\mathcal{C}^{(k_i)}(m_i)} \mathbb{S}_{g_{2,i}}(m_i)$ and $\mathcal{C}_i := \mathcal{C} \setminus \mathcal{C}^{(k_i)}(m_i)$, we now define

$$S_g := \overline{\#}_{i=1}^s \left(S_{g_1}\#_{\mathcal{C}^{(k_i)}(m_i)} \mathbb{S}_{g_{2,i}}(m_i) \right) := \bigcup_{i=1}^s \widehat{S_{g_i'}(\mathcal{C}_i)}.$$

If $s = 2$, we simply write $S_g = \mathbb{S}_{g_{2,1}}(m_1)\#_{\mathcal{C}^{(k_1)}(m_1)} S_{g_1}\#_{\mathcal{C}^{(k_2)}(m_2)} \mathbb{S}_{g_{2,2}}(m_2)$. In Figure 3, we give an example of a such a surface S_{22} with a multicurve $\mathcal{C} = \mathcal{C}^{(2)}(2) \sqcup \mathcal{C}^{(1)}(3)$.

DEFINITION 2.4. Let $\mathcal{C}$ be a multicurve in S_g. Suppose that there exists an integer $k \geq 3$ such that $S_g(\mathcal{C}) = \sqcup_{i=1}^k S_{g_i}$, and that there exists submulticurves $\mathcal{C}_i$ of $\mathcal{C}$ such that

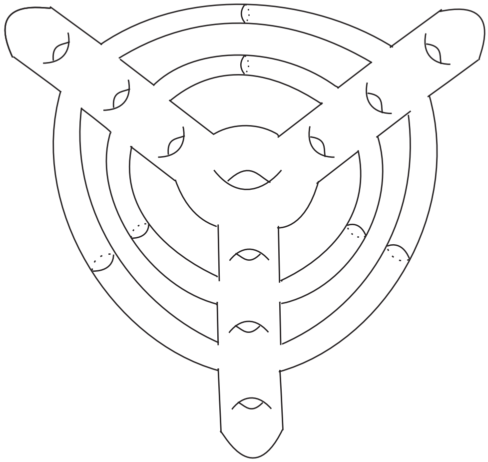

Figure 4. A cyclical multicurve of size 6 in S_{16}.

$\mathcal{C} = \sqcup_{i=1}^{k} \mathcal{C}_i$, and

$$S_g = \bigcup_{i=1}^{k} \widehat{\Sigma_i(\mathcal{C} \setminus \mathcal{C}_i)},$$

where $\Sigma_i = S_{g_i} \#_{\mathcal{C}_i} S_{g_j}$ and $j \equiv (i+1) \pmod{k}$. Then, $\mathcal{C}$ is a said to be a *cyclical* multicurve (see Figure 4).

If $\mathcal{C}$ denotes a multicurve in S_g, our immediate goal is to show that any root of $t_{\mathcal{C}}$ must preserve $\mathcal{C}$. In order to do this, we use the geometric intersection number $i(a, b)$ between the isotopy classes of two essential simple closed curves in a and b in S_g. In particular, if $\varphi \in \mathrm{Mod}(S_g)$ is expressed as a product of Dehn twists, then its effect on the geometric intersection number can be described using the following result from [**2**].

LEMMA 2.5 [**2**, Proposition 3.4]. *Let $a_1, a_2, \ldots, a_n$ be a collection of pairwise disjoint, simple closed curves in a surface S and let $M = \prod_{i=1}^{n} t_{a_i}^{e_i}$. Suppose that $e_i > 0$ for all i, or $e_i < 0$ for all i. If b and c are arbitrary isotopy classes of simple closed curves in S, then*

$$\left| i(M(b), c) - \sum_{i=1}^{n} |e_i| i(a_i, b) i(a_i, c) \right| \leq i(b, c)$$

This leads us to the following proposition.

PROPOSITION 2.6. *Let $\mathcal{C}$ be a multicurve in S_g, and h be a root of $t_{\mathcal{C}}$. Then, we can modify h by an isotopy so that it preserves $\mathcal{C}$.*

Proof. Let $\mathcal{C} = \{c_1, c_2, \ldots, c_m\}$, then $h(\mathcal{C}) = \{h(c_1), h(c_2), \ldots, h(c_m)\}$ consists of disjoint non-isotopic simple closed curves. Since $h^n = t_{\mathcal{C}}$, it follows that $t_{\mathcal{C}} = h t_{\mathcal{C}} h^{-1} = \prod_{i=1}^{m} h t_{c_i} h^{-1} = \prod_{i=1}^{m} t_{h(c_i)}$. By Lemma 2.5, for each $1 \leq j \leq m$, we have

$$0 = \sum_{i=1}^{m} i(c_i, c_j)^2 = i(t_{\mathcal{C}}(c_j), c_j) = i\left(\left(\prod_{i=1}^{m} t_{h(c_i)}\right)(c_j), c_j\right) = \sum_{i=1}^{m} i(h(c_i), c_j)^2,$$

and so it follows that $i(h(c_i), c_j) = 0$ for all $1 \leq i, j \leq m$. Now suppose that $h(c_i) \nsim c_1$ for all $1 \leq i \leq m$; then there exists a neighbourhood N of c_1 such that $t_{h(c_i)}|_N = \mathrm{id}_N$ and $t_{c_i}|_N = \mathrm{id}_N$ for all $i \neq 1$. However,

$$t_{c_1}|_N = t_{\mathcal{C}}|_N = t_{h(c_1)} t_{h(c_2)} \dots t_{h(c_m)}|_N = \mathrm{id}_N,$$

which is a contradiction. So there exists $1 \leq i \leq m$ such that $h(c_i) \sim c_1$. Hence, up to isotopy, we may assume that $h(c_i) = c_1$. Now note that $h(c_j) \nsim c_1$ for all $j \neq i$, which allows us to proceed by induction on $|\mathcal{C}|$ to conclude that, up to isotopy, $h(\mathcal{C}) = \mathcal{C}$. □

Definition 2.7. Let $\mathcal{C}$ be a multicurve of size m in S_g. Then, for integers $r, k \geq 0$, an (r, k)-*partition* of $\mathcal{C}$ is a partition $\mathbb{P}_{r,k}(\mathcal{C}) = \{\mathcal{C}'_1, \dots, \mathcal{C}'_r, \mathcal{C}_1, \dots, \mathcal{C}_k\}$ of the set $\mathcal{C}$ into subsets such that for all i,

(i) $|\mathcal{C}'_i| = 1$, $|\mathcal{C}_i| > 1$, and
(ii) $\mathcal{C}_i$ comprises only separating or only non-separating curves.

Note that by Proposition 2.6, any root of $t_{\mathcal{C}}$ partitions $\mathcal{C}$ into a collection of orbits that form an (r, k)-partition of $\mathcal{C}$.

Definition 2.8. Let $\mathcal{C}$ be a multicurve in S_g. Then, for integers $r, k \geq 0$, a root h of $t_{\mathcal{C}}$ of is said to be (r, k)-*permuting* if it induces an (r, k)-partition of $\mathcal{C}$.

3. Permuting actions and permuting data sets. In this section, we shall introduce permuting (n, r, k)-actions, which are generalizations of the nestled (n, ℓ)-actions from [**9**]. We shall also introduce the notion of a permuting (n, r, k)-data set, which is an abstract tuple involving non-negative integers that algebraically encodes a permuting (n, r, k)-action.

Definition 3.1. For integers $n \geq 1$, and $r, k \geq 0$, an orientation-preserving C_n-action t on S_g is called a *permuting* (n, r, k)-*action* if

(i) there is a set $\mathbb{P}(t)$ of r distinguished fixed points of t, which correspond to r distinguished cone points of order n in the quotient orbifold, and
(ii) there is a set $\mathbb{O}(t)$ of k distinguished non-trivial orbits of t.

Notation 3.2. Let t be a permuting (n, r, k)-action on S_g.

(i) Fix a point $P \in S_g$, and consider $t_* : T_P(S_g) \to T_{t(P)}(S_g)$, where $T_x(S_g)$ denotes the tangent space at x. By the Nielsen realisation theorem [**4**], we may change t by isotopy so that t_* is an isometry. Hence, t_* induces a local rotation by an angle, which we shall denote by $\theta_P(t)$. Note that if $P \in \mathbb{P}(t)$, then $\theta_P(t) = 2\pi a/n$, where $\gcd(a, n) = 1$.
(ii) Fix an orbit $\mathbb{O} = \{Q_1, \dots, Q_s\} \in \mathbb{O}(t)$. If $s < n$, then $s \mid n$, and there exists a cone point in the quotient orbifold of degree n/s. Each Q_i has stabilizer generated by t^s and the rotation induced by t^s around each Q_i must be the same, since its action at one point is conjugate by a power of t to its action at each other point in the orbit. So, the rotation angle is of the form $2\pi c^{-1}/(n/s) \pmod{2\pi}$, where $\gcd(c, n/s) = 1$ and c^{-1} denotes the inverse of $c \pmod{n/s}$. We now associate to this orbit a pair $p(\mathbb{O})$ as follows:

$$p(\mathbb{O}) := \begin{cases} (c, n/s), & \text{if } s < n, \text{ and} \\ (0, 1), & \text{if } s = n. \end{cases}$$

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(iii) For any orbit $\mathbb{O} \in \mathbb{O}(t)$, if $p(\mathbb{O}) = (a, b)$, then we define

$$\theta_{\mathbb{O}}(t) := \begin{cases} 2\pi a^{-1}/b, & \text{if } a \neq 0, \text{ and} \\ 0, & \text{otherwise.} \end{cases}$$

DEFINITION 3.3. Consider a permuting (n, r, k)-action t on S_g with $\mathbb{P}(t) = \{P_1, \ldots, P_r\}$ and $\mathbb{O}(t) = \{\mathbb{O}_1, \mathbb{O}_2, \ldots, \mathbb{O}_k\}$.

(i) We write $S(t) = \left\{\left\{| \mathbb{O}_1 |, | \mathbb{O}_2 |, \ldots, | \mathbb{O}_k |\right\}\right\}$. Note that we will henceforth use the symbol $\left\{\{\}\right\}$ to denote a multiset.
(ii) For each $p \in \{p(\mathbb{O}_i) : 1 \leq i \leq k\}$, define $m_p = |\{j : p(\mathbb{O}_j) = p\}|$. We define the *orbit distribution* of t to be the set $\mathbb{O}_t = \{(p, m_p) : p \in \{p(\mathbb{O}_i) : 1 \leq i \leq k\}\}$.

DEFINITION 3.4. Let t_1 and t_2 be two permuting (n, r, k)-actions on S_g with $\mathbb{P}(t_s) = \{P_{s,1}, P_{s,2}, \ldots, P_{s,r}\}$ and $\mathbb{O}(t_s) = \{\mathbb{O}_{s,1}, \mathbb{O}_{s,2}, \ldots, \mathbb{O}_{s,k}\}$, for $s = 1, 2$. We say t_1 is *equivalent* to t_2 if $\mathbb{O}_{t_1} = \mathbb{O}_{t_2}$ and there is an orientation-preserving homeomorphism ϕ on S_g such that

(i) $\phi(P_{1,i}) = P_{2,i}$ for $1 \leq i \leq r$, and
(ii) for each $1 \leq j \leq k$, if $\mathbb{O}_{s,j} = \{Q^s_{j,1}, Q^s_{j,2}, \ldots, Q^s_{j,m_{s_j}}\}$, then $m_{1_j} = m_{2_j}$ and $\phi(Q^1_{j,i}) = Q^2_{j,i}$ for all $1 \leq i \leq m_{1_j}$, and
(iii) $\phi t_1 \phi^{-1}$ is isotopic to t_2 relative to $\mathbb{P}(t_2) \sqcup (\cup_{j=1}^k \mathbb{O}_{2,j})$.

The equivalence class of a permuting (n, r, k)-action will be denoted by $[\![t]\!]$.

We now introduce the notion of an (n, r)-*data set*, which encodes the signature of the quotient orbifold of a permuting (n, r, k)-action and the turning angles around its distinguished fixed points. Furthermore, the (n, r)-data set will be combined with an orbit distribution of the action to form a pair, which we will call a *permuting* (n, r, k)-*data set*.

DEFINITION 3.5. Given $n \geq 1$ and $r \geq 0$, an (n, r)-*data set* is a tuple

$$\mathcal{D} = (n, g_0, \ell, (a_1, a_2, \ldots, a_r); (c_1, n_1), (c_2, n_2), \ldots, (c_s, n_s)),$$

where $n \geq 1$, $g_0 \geq 0$, and $\ell \geq 0$ are integers, each a_i is a residue class modulo n, and each c_i is a residue class modulo n_i such that:

(i) $0 \leq \ell \leq n - 1$, and $\ell > 0$ if, and only if $r = s = 0$,
(ii) each $n_i \mid n$,
(iii) for each i, $\gcd(a_i, n) = \gcd(c_i, n_i) = 1$, and
(iv) $\displaystyle\sum_{i=1}^{r} a_i + \sum_{j=1}^{s} \frac{n}{n_i} c_i \equiv 0 \pmod{n}$.

The number g determined by the equation

$$\frac{2 - 2g}{n} = 2 - 2g_0 + r\left(\frac{1}{n} - 1\right) + \sum_{j=1}^{s}\left(\frac{1}{n_j} - 1\right)$$

is called the *genus* of the data set.

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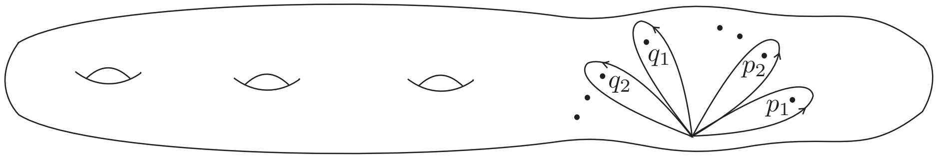

Figure 5. The quotient orbifold $\mathcal{O}$.

REMARK 3.6. The data set in Definition 3.5 above is a generalisation of the notion of a data set from **[9]**. As in **[9]**, the data set here will correspond to the equivalence class $[\![t]\!]$ of a permuting (n, r, k)-action. The quantity ℓ in the data set $\mathcal{D}$ will be non-zero if and only if the action is a free rotation of S_g by $2\pi\ell/n$.

DEFINITION 3.7. Fix an (n, r)-data set $\mathcal{D}$ of genus g as above.

(i) For each $(a, b) \in \{(0, 1), (c_1, n_1), \ldots, (c_s, n_s)\}$, we write

$$\theta((a, b)) := \begin{cases} 0, & \text{if } a = 0, \text{ and} \\ 2\pi a^{-1}/b, & \text{otherwise.} \end{cases}$$

(ii) For each $p \in \{(0, 1), (c_1, n_1), \ldots, (c_s, n_s)\}$, choose a non-negative integer m_p, with the caveat that $m_{(0,1)} > 0$ only if one of the following conditions hold:
  (a) $\ell > 0$,
  (b) $r > 0$, or
  (c) $n_i = n$ for some $1 \leq i \leq s$.

Then, the set $\mathbb{O}_{\mathcal{D}} = \{(p, m_p) : m_p > 0\}$ is called an *orbit distribution* of $\mathcal{D}$.

(iii) Given an orbit distribution $\mathbb{O}_{\mathcal{D}}$ associated with an (n, r)-data set $\mathcal{D}$, the pair $(\mathcal{D}, \mathbb{O}_{\mathcal{D}})$ is called a *permuting* (n, r, k)*-data set* of genus g, where $k = \sum_p m_p$.

DEFINITION 3.8. Let $\mathcal{D} = (n, g_0, \ell, (a_1, \ldots, a_r); (c_1, n_1), \ldots, (c_s, n_s))$ and $\mathcal{D}' = (n, g_0', \ell', (a_1', \ldots, a_r'); (c_1', n_1'), \ldots, (c_s', n_s'))$ be two (n, r)-data sets as in Definition 3.5.

(i) $\mathcal{D}$ and $\mathcal{D}'$ are said to be *equivalent* if $\ell = \ell'$, $\left\{\!\left\{a_1, a_2, \ldots, a_r\right\}\!\right\} = \left\{\!\left\{a_1', a_2', \ldots, a_r'\right\}\!\right\}$ and $\left\{\!\left\{(c_1, n_1), \ldots, (c_s, n_s)\right\}\!\right\} = \left\{\!\left\{(c_1', n_1'), \ldots (c_s', n_s')\right\}\!\right\}$.

(ii) Two permuting (n, r, k)-data sets $(\mathcal{D}, \mathbb{O}_{\mathcal{D}})$ and $(\mathcal{D}', \mathbb{O}_{\mathcal{D}'})$ are said to be *equivalent* if $\mathcal{D}$ and $\mathcal{D}'$ are equivalent as above, and $\mathbb{O}_{\mathcal{D}} = \mathbb{O}_{\mathcal{D}'}$.

Note that equivalent data sets have the same genus.

THEOREM 3.9. *Given $n \geq 1$ and $g \geq 0$, there exists a bijective correspondence from the set of equivalence classes of permuting (n, r, k)-data sets of genus g to the set of equivalence classes of permuting (n, r, k)-actions on S_g.*

Proof. Let t be a permuting (n, r, k)-action on S_g with quotient orbifold $\mathcal{O}$, whose underlying surface has genus g_0. If t is a free rotation of S_g by $2\pi\ell/n$, for some $0 \leq \ell \leq n - 1$, then $\mathcal{O} = S_{g_0}$, where $g_0 = ((g - 1)/n) + 1$, and we simply write $\mathcal{D} = (n, g_0, \ell;)$ and $\mathbb{O}_{\mathcal{D}} = \{((0, 1), k)\}$. Otherwise, let p_j be the image in $\mathcal{O}$ of the P_j, for $1 \leq j \leq r$, and let $q_1, q_2, \ldots, q_s$ be the other possible cone points of $\mathcal{O}$ as in Figure 5.

Let α_i be the generator of the orbifold fundamental group $\pi_1^{orb}(\mathcal{O})$ that goes around the point p_i, $1 \leq i \leq r$, and let γ_j be the generators going around q_j, $1 \leq j \leq s$. Let x_p and y_p, $1 \leq p \leq g_0$, be the standard generators of the 'surface part' of $\mathcal{O}$, chosen to

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give the following presentation of $\pi_1^{orb}(\mathcal{O})$:

$$\pi_1^{orb}(\mathcal{O}) = \langle \alpha_1, \alpha_2, \ldots, \alpha_r, \gamma_1, \gamma_2, \ldots, \gamma_s, x_1, y_1, x_2, y_2, \ldots x_{g_0}, y_{g_0} \mid$$
$$\alpha_1^n = \ldots = \alpha_r^n = \gamma_1^{n_1} = \ldots = \gamma_s^{n_s} = 1,\ \alpha_1 \ldots \alpha_r \gamma_1 \ldots \gamma_s = \prod_{p=1}^{g_0} [x_p, y_p] \rangle.$$

From orbifold covering space theory **[11]**, we have the following exact sequence:

$$1 \to \pi_1(S_g) \to \pi_1^{orb}(\mathcal{O}) \xrightarrow{\rho} C_n \to 1,$$

where $C_n = \langle t \rangle$. The homomorphism ρ is obtained by lifting path representatives of elements of $\pi_1^{orb}(\mathcal{O})$. Since these do not pass through the cone points, the lifts are uniquely determined.

For $1 \le i \le s$, the preimage of q_i consists of n/n_i points cyclically permuted by t. As in Notation 3.2, the rotation angle at each point is of the form $2\pi c_i^{-1}/n_i$ where c_i is a residue class modulo n_i and $\gcd(c_i, n_i) = 1$. Lifting the γ_i, we have that $\rho(\gamma_i) = t^{(n/n_i)c_i}$. Similarly, lifting the α_i gives $\rho(\alpha_i) = t^{a_i}$, where $\gcd(a_i, n) = 1$. Finally, we have

$$\rho\left(\prod_{p=1}^{g_0} [x_p, y_p]\right) = 1,$$

since C_n is abelian,

$$1 = \rho(\alpha_1 \ldots \alpha_r \gamma_1 \ldots \gamma_s) = t^{a_1 + \cdots + a_r + (n/n_1)c_1 + \cdots + (n/n_s)c_s},$$

giving

$$\sum_{i=1}^{r} a_i + \sum_{j=1}^{s} \frac{n}{n_j} c_j \equiv 0 \pmod{n}.$$

The fact that the data set $\mathcal{D}$ has genus g follows easily from the multiplicativity of the orbifold Euler characteristic for the orbifold covering $S_g \to \mathcal{O}$:

$$\frac{2-2g}{n} = 2 - 2g_0 + r\left(\frac{1}{n} - 1\right) + \sum_{j=1}^{s} \left(\frac{1}{n_j} - 1\right).$$

Thus, t gives a (n, r)-data set

$$\mathcal{D} = (n, g_0, 0; (a_1, a_2, \ldots, a_r); (c_1, n_1), (c_2, n_2), \ldots, (c_s, n_s))$$

of genus g, and hence $(\mathcal{D}, \mathbb{O}_t)$ forms a permuting (n, r, k)-data set.

Consider another permuting (n, r, k)-action t' in the equivalence class of t with a distinguished fixed point set $\mathbb{P}(t') = \{P_1', P_2', \ldots, P_r'\}$. Then, by definition, there exists an orientation-preserving homeomorphism ϕ on S_g such that $\phi(P_j) = P_j'$, for all j, and $\phi t \phi^{-1}$ is isotopic to t' relative to $\mathbb{P}(t')$. Therefore, $\theta_{P_j}(t) = \theta_{P_j'}(t')$, for $1 \le j \le r$, and since $\mathbb{O}_t = \mathbb{O}_{t'}$, the two actions will produce the same permuting (n, r, k)-data sets.

Conversely, given a permuting (n, r, k)-data set $(\mathcal{D}, \mathbb{O}_{\mathcal{D}})$, we construct the orbifold $\mathcal{O}$ and a representation $\rho : \pi_1^{orb}(\mathcal{O}) \to C_n$. Any finite subgroup of $\pi_1^{orb}(\mathcal{O})$ is conjugate to one of the cyclic subgroups generated by α_j or γ_i, so condition (iii) in the definition of the data set ensures that the kernel of ρ is torsion free. Therefore, the orbifold covering $S \to \mathcal{O}$ corresponding to the kernel is a manifold, and calculation of the Euler characteristic shows that $S = S_g$. Thus, we obtain a C_n-action t on S_g with r distinguished fixed points $\mathbb{P}(t)$. We now construct $\mathbb{O}_t$ from $\mathbb{O}_{\mathcal{D}}$ in the following manner. For each pair $(p, m_p) \in \mathbb{O}_{\mathcal{D}}$, write $p = (a, b)$. If $a = 0$, then choose m_p orbits of size n (this is permitted by the conditions of Definition 3.7(ii)). If $a \neq 0$, then there exists a cone point in $\mathcal{O}$ of degree b, so there exists an orbit of t of size n/b in S_g. Once again, by considering a small neighbourhood of this orbit, we may choose m_p distinct orbits $\{\mathbb{O}_p^1, \mathbb{O}_p^2, \ldots, \mathbb{O}_p^{m_p}\}$ and set $\mathbb{O}(t) := \bigsqcup_{(p,m_p)\in\mathbb{O}_{\mathcal{D}}} \{\mathbb{O}_p^1, \mathbb{O}_p^2, \ldots, \mathbb{O}_p^{m_p}\}$, which in turn gives $\mathbb{O}_t = \mathbb{O}_{\mathcal{D}}$.

It remains to show that the resulting action on S_g is determined up to our equivalence. Suppose that two permuting (n, r, k)-actions t and t' have the same permuting (n, r, k)-data set $(\mathcal{D}, \mathbb{O}_{\mathcal{D}})$. $\mathcal{D}$ encodes the fixed point data of the periodic transformation t, so by a result of Nielsen [**8**] (or by a subsequent result of Edmonds [**1**, Theorem 1.3]), t and t' have to be conjugate by an orientation-preserving homeomorphism ϕ. Let $\mathcal{O}'$ be the quotient orbifold of the action t', and $\rho' : \pi_1^{orb}(\mathcal{O}') \to C_n$ be the induced representations. Then ϕ induces a map $\phi_\# : \pi_1^{orb}(\mathcal{O}) \to \pi_1^{orb}(\mathcal{O}')$ such that $\rho' \circ \phi_\# = \rho$ as in [**6**, Theorem 2.1]. If γ is a loop around a cone point in $\mathcal{O}$, then $\phi_\#(\gamma)$ is a loop around a cone point in $\mathcal{O}'$, and these cone points are associated to the same pair in $\mathcal{D}$ since $\rho'(\phi_\#(\gamma)) = \rho(\gamma)$. Once again, as in [**6**, Theorem 2.1], a careful choice of ϕ will ensure that it maps $\mathbb{P}(t)$ to $\mathbb{P}(t')$ and $\mathbb{O}(t)$ to $\mathbb{O}(t')$. Furthermore, $\mathbb{O}_t = \mathbb{O}_{\mathcal{D}} = \mathbb{O}_{t'}$ by construction, and hence the permuting data set determines t up to equivalence. □

4. Non-separating multicurves. Recall that a multicurve $\mathcal{C}$ is said to be non-separating if $S_g(\mathcal{C})$ is connected. In this section, we establish that a root of $t_{\mathcal{C}}$ corresponds to a special kind of permuting action on $S_g(\mathcal{C})$.

DEFINITION 4.1. Let t_i be a permuting (n_i, r_i, k_i)-action on S_{g_i} for $i = 1, 2$. Two orbits $\mathbb{O}_i \in \mathbb{O}(t_i)$ are said to be *equivalent* (in symbols, $\mathbb{O}_1 \sim \mathbb{O}_2$) if

(i) $|\,\mathbb{O}_1\,| = |\,\mathbb{O}_2\,|$, and
(ii) if $|\,\mathbb{O}_1\,| < n := \mathrm{lcm}(n_1, n_2)$, then we further require that

$$\theta_{\mathbb{O}_1}(t_1) + \theta_{\mathbb{O}_2}(t_2) \equiv 2\pi/n \pmod{2\pi}.$$

In this section, we will only need the case when $t_1 = t_2$, but we will need the general case in Section 5.

DEFINITION 4.2. Let $\mathcal{C}$ be a non-separating multicurve in S_g. A permuting $(n, 2r, 2k)$-action t on $S_g(\mathcal{C})$ is said to be *non-separating with respect to* $\mathcal{C}$ if

(i) there exists r mutually disjoint pairs $\{P_i, P_i'\}$ of distinguished fixed points in $\mathbb{P}(t)$ such that $\theta_{P_i}(t) + \theta_{P_i'}(t) \equiv 2\pi/n \pmod{2\pi}$, for $1 \le i \le r$,
(ii) there exists k mutually disjoint pairs $\{\mathbb{O}_i, \mathbb{O}_i'\}$ of distinguished non-trivial orbits in $\mathbb{O}_t$ such that $\mathbb{O}_i \sim \mathbb{O}_i'$, for $1 \le i \le k$, and
(iii) $r + \sum_{i=1}^{k} |\,\mathbb{O}_i\,| = |\,\mathcal{C}\,|$.

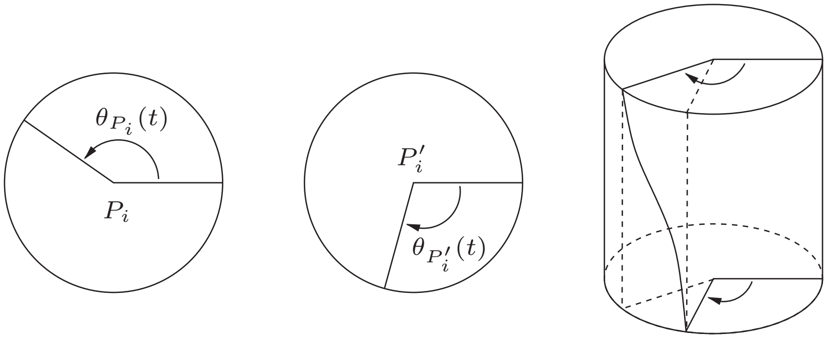

Figure 6. Angle compatibility at each pair $\{P_i, P_i'\} \subset \mathbb{P}(t)$.

THEOREM 4.3. *Let $\mathcal{C}$ be a non-separating multicurve in S_g. Then, for $n \geq 1$, equivalence classes of permuting $(n, 2r, 2k)$-actions on $S_g(\mathcal{C})$ that are non-separating with respect to $\mathcal{C}$ correspond to the conjugacy classes in* $\mathrm{Mod}(S_g)$ *of (r, k)-permuting roots of $t_{\mathcal{C}}$ of degree n.*

Proof. First, we shall prove that a conjugacy class of an (r, k)-permuting root h of $t_{\mathcal{C}}$ of degree n yields an equivalence class of a permuting $(n, 2r, 2k)$-action that is non-separating with respect to $\mathcal{C}$. We assume that $r, k > 0$, with the implicit understanding that, when either of them is zero, the corresponding arguments may be disregarded.

Let $\mathbb{P}_{r,k}(\mathcal{C}) = \{\mathcal{C}_1', \mathcal{C}_2', \ldots, \mathcal{C}_r', \mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_k\}$ be the partition of $\mathcal{C}$ induced by h as in Definition 2.7. Choose a closed tubular neighborhood N of $\mathcal{C}$, and consider $S_g(\mathcal{C})$ as in Definition 2.1. By isotopy, we may assume that $t_{\mathcal{C}}(\mathcal{C}) = \mathcal{C}$, $t_{\mathcal{C}}(N) = N$, and $t_{\mathcal{C}}|_{\widehat{S_g(\mathcal{C})}} = \mathrm{id}_{\widehat{S_g(\mathcal{C})}}$. Suppose that h is a root of $t_{\mathcal{C}}$ of degree n, then by Proposition 2.6, we may assume that h preserves $\mathcal{C}$ and takes N to N.

By the Nielsen–Kerckhoff theorem [**4**], $\hat{t} := h|_{\widehat{S_g(\mathcal{C})}}$ is isotopic to a homeomorphism whose nth power is $\mathrm{id}_{\widehat{S_g(\mathcal{C})}}$. So, we may change h by isotopy so that $\hat{t}^n = \mathrm{id}_{\widehat{S_g(\mathcal{C})}}$. We fill in the $2m$ boundary circles of $\widehat{S_g(\mathcal{C})}$ with disks and extend $\hat{t}$ to a homeomorphism t on $S_g(\mathcal{C})$ by coning. Thus, t defines a C_n-action on $S_g(\mathcal{C})$, where $C_n = \langle t \mid t^n = 1\rangle$.

The C_n-action t fixes the centres P_i and P_i' of the $2r$ disks D_i and D_i' of $\overline{S_g \setminus \widehat{S_g(\mathcal{C})}}$, for $1 \leq i \leq 2r$, whose boundaries are the components of ∂N which are preserved by t. The orientation of S_g determines one for $S_g(\mathcal{C})$, so we may speak of directed angles of rotation about the centres of these disks. Since $h^n = t_{\mathcal{C}}$, it follows from [**6**, Theorem 2.1] that $\theta_{P_i}(t) + \theta_{P_i'}(t) \equiv 2\pi/n \pmod{2\pi}$, as illustrated in Figure 6.

The remaining disks occurring in $\overline{S_g \setminus \widehat{S_g(\mathcal{C})}}$ form k pairs of orbits, whose sizes we denote by $m_1, m_2, \ldots, m_k$. For $1 \leq j \leq k$, we denote the centres of these pairs of disks by $Q_{i,j}$ and $Q_{i,j}'$, and the orbits of these centres by $\mathbb{O}_j$ and $\mathbb{O}_j'$. Thus, t is a permuting $(n, 2r, 2k)$-action with $\mathbb{P}(t) = \{P_1, P_1', \ldots, P_r, P_r'\}$ and $\mathbb{O}(t) = \{\mathbb{O}_1, \mathbb{O}_1', \ldots, \mathbb{O}_k, \mathbb{O}_k'\}$.

It remains to show that $\mathbb{O}_i \sim \mathbb{O}_i'$ for each i. By construction, $|\,\mathbb{O}_i\,| = |\,\mathbb{O}_i'\,| = m_i$, and if $m_i = n$, then $\mathbb{O}_i \sim \mathbb{O}_i'$ holds trivially. If not, we write $\mathbb{O}_i = \{Q_{i,1}, Q_{i,2}, \ldots, Q_{i,m_i}\}$, $\mathbb{O}_i' = \{Q_{i,1}', Q_{i,2}', \ldots, Q_{i,m_i}'\}$ and note that h^{m_i} is an (n/m_i)th root of $t_{\mathcal{C}}$ such that $h^{m_i}(c_{i,1}) = c_{i,1}$, where $C_{i,1}$ is a curve in $\mathcal{C}_i$. Hence, $\theta_{Q_{i,1}}(t^{m_i}) + \theta_{Q_{i,1}'}(t^{m_i}) \equiv 2\pi/(n/m_i) \pmod{2\pi}$, which implies that $m_i\theta_{\mathbb{O}_i}(t) + m_i\theta_{\mathbb{O}_i'}(t) \equiv 2\pi/(n/m_i) \pmod{2\pi}$. Since t^n is induced by $t_{\mathcal{C}}|_{\widehat{S_g(\mathcal{C})}}$, it follows that $\theta_{\mathbb{O}_i}(t) + \theta_{\mathbb{O}_i'}(t) \equiv 2\pi/n \pmod{2\pi}$. Hence $\mathbb{O}_i \sim \mathbb{O}_i'$, and since

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$r + \sum_{i=1}^{k} m_i = |\mathcal{C}|$ clearly holds by construction, we obtain a permuting $(n, 2r, 2k)$-action t on $S_g(\mathcal{C})$ that is non-separating with respect to $\mathcal{C}$.

Now suppose $h_1, h_2 \in \mathrm{Mod}(S_g)$ are two roots of $t_{\mathcal{C}}$ that are conjugate in $\mathrm{Mod}(S_g)$ via $\Phi \in \mathrm{Mod}(S_g)$, and let t_s denote the finite order homeomorphisms on $S_g(\mathcal{C})$ induced by h_s, for $s = 1, 2$. Then, $t_{\mathcal{C}} = \Phi t_{\mathcal{C}} \Phi^{-1} = t_{\Phi(\mathcal{C})}$, so we may assume up to isotopy that $\Phi(\mathcal{C}) = \mathcal{C}$ (as in Proposition 2.6) and that $\Phi(N) = N$. We extend $\Phi\,|_{\widehat{S_g(\mathcal{C})}}$ to an element $\phi \in \mathrm{Mod}(S_g(\mathcal{C}))$ by coning. Now, ϕ maps $\mathbb{P}(t_1)$ to $\mathbb{P}(t_2)$, and $\mathbb{O}(t_1)$ to $\mathbb{O}(t_2)$ bijectively as in Definition 3.4. Since h_s and Φ all preserve N, $\phi t_1 \phi^{-1}$ is isotopic to t_2 preserving $\mathbb{P}(t_2)$ and $\mathbb{O}(t_2)$. Furthermore, for each $\mathbb{O} \in \mathbb{O}(t_1)$, $p(\phi(\mathbb{O})) = p(\mathbb{O})$. Hence, $\mathbb{O}_{t_1} = \mathbb{O}_{t_2}$ and so t_1 and t_2 will be equivalent permuting $(n, 2r, 2k)$-actions.

Conversely, given a permuting $(n, 2r, 2k)$-action t on $S_g(\mathcal{C})$ that is non-separating with respect to $\mathcal{C}$, we can reverse the argument to produce the (r, k)-permuting root h. If $\mathbb{P}(t) = \{P_1, P_1', \ldots, P_r, P_r'\}$, then for $1 \le i \le r$, we remove disks D_i and D_i' invariant under the action of t around the P_i and P_i' and attaching r annuli to obtain the surface $S_g(\mathcal{C} \setminus \mathcal{C}')$. The condition on the angles $\{\theta_{P_i}(t), \theta_{P_i'}(t)\}$ ensures that the rotation angles work correctly to allow an extension of t to obtain an h_0 with $h_0^n = t_{\mathcal{C}'}$ in $\mathrm{Mod}(S_g(\mathcal{C} \setminus \mathcal{C}'))$, where $\mathcal{C}' = \cup_{i=1}^{r} \mathcal{C}_i'$.

If $\mathbb{O}(t) = \{\mathbb{O}_1, \mathbb{O}_1', \ldots, \mathbb{O}_k, \mathbb{O}_k'\}$, we write $\mathbb{O}_1 = \{Q_{1,1}, Q_{1,2}, \ldots, Q_{1,m_1}\}$, and consider disks $D_{1,i}$ around $Q_{1,i}$ such that $t(D_{1,i}) = D_{1,i+1}$. Similarly, write $\mathbb{O}_1' = \{Q_{1,1}', Q_{1,2}', \ldots, Q_{1,m_1}'\}$ and consider disks $D_{1,i}'$ as earlier. Then, we attach m_1 annuli connecting $\partial D_{1,i}$ to $\partial D_{1,i}'$. Each such annulus contains a non-separating curve $c_{1,i}$, which is unique unto isotopy. Repeating this process for $1 \le i \le m_1$, we obtain the surface $S_g(\mathcal{C} \setminus (\mathcal{C}' \cup \mathcal{C}_1))$. Since $t(D_{1,i}) = D_{1,i+1}$, we may extend the homeomorphism h_0 to a homeomorphism $\widetilde{h_0} \in \mathrm{Mod}(S_g(\mathcal{C}' \cup \mathcal{C}_1))$, which cyclically permutes the $c_{1,i}$. If $|\mathbb{O}_1| = |\mathbb{O}_1'| = n$, then define $h_1 := \widetilde{h_0} t_{c_{1,1}}$. Otherwise, since $\mathbb{O}_1 \sim \mathbb{O}_1'$, the difference in the turning angles around $Q_{1,i}$ and $Q_{1,i}'$ is $2\pi/n$. Let $\widetilde{h_1}$ be the $(1/n)$th-twist around $c_{1,1}$. Now $h_1 := \widetilde{h_0}\widetilde{h_1}$ is an $(r, 1)$-permuting root of $t_{\mathcal{C}' \cup \mathcal{C}_1}$ of degree n in $\mathrm{Mod}(S_g(\mathcal{C} \setminus (\mathcal{C}' \cup \mathcal{C}_1)))$. We now repeat this process inductively to obtain an (r, k)-permuting root $h := h_k$ of $t_{\mathcal{C}}$ of degree n.

It remains to show that the resulting root h of $t_{\mathcal{C}}$ is determined up to conjugacy. Suppose t_1 and t_2 are two equivalent $(n, 2r, 2k)$-actions on $S_g(\mathcal{C})$ that are non-separating with respect to $\mathcal{C}$ with $\mathbb{P}(t_s) = \{P_{s,1}, P_{s,2}, \ldots, P_{s,r}\}$ and $\mathbb{O}(t_s) = \{\mathbb{O}_{s,1}, \mathbb{O}_{s,2}, \ldots, \mathbb{O}_{s,k}\}$, for $s = 1, 2$. Let ϕ be an orientation-preserving homeomorphism on $S_g(\mathcal{C})$ satisfying the conditions in Definition 3.4. Then, repeating the argument from [**6**, Theorem 2.1], ϕ extends to a homeomorphism Φ_0 on $S_g(\mathcal{C} \setminus \mathcal{C}'))$ such that $\Phi_0 h_{1,0} \Phi_0^{-1} = h_{2,0}$, where $h_{s,0}$ is the root of $t_{\mathcal{C}'}$ obtained from t_s, for $s = 1, 2$, as above. Furthermore, since ϕ maps $\mathbb{O}_{1,i}$ to $\mathbb{O}_{2,i}$ as in Definition 3.4, we may once again extend Φ_0 to a homeomorphism Φ satisfying $\Phi h_1 \Phi^{-1} = h_2$, where h_s is the root of $t_{\mathcal{C}}$ obtained from t_s, for $s = 1, 2$. □

Note that if $\mathcal{C}$ is a non-separating multicurve of size m, then $S_g(\mathcal{C}) \approx S_{g-m}$. A result of Wiman [**3**, Theorem 6] states that, if $g \ge 1$, then the highest order of a cyclic action on S_g is $(4g + 2)$. Furthermore, if $|\mathcal{C}| = 1$ and $g \ge 2$, then it was shown in [**6**, Corollary 2.2] that the degree n of a root of $t_{\mathcal{C}}$ is necessarily odd, and that $n \le 2g - 1$. These results, together with Theorems 3.9 and 4.3, lead to the following corollary.

Corollary 4.4. *Let $\mathcal{C}$ be a non-separating multicurve in S_g of size m, and let h be an (r, k)-permuting root of $t_{\mathcal{C}}$ of degree n.*

(*i*) *If* $r \geq 0$*, then*

$$n \leq \begin{cases} 4(g-m)+2, & \text{if } g-m \geq 1, \text{ and} \\ g, & \text{if } g = m. \end{cases}$$

Furthermore, if $g = m$*, then this upper bound is realizable.*

(*ii*) *If* $r \geq 1$*, then* n *is odd.*

(*iii*) *If* $r = 1$*, then* $n \leq 2(g-m)+1$.

(*iv*) *If* $r \geq 2$*, then* $n \leq \dfrac{g-m+r-1}{r-1}$.

Proof. If $g - m \geq 1$, then by Wiman's result, the highest order of a cyclic action on $S_g(\mathcal{C})$ is $4(g-m)+2$. If $g = m$, then consider the permuting $(n, 2r, 2k)$-action t on S_0 that is non-separating with respect to $\mathcal{C}$ guaranteed by Theorem 4.3. Since t is a cyclic action of order n on S_0, it must be a rotation by $2\pi\ell/n$ radians, where $\gcd(\ell, n) = 1$. Since the two fixed points of this action are not compatible in the sense of Definition 4.2, $r = 0$ and every non-trivial orbit has size n. Hence, $m = nk$ and so $n \mid m$, and in particular, $n \leq m = g$. Furthermore, if $g = m$, let t be the rotation of S_g by $2\pi/m$. Then, $t \circ t_c$, for some $c \in \mathcal{C}$, is a root of $t_{\mathcal{C}}$ of degree m (as in Figure 1), so (i) follows.

Parts (ii) and (iii) follow from [**6**, Corollary 2.2] that was stated in the discussion preceding this Corollary. If $r \geq 2$, then consider the corresponding permuting $(n, 2r, 2k)$ data set from Theorem 3.9. Note that the genus $(g-m)$ of the permuting data set is given by

$$\frac{2-2(g-m)}{n} = 2 - 2g_0 + 2r\left(\frac{1}{n} - 1\right) + \sum_{j=1}^{s}\left(\frac{1}{n_j} - 1\right).$$

Since $(1 - 1/n_j) \geq 0$ and $g_0 \geq 0$, we have $(g-m) - 1 + r \geq n(-1+r)$, from which (iv) follows. □

As mentioned earlier, the bound obtained in Corollary 4.4 (i) is not realizable in general, as we will show in Section 6.

5. Separating multicurves. A separating multicurve $\mathcal{C}$ in S_g is one where $S_g(\mathcal{C})$ is disconnected. In this case, we will require multiple finite order actions on the individual components of $S_g(\mathcal{C})$ to come together to form a root of $t_{\mathcal{C}}$ on S_g.

To begin with, a root h of $t_{\mathcal{C}}$ induces a non-trivial permutation of the components of $S_g(\mathcal{C})$. Since h is a homeomorphism, it maps one component to another of the same genus. Thus, we obtain a decomposition $S_g(\mathcal{C}) = \sqcup_{i=1}^{s} \mathbb{S}_{g_i}(m_i)$, where $h|_{\mathbb{S}_{g_i}(m_i)}$ induces a transitive action on each $\mathbb{S}_{g_i}(m_i)$. The number s above will be referred to as the *number of surface orbits* of h.

To improve the exposition, we consider simpler separating multicurves first, and then combine them inductively to obtain the general theory. We first consider the case when $\mathcal{C} = \mathcal{C}^{(m)}$, so $S_g(\mathcal{C})$ has exactly two components, which leads to the following two possibilities.

DEFINITION 5.1. Suppose that $S_g = S_{g_1} \#_{\mathcal{C}} S_{g_2}$, where $\mathcal{C} = \mathcal{C}^{(m)}$. Then a root h of $t_{\mathcal{C}}$ is said to be *side-preserving* if $h(S_{g_i}) = S_{g_i}$, for $i = 1, 2$, and *side-reversing* otherwise.

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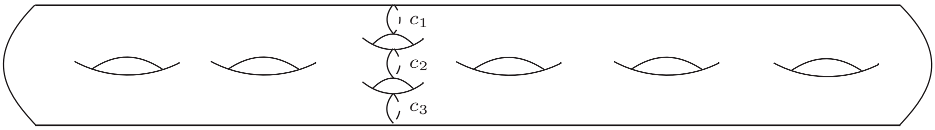

Figure 7. A multicurve $\mathcal{C} = \mathcal{C}^{(3)} = \{c_1, c_2, c_3\}$ in $S_7 = S_2 \#_{\mathcal{C}} S_3$.

5.1. Case 1: $\mathcal{C} = \mathcal{C}^{(m)}$ and the root is side-preserving. In this case, we will require a pair of compatible actions on two components of $S_g(\mathcal{C})$ that need to come together to yield a root of $t_{\mathcal{C}}$. For example, in Figure 7, a root of $t_{\mathcal{C}}$ would require a pair of compatible actions (t_1, t_2), where t_1 acts on S_2 and t_2 acts on S_3.

DEFINITION 5.2. Equivalence classes $[\![t_i]\!]$ of permuting (n_i, r_i, k_i)-actions on S_{g_i}, for $i = 1, 2$, are said to form an *(u,v)-compatible pair* $([\![t_1]\!], [\![t_2]\!])$ *of degree n* for integers $0 \le u \le r_i$ and $0 \le v \le k_i$, if

(i) $n = \mathrm{lcm}(n_1, n_2)$,
(ii) there exists $\{P_{i,1}, P_{i,2}, \dots, P_{i,u}\} \subset \mathbb{P}(t_i)$ such that for $1 \le j \le u$,

$$\theta_{P_{1,j}}(t_1) + \theta_{P_{2,j}}(t_2) \equiv \frac{2\pi}{n} \pmod{2\pi}, \text{ and}$$

(iii) there exists $\{\mathbb{O}_{i,1}, \mathbb{O}_{i,2}, \dots, \mathbb{O}_{i,v}\} \subset \mathbb{O}(t_i)$ for $i = 1, 2$ such that $\mathbb{O}_{1,j} \sim \mathbb{O}_{2,j}$ as in Definition 4.1, for $1 \le j \le v$.

The number $g := g_1 + g_2 + u + \sum_{j=1}^{v} |\mathbb{O}_{1,j}| - 1$ is called the *genus* of the pair $([\![t_1]\!], [\![t_2]\!])$. Note that two actions are $(1, 0)$-compatible if they are compatible as nestled actions in the sense of [**9**, Definition 3.2]. We write $\mathbb{P}(t_1, t_2) := \{P_{1,1}, P_{1,2}, \dots, P_{1,u}\}$ and $\mathbb{O}(t_1, t_2) := \{\mathbb{O}_{1,1}, \mathbb{O}_{1,2}, \dots, \mathbb{O}_{1,v}\}$. We define $\mathbb{P}(t_2, t_1)$ and $\mathbb{O}(t_2, t_1)$ similarly.

LEMMA 5.3. *Let $\mathcal{C} = \{c_1, c_2, \dots, c_m\}$ be a multicurve on S_g, and N_i be annular neighbourhood of c_i. Write $N = \sqcup_{i=1}^{m} N_i$, and suppose $t \in \mathrm{Mod}(S_g)$ is such that $t|_{S_g \setminus N} = id_{S_g \setminus N}$, then there exists $d_1, d_2, \dots, d_m \in \mathbb{N} \cup \{0\}$ such that $t = t_{c_1}^{d_1} \dots t_{c_m}^{d_m}$.*

Proof. Since $t|_{S_g \setminus N} = \mathrm{id}_{S_g \setminus N}$, t fixes ∂N. By definition, each $h \in \mathrm{Mod}(N_i)$ fixes ∂N_i pointwise, and $\mathrm{Mod}(N_i)$ is a cyclic group generated by t_{c_i} (as stated in [**2**, Proposition 2.4]). So, the lemma now follows from the fact that

$$\mathrm{Mod}(N) \cong \oplus_{i=1}^{m} \mathrm{Mod}(N_i) = \oplus_{i=1}^{m} \langle t_{c_i} \rangle.$$

□

THEOREM 5.4. *Suppose that $S_g = S_{g_1} \#_{\mathcal{C}} S_{g_2}$, where $\mathcal{C} = \mathcal{C}^{(m)}$. Then, (u, v)-permuting, side-preserving roots of $t_{\mathcal{C}}$ of degree n correspond to the (u, v)-compatible pairs $([\![t_1]\!], [\![t_2]\!])$ of equivalence classes of permuting (n_i, r_i, k_i)-actions on the S_{g_i}, of degree n.*

Proof. As before, we assume $m > 1$, and first show that every (u, v)-permuting root h of $t_{\mathcal{C}}$ of degree n yields a compatible pair $([\![t_1]\!], [\![t_2]\!])$ of degree n. Consider the (u, v)-partition $\mathbb{P}_{u,v}(\mathcal{C}) = \{\mathcal{C}'_1, \dots \mathcal{C}'_u, \mathcal{C}_1, \dots, \mathcal{C}_v\}$ of $\mathcal{C}$ induced by h as in Definition 2.7. Let $\widehat{S_{g_s}}$, for $s = 1, 2$, denote the two components of $\widehat{S_g(\mathcal{C})}$. Let N be a closed annular neighborhood of $\mathcal{C}$. By isotopy, we may assume that $t_{\mathcal{C}}(\mathcal{C}) = \mathcal{C}$, $t_{\mathcal{C}}(N) = N$, and $t_{\mathcal{C}}|_{\widehat{S_{g_s}}} = id_{\widehat{S_{g_s}}}$. Putting $\widehat{t_s} = h|_{\widehat{S_{g_s}}}$, we may assume up to isotopy that $\widehat{t_s}^n|_{\widehat{S_{g_s}}} = \mathrm{id}_{\widehat{S_{g_s}}}$ for $s = 1, 2$.

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Let n_s be the smallest positive integer such that $\widehat{t_s}^{n_s} = \mathrm{id}_{\widehat{S_{g_s}}}$, for $s = 1, 2$, and let $q = \mathrm{lcm}(n_1, n_2)$. Then, $t := h^q$ satisfies the hypotheses of Lemma 5.3. Hence, there exists $d_c \in \mathbb{N} \cup \{0\}$ such that $h^q = \prod_{c \in \mathcal{C}} t_c^{d_c}$. Since $h^n|_{\widehat{S_{g_1}}} = \mathrm{id}_{\widehat{S_{g_1}}}$ it follows that $n_1 \mid n$, and similarly $n_2 \mid n$. Hence, $q \mid n$ and so $\prod_{c \in \mathcal{C}} t_c = t_{\mathcal{C}} = (h^q)^{n/q} = \prod_{c \in \mathcal{C}} t_c^{nd_c/q}$. Fix $c \in \mathcal{C}$ and restrict the functions on both sides of this equation to a closed annular neighbourhood of c disjoint from other curves in $\mathcal{C}$. As in Proposition 2.6, we see that $nd_c/q = 1$, and hence $n = q = \mathrm{lcm}(n_1, n_2)$. We fill in $\partial \widehat{S_{g_s}}$ with disks to obtain the closed oriented surfaces S_{g_s} for $s = 1, 2$. We then extend $\widehat{t_s}$ to a permuting (n_s, r_s, k_s)-action t_s on S_{g_s}, where $n_s \mid n$, for $s = 1, 2$, and $n = \mathrm{lcm}(n_1, n_2)$.

When $u > 0$, the homeomorphism t_s fixes the centre points $\{P_{s,1}, \dots, P_{s,u}\}$ of u disks in $\overline{S_{g_s} \setminus \widehat{S_{g_s}}}$, for $s = 1, 2$. Hence, we may write $\mathbb{P}(t_1, t_2) = \{P_{1,1}, \dots, P_{1,u}\}$ and $\mathbb{P}(t_2, t_1) = \{P_{2,1}, \dots, P_{2,u}\}$. For $1 \le j \le u$, the proof of [**9**, Theorem 3.4] implies that the corresponding turning angles around $P_{1,j}$ and $P_{2,j}$ must be compatible in the sense of condition (ii) of Definition 5.2.

When $v > 0$, let $\mathcal{C}_i = \{c_{i,1}, c_{1,2}, \dots, c_{i,m_i}\}$, for $1 \le i \le v$. Associated with each curve $c_{i,j} \in \mathcal{C}_i$, is a disk $D^s_{i,j}$ in each $\overline{S_{g_s} \setminus \widehat{S_{g_s}}}$ for $s = 1, 2$. The centres $Q^s_{i,j}$ of the m_i disks $D^s_{i,j}$ for $1 \le j \le m_i$ form an orbit $\mathbb{O}_{s,i}$ in S_{g_s} for $s = 1, 2$. Thus, we obtain a collection $\mathbb{O}_{t_s} = \{\mathbb{O}_{s,1}, \dots, \mathbb{O}_{s,v}\}$ of v distinguished non-trivial orbits on S_{g_s} for $s = i, j$. It remains to show that $\mathbb{O}_{1,i} \sim \mathbb{O}_{2,i}$, for $1 \le i \le v$, but the argument for this is similar to that of Theorem 4.3. Hence, the pair $([\![t_1]\!], [\![t_2]\!])$ forms a (u, v)-compatible pair of degree n.

The arguments for the converse, and the fact that the resulting root is determined up to conjugacy, are analogous to that of Theorem 4.3. $\square$

We are now in a position to adapt the arguments from [**9**] to obtain an upper bound on the degree of a root of $t_{\mathcal{C}}$. In [**9**, Proposition 8.4], it was proved that if c is a separating curve and $S_g = S_{g_1} \#_c S_{g_2}$, then the degree n of a root of t_c is bounded above by $16g_1g_2 + 4(2g_1 - g_2) - 2$. Furthermore, if $g \ge 2$, then it was proved in [**9**, Proposition 8.6] that $n \le 4g^2 + 2g$. These results and their proofs will be used below.

COROLLARY 5.5. *Suppose that $S_g = S_{g_1} \#_{\mathcal{C}} S_{g_2}$, where $\mathcal{C} = \mathcal{C}^{(m)}$. If n denotes the degree of a side-preserving root of $t_{\mathcal{C}}$, then*

$$n \le 4(g - m)^2 + 10(g - m) + \frac{25}{4}.$$

Proof. From Theorem 5.4, an (r, k)-permuting root of $t_{\mathcal{C}}$ of degree n corresponds to a (r, k)-compatible pair $([\![t_1]\!], [\![t_2]\!])$ of equivalence classes of permuting (n_i, r_i, k_i)-actions on the S_{g_i}, of degree n. We shall first establish that, if $g_1 \ge g_2$, then

$$n \le 16g_1g_2 + 4(2g_1 - g_2) - 2. \tag{*}$$

If $r > 0$, (*) follows from [**9**, Proposition 8.4]. By an analogous argument, it can be shown (*) also holds when $\mathbb{O}(t_1, t_2)$ (and hence $\mathbb{O}(t_2, t_1)$) contains at least one orbit whose size is a proper divisor of both n_1 and n_2.

Suppose that $r = 0$ and every orbit of $\mathbb{O}(t_1, t_2)$ is of size n_1, and every orbit of $\mathbb{O}(t_2, t_1)$ is of size n_2. Since $\mathbb{O}(t_1, t_2) \neq \emptyset$, it follows that $n_1 = n_2$, and hence $n = \mathrm{lcm}(n_1, n_2) = n_1 \le 4g_1 + 2$, by Wiman's result [**3**, Theorem 6] (as stated before Corollary 4.4 in Section 4). Once again, we conclude that (*) holds.

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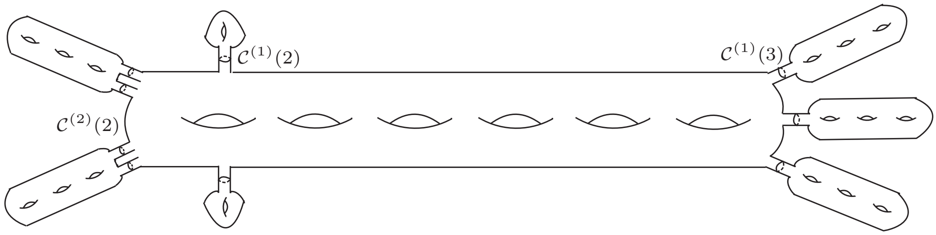

Figure 8. A multicurve $\mathcal{C} = \mathcal{C}^{(2)}(2) \sqcup \mathcal{C}^{(1)}(2) \sqcup \mathcal{C}^{(1)}(3)$ in S_{25} with $S_{25}(\mathcal{C}) = S_6 \sqcup \mathbb{S}_3(2) \sqcup \mathbb{S}_1(2) \sqcup \mathbb{S}_3(3)$.

Denoting the expression on the right hand side of (*) by $M(g_1, g_2)$ and putting $g_2 = g - g_1 - m + 1$, we obtain a quadratic polynomial in g_1 as in [**9**, Theorem 8.6], which attains its maximum at $g_1 = \frac{1}{2}(g - m) + \frac{7}{8}$, and consequently, $g_2 = \frac{1}{2}(g - m) + \frac{1}{8}$. Upon substituting these values of g_1 and g_2 in the expression for $M(g_1, g_2)$, we get the required result. □

5.2. Case 2: $\mathcal{C}$ is non-cyclical and the root has exactly one surface orbit of cardinality 1. In this case, the action induced by h on $S_g(\mathcal{C})$ has one distinguished surface orbit of cardinality one, and so it decomposes $S_g(\mathcal{C})$ in the form $S_g(\mathcal{C}) = S_{g_1} \sqcup_{i=1}^{s} \mathbb{S}_{g_{2,i}}(m_i)$ where $m_i > 1$, for $1 \leq i \leq s$. Note that h has s non-trivial surface orbits. Furthermore, h partitions $\mathcal{C}$ as $\mathcal{C} = \sqcup_{i=1}^{s} \mathcal{C}^{(k_i)}(m_i)$ as in Notation 2.3(v). (We refer the reader to Figure 8 for an example.) We thus require an action on S_{g_1} that is pairwise compatible with actions on each $\mathbb{S}_{g_{2,j}}(m_j)$. In order to classify roots in this case, we generalize the notion of a permuting (n, r, k)-action to encompass the action on $\mathbb{S}_g(m)$ induced by h.

Definition 5.6. Fix integers $g \geq 0$ and $m \geq 1$.

(i) An orientation-preserving C_n-action t on $\mathbb{S}_g(m)$ is said to be a *permuting* (n, r, k)-*action* if $m \mid n$ and $t = \sigma_m \circ \widetilde{t}$, where $\widetilde{t}$ is a permuting (n, r, k)-action on each S_g^1 and σ_m is a cyclical permutation of the components of $\mathbb{S}_g(m)$, which may be viewed as an m-cycle $(1\,2\,\ldots m)$.

(ii) Let t_1 and t_2 be two permuting (n, r, k)-actions on $\mathbb{S}_g(m)$. Then we say t_1 is *equivalent* to t_2 if for all i, ${t_1}^m|_{S_g^i}$ and ${t_2}^m|_{S_g^i}$ are equivalent as permuting $(n/m, \widetilde{r}, \widetilde{k})$-actions on S_g^i in the sense of Definition 3.4.

Remark 5.7. Suppose that $\widetilde{t} \in \mathrm{Mod}(S_g)$ defined a permuting (n, r, k)-action and $t = \sigma_m \circ \widetilde{t}$, then $t_i := t^m|_{S_g^i} \in \mathrm{Mod}(S_g^i)$ defines a permuting $(n/m, \widetilde{r}, \widetilde{k})$-action on S_g^i. Furthermore, all the t_i are conjugate to each other via σ_m.

Conversely, if $t' \in \mathrm{Mod}(S_g)$ is a permuting $(n/m, \widetilde{r}, \widetilde{k})$-action on S_g that has an mth root $\widetilde{t} \in \mathrm{Mod}(S_g)$, then the map $t := \sigma_m \circ \widetilde{t}$ defines a permuting (n, r, k)-action on $\mathbb{S}_g(m)$. Thus, a permuting (n, r, k)-action on $\mathbb{S}_g(m)$ corresponds to a permuting $(n/m, \widetilde{r}, \widetilde{k})$-action on S_g that has an mth root in $\mathrm{Mod}(S_g)$.

We begin with the simple case when the root induces a single non-trivial surface orbit.

Definition 5.8. Let t_1 be a permuting (n_1, r_1, k_1)-action on S_{g_1}, and let t_2 be a permuting (n_2, r_2, k_2)-action on $\mathbb{S}_{g_2}(m)$ such that $t_2 = \sigma_m \circ \widetilde{t_2}$ as in Remark 5.7. Then, for fixed integers $u, v \geq 0$, $(\llbracket t_1 \rrbracket, \llbracket t_2 \rrbracket)$ forms a *(u,v)-compatible pair of degree n* if

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(i) $n = \mathrm{lcm}(n_1, n_2)$,
(ii) for $1 \le i \le m$, $(\llbracket {t_1}^m \rrbracket, \llbracket t_2^m|_{S_{g_2}^i} \rrbracket)$ is a (u, v)-compatible pair of degree n/m,
(iii) for $1 \le i \le m$ and $1 \le j \le u$, there exists mutually disjoint pairs $\{P_{i,j}^1, P_{i,j}^2\}$, where $P_{i,j}^1 \in \mathbb{P}(t_1^m)$ and $P_{i,j}^2 \in \mathbb{P}(t_2^m|_{S_{g_2}^i})$ such that

$$\theta_{P_{i,j}^1}(t_1) + \theta_{P_{i,j}^2}(\widetilde{t_2}) \equiv 2\pi/n \pmod{2\pi}, \text{ and}$$

(iv) for $1 \le i \le m$ and $1 \le j \le v$, there exists mutually disjoint pairs $\{\mathbb{O}_{i,j}^1, \mathbb{O}_{i,j}^2\}$, where $\mathbb{O}_{i,j}^1 \in \mathbb{O}(t_1)$ and $\mathbb{O}_{i,j}^2 \in \mathbb{O}(\widetilde{t_2})$, such that $\mathbb{O}_{i,j}^1 \sim \mathbb{O}_{i,j}^2$, as in Definition 4.1.

Let $(\llbracket t_1 \rrbracket, \llbracket t_2 \rrbracket)$ be a (u, v)-compatible pair of degree n as above.

(i) We write $\mathbb{P}(t_1, t_2) = \{P_{i,j}^1 : 1 \le i \le m, 1 \le j \le u\}$ and $\mathbb{P}(t_2, t_1) = \{P_{i,j}^2 : 1 \le i \le m, 1 \le j \le u\}$.
(ii) Similarly, we define $\mathbb{O}(t_1, t_2) = \{\mathbb{O}_{i,j}^1 : 1 \le i \le m, 1 \le j \le v\}$ and $\mathbb{O}(t_1, t_2) = \{\mathbb{O}_{i,j}^2 : 1 \le i \le m, 1 \le j \le v\}$.

Definition 5.9. Let t_1 be a permuting (n_1, r_1, k_1)-action on S_{g_1}, and let $t_{2,j}$ be a permuting $(n_{2,j}, r_{2,j}, k_{2,j})$-action on $\mathbb{S}_{g_{2,j}}(m_j)$, for $1 \le j \le s$. Then, $(\llbracket t_1 \rrbracket, \llbracket t_{2,1} \rrbracket, \ldots, \llbracket t_{2,s} \rrbracket)$ forms a *(s+1)-compatible tuple of degree n* if:

(i) for each $1 \le j \le s$, $(\llbracket t_1 \rrbracket, \llbracket t_{2,j} \rrbracket)$ forms an (u_j, v_j)-compatible pair of degree n, for some $u_j, v_j \ge 0$ such that $k_{2,j} = u_j + v_j$, and
(ii) for each $i \ne j$, $\mathbb{O}(t_1, t_{2,i}) \cap \mathbb{O}(t_1, t_{2,j}) = \emptyset = \mathbb{P}(t_1, t_{2,i}) \cap \mathbb{P}(t_1, t_{2,j})$.

The number $g = g_1 + \sum_{j=1}^{s} m_j(g_{2,j} + k_{2,j} - 1)$ is called the *genus* of the $(s+1)$-tuple. The number $k = \sum_{j=1}^{s} k_{2,j}$ is called the *orbit number* of the tuple.

Theorem 5.10. *Let $\mathcal{C}$ be a non-cyclical separating multicurve. Then, conjugacy classes of $(0, k)$-permuting roots of $t_{\mathcal{C}}$ of degree n with s non-trivial surface orbits correspond to $(s+1)$-compatible tuples of degree n, genus g and orbit number k.*

Proof. Let $h \in \mathrm{Mod}(S_g)$ be a $(0, k)$-permuting root of degree n on S_g. Since the argument easily generalizes, we assume that h has exactly one non-trivial surface orbit, so that $C = \mathcal{C}^{(k)}(m)$ and $S_g = S_{g_1} \#_{\mathcal{C}} \mathbb{S}_{g_2}(m)$. Then, as in Theorem 5.4, we obtain a permuting (n_1, r_1, k_1)-action t_1 on S_{g_1} and a C_{n_2}-action t_2 on $\mathbb{S}_{g_2}(m)$, where $n = \mathrm{lcm}(n_1, n_2)$. Furthermore, h restricts to $\sigma_m : \mathbb{S}_{g_2}(m) \to \mathbb{S}_{g_2}(m)$ such that $\sigma_m \circ t_2 = t_2 \circ \sigma_m$. Hence, the maps $t_{2,i} := \sigma_m^{-1} t|_{S_{g_2}^i} : S_{g_2}^i \to S_{g_2}^i$ are conjugate to each other, and so t_2 is a permuting (n_2, r_2, k_2)-action on $\mathbb{S}_{g_2}(m)$, as in Definition 5.6.

Since h^m is a root of $t_{\mathcal{C}}$ of degree (n/m) that preserves m submulticurves $\mathcal{C}_i^{(k)}$ of $\mathcal{C}$, for $1 \le i \le m$, it induces $h_i \in \mathrm{Mod}(\Sigma_i)$, where $\Sigma_i := S_{g_1} \#_{\mathcal{C}_i^{(k)}} S_{g_2}^i$, and these h_i are pairwise conjugate to each other via h. Thus, it follows from Theorem 5.4 that there exist integers $u, v \ge 0$ such that $(\llbracket t_1^m \rrbracket, \llbracket t_2^m|_{S_{g_2}^i} \rrbracket)$ forms a (u, v)-compatible pair of degree n/m, for $1 \le i \le m$.

By condition (ii) of Definition 5.2, there exists mutually disjoint pairs $\{P_{i,j}^1, P_{i,j}^2\}$, where $P_{i,j}^1 \in \mathbb{P}(t_1^m)$ and $P_{i,j}^2 \in \mathbb{P}(t_2^m|_{S_{g_2}^i})$, such that

$$\theta_{P_{i,j}^1}(t_1^m) + \theta_{P_{i,j}^2}(\widetilde{t_2}^m) \equiv \frac{2\pi}{(n/m)} \pmod{2\pi}.$$

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Once again, since h^m is a root of $t_{\mathcal{C}}$, it follows that

$$\theta_{P^1_{i,j}}(t_1) + \theta_{P^2_{i,j}}(\widetilde{t_2}) \equiv 2\pi/n \pmod{2\pi}.$$

Similarly, one obtains condition (iv) of Definition 5.8 as well. Hence, h yields a compatible pair $(\llbracket t_1 \rrbracket, \llbracket t_2 \rrbracket)$ of degree n, genus g, and orbit number $k = k_2$. The converse is a just a matter of reversing this argument. $\square$

DEFINITION 5.11. Let $\mathcal{C}$ be a non-cyclical separating multicurve on S_g. We say that a tuple of non-negative integers

$$\psi = (g_1, (g_{2,1}, k_{2,1}, m_1), (g_{2,2}, k_{2,2}, m_2), \ldots, (g_{2,s(\psi)}, k_{2,s(\psi)}, m_{s(\psi)}))$$

is said to be *admissible with respect to* $\mathcal{C}$ if there is a decomposition of $\mathcal{C}$ in the form $\mathcal{C} = \sqcup_{j=1}^{s(\psi)} \mathcal{C}^{(k_{2,j})}(m_j)$ and $S_g(\mathcal{C})$ in the form $S_g(\mathcal{C}) = S_{g_1} \sqcup_{i=1}^{s(\psi)} \mathbb{S}_{g_{2,i}}(m_i)$.

With a view towards computing bounds, to an admissible tuple ψ as above, we associate the number

$$M(\psi) := \min_{1 \le i \le s(\psi)} m_i \left[4(g_1 + g_{2,i} - 1)^2 + 10(g_1 + g_{2,i} - 1) + \frac{25}{4} \right].$$

Note that every root of $t_{\mathcal{C}}$ yields an admissible tuple arising from the associated $(s+1)$-compatible tuple of actions from Theorem 5.10. While the converse is not necessarily true, it is clear that, given a surface S_g and a multicurve $\mathcal{C}$ in S_g, there are only finitely many such tuples that are admissible with respect to $\mathcal{C}$. Hence, the supremum described in the next corollary is taken over a finite set, and can thus be computed by elementary means.

COROLLARY 5.12. *Let $\mathcal{C}$ is a non-cyclical separating multicurve, and let h be a $(0, k)$-permuting root of $t_{\mathcal{C}}$ of degree n. Then, $n \le \sup_\psi M(\psi)$, where the supremum is taken over all tuples ψ that are admissible with respect to $\mathcal{C}$.*

Proof. By Theorem 5.10, we obtain an $(s+1)$-compatible tuple $(\llbracket t_1 \rrbracket, \llbracket t_{2,1} \rrbracket, \ldots, \llbracket t_{2,s} \rrbracket)$. For each $1 \le i \le s$, $(\llbracket t_1^{m_i} \rrbracket, \llbracket t_{2,i}^{m_i}|_{S^1_{g_{2,i}}} \rrbracket)$ forms a compatible pair of degree n/m_i as in Definition 5.2. By Corollary 5.5, it follows that $n/m_i \le [4(g_1 + g_{2,i} - 1)^2 + 10(g_1 + g_{2,i} - 1) + \frac{25}{4}]$, and the result follows. $\square$

5.3. Case 3: $\mathcal{C} = \mathcal{C}^{(m)}$ and the root is side-reversing. Let $S_g = S_{g_1} \#_{\mathcal{C}} S_{g_2}$ and h be a root of $t_{\mathcal{C}}$. Then, as shown in Figure 9, we should have $g_1 = g_2$, and for $i = 1, 2$, the actions $h_i = h^2|_{S_{g_i}}$ are now compatible $(n/2, r_i, k_i)$-actions in the sense of Definition 5.2. Furthermore, they are conjugate by h, so they define the same equivalence class. Thus, the proof of Theorem 5.10 can be adapted to obtain side-reversing version of Theorem 5.4 and its corollaries.

THEOREM 5.13. *Suppose that $S_g = S_{g_1} \#_{\mathcal{C}} S_{g_2}$, where $\mathcal{C} = \mathcal{C}^{(m)}$ and $g_1 = g_2$. Then, (r, k)-permuting, side-reversing roots of $t_{\mathcal{C}}$ of degree n correspond to the (r, k)-compatible pairs $(\llbracket t_1 \rrbracket, \llbracket t_2 \rrbracket)$ of equivalence classes of permuting (n, r, k)-actions on the S_{g_1}, of degree n, where $\llbracket t_1 \rrbracket = \llbracket t_2 \rrbracket$*

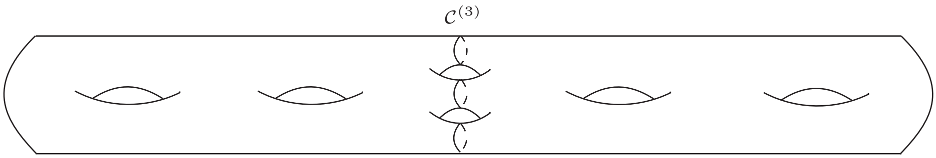

Figure 9. A multicurve $\mathcal{C} = \mathcal{C}^{(3)}$ in $S_6 = S_2 \#_{\mathcal{C}} S_2$.

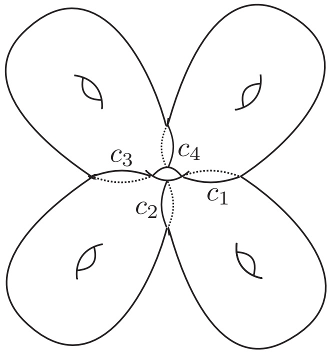

Figure 10. A cyclical multicurve $\mathcal{C} = \{c_1, c_2, c_3, c_4\}$ in S_5.

Since side-reversing roots of degree n can be squared to obtain side-preserving roots of degree $n/2$, the following corollary can be obtained by putting $g_1 = g_2 = \frac{g-m+1}{2}$ in Equation (*) of Corollary 5.5.

COROLLARY 5.14. *Suppose that $S_g = S_{g_1} \#_{\mathcal{C}} S_{g_2}$, where $\mathcal{C} = \mathcal{C}^{(m)}$. If n denotes the degree of a side-reversing root of $t_{\mathcal{C}}$, then*

$$n \leq 8(g-m)^2 + 20(g-m) + 8.$$

5.4. Case 4: $\mathcal{C}$ is cyclical and the root is $(0, k)$-permuting. In this case, a root h of $t_{\mathcal{C}}$ decomposes $S_g(\mathcal{C})$ as $S_g(\mathcal{C}) = \sqcup_{i=1}^{s} \mathbb{S}_{g_i}(m_i)$, where each $m_i > 1$. Note that, as in Definition 2.4, we assume that $S_g(\mathcal{C})$ has at least 3 components. For example, in Figure 10, a root of $t_{\mathcal{C}}$ may induce a partition of $S_5(\mathcal{C})$ in two different ways. It may happen that h is a (0, 1)-permuting root, in which case $S_5(\mathcal{C}) = \mathbb{S}_1(4)$. Alternatively, if h is a (0, 2)-permuting root, then we would write $S_5(\mathcal{C}) = \mathbb{S}_1(2) \sqcup \mathbb{S}_1(2)$.

REMARK 5.15. We claim that all the m_i in the above decomposition of $S_g(\mathcal{C})$ are equal. To see this, consider the decomposition (as in Definition 2.4)

$$S_g = \bigcup_{i=1}^{k} \widehat{\Sigma_i(\mathcal{C} \setminus \mathcal{C}_i)},$$

where $\Sigma_i = S_{g_i} \#_{\mathcal{C}_i} S_{g_j}$ with $j \equiv (i+1) \pmod{k}$. The homeomorphism h induces a homeomorphism $h' : \Sigma_i \to \Sigma_i'$, where $\Sigma_i' = S_{g_i'} \#_{h(\mathcal{C}_i)} S_{g_j'}$ and $\{g_i, g_j\} = \{g_i', g_j'\}$. Hence, $m_j = m_i$.

DEFINITION 5.16. Let $m \geq 2$, $u, v \geq 0$ be fixed integers. Let t_i be a permuting (n_i, r_i, k_i)-action on $\mathbb{S}_{g_i}(m)$ such that $t_i = \sigma_{m,i} \circ \widetilde{t_i}$, for $i = 1, 2$ as in Remark 5.7. Then, the equivalence classes $(\llbracket t_1 \rrbracket, \llbracket t_2 \rrbracket)$ are said to form a *cyclical (u,v)-compatible pair of degree n* if

(i) $n = \mathrm{lcm}(n_1, n_2)$,
(ii) for $1 \leq i \leq m$, $(\llbracket {t_1}^m|_{S^i_{g_1}} \rrbracket, \llbracket t_2^m|_{S^i_{g_2}} \rrbracket)$ is a (u, v)-compatible pair of degree n/m,
(iii) for $1 \leq i \leq m, 1 \leq j \leq u$, there exists mutually disjoint pairs $\{P^1_{i,j}, P^2_{i,j}\}$, where $P^1_{i,j} \in \mathbb{P}(t_1^m|_{S^i_{g_1}})$ and $P^2_{i,j} \in \mathbb{P}(t_2^m|_{S^i_{g_2}})$ such that

$$\theta_{P^1_{i,j}}(\widetilde{t_1}) + \theta_{P^2_{i,j}}(\widetilde{t_2}) \equiv \begin{cases} 0, & \text{if } \widetilde{t_i} = \mathrm{id}_{S^j_{g_i}} \text{ for } i = 1, 2, \text{ and} \\ 2\pi/n \pmod{2\pi}, & \text{otherwise, and} \end{cases}$$

(iv) for $1 \leq i \leq m$ and $1 \leq j \leq v$, mutually disjoint pairs $\{\mathbb{O}^1_{i,j}, \mathbb{O}^2_{i,j}\}$, where $\mathbb{O}^1_{i,j} \in \mathbb{O}(\widetilde{t_1})$ and $\mathbb{O}^2_{i,j} \in \mathbb{O}(\widetilde{t_2})$, such that $\mathbb{O}^1_{i,j} \sim \mathbb{O}^2_{i,j}$, as in Definition 4.1.

Let $(\llbracket t_1 \rrbracket, \llbracket t_2 \rrbracket)$ be a cyclical (u, v)-compatible pair of degree n as in Definition 5.8. We write

(i) $\mathbb{P}(t_1, t_2) = \{P^1_{i,j} : 1 \leq i \leq m, 1 \leq j \leq u\}$ and $\mathbb{P}(t_2, t_1) = \{P^2_{i,j} : 1 \leq i \leq m, 1 \leq j \leq u\}$.
(ii) Similarly, we define $\mathbb{O}(t_1, t_2) = \{\mathbb{O}^1_{i,j} : 1 \leq i \leq m, 1 \leq j \leq v\}$ and $\mathbb{O}(t_1, t_2) = \{\mathbb{O}^2_{i,j} : 1 \leq i \leq m, 1 \leq j \leq v\}$.
(iii) Putting $\beta(t_1, t_2) := u + \sum_{j=1}^{v} |\mathbb{O}^1_{1,j}| - 1 = u + \sum_{j=1}^{v} |\mathbb{O}^2_{1,j}| - 1$, we define the number $g := m(g_1 + g_2 + \beta(t_1, t_2))$ to be the *genus* of the pair.

DEFINITION 5.17. Let $m \geq 2$ be a fixed integer. Let t_i be permuting (n_i, r_i, k_i)-actions on $\mathbb{S}_{g_i}(m)$, for $1 \leq i \leq s$. Then, $(\llbracket t_1 \rrbracket, \llbracket t_2 \rrbracket, \ldots, \llbracket t_s \rrbracket)$ forms a *cyclical s-compatible tuple of degree n* if:

(a) For each $1 \leq i \leq s$, $n = \mathrm{lcm}(n_i, n_j)$, where $j \equiv (i + 1) \pmod{s}$.
(b) When $s = 1$,
  (i) for $1 \leq i \leq m$, $(\llbracket t_1^m|_{S^i_{g_1}} \rrbracket, \llbracket t_1^m|_{S^k_{g_1}} \rrbracket)$ is a (u, v)-compatible pair of degree n/m, where $k \equiv (i + 1) \pmod{m}$,
  (ii) for $1 \leq i \leq m$ and $1 \leq j \leq u$, there exists mutually disjoint pairs $\{P^1_{i,j}, P^2_{i,j}\}$, where $P^1_{i,j} \in \mathbb{P}(t_1^m|_{S^i_{g_1}})$ and $P^2_{i,j} \in \mathbb{P}(t_1^m|_{S^k_{g_1}})$ such that

$$\theta_{P^1_{i,j}}(\widetilde{t_1}) + \theta_{P^2_{i,j}}(\widetilde{t_1}) \equiv \begin{cases} 0, & \text{if } \widetilde{t_1} = \mathrm{id}_{S^j_{g_1}}, \text{ and} \\ 2\pi/n \pmod{2\pi}, & \text{otherwise,} \end{cases}$$

  (iii) for $1 \leq i \leq m$ and $1 \leq j \leq v$, there exists mutually disjoint pairs $\{\mathbb{O}^1_{i,j}, \mathbb{O}^2_{i,j}\}$, where $\mathbb{O}^1_{i,j} \in \mathbb{O}(\widetilde{t_1})$ and $\mathbb{O}^2_{i,j} \in \mathbb{O}(\widetilde{t_1})$, such that $\mathbb{O}^1_{i,j} \sim \mathbb{O}^2_{i,j}$, as in Definition 4.1, and
  (iv) for $1 \leq i \leq m$, if $\ell \equiv (i - 1) \pmod{m}$, then with Notation as in Definition 5.2,

$$\mathbb{P}(t_1^m|_{S^i_{g_1}}, t_1^m|_{S^k_{g_1}}) \cap \mathbb{P}(t_1^m|_{S^i_{g_1}}, t_1^m|_{S^\ell_{g_1}}) = \emptyset = \mathbb{O}(t_1^m|_{S^i_{g_1}}, t_1^m|_{S^k_{g_1}}) \cap \mathbb{O}(t_1^m|_{S^i_{g_1}}, t_1^m|_{S^\ell_{g_1}}), \text{ and}$$

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(v) the numbers $k := u + v$ and $g := m(g_1 + u + v - 1)$ are called the *orbit number* and *genus* of the tuple respectively.

(c) When $s > 1$, for $1 \leq i \leq s$,

(i) $(\llbracket t_i \rrbracket, \llbracket t_j \rrbracket)$ forms a (u_i, v_i)-compatible pair of degree n for some $u_i, v_i \geq 0$, where $j \equiv (i + 1) \pmod{s}$,

(ii) $\mathbb{O}(t_i, t_j) \cap \mathbb{O}(t_i, t_\ell) = \emptyset = \mathbb{P}(t_i, t_j) \cap \mathbb{P}(t_i, t_\ell)$, where $\ell \equiv (i - 1) \pmod{s}$, and

(iii) if $\widetilde{g}_i$ denotes the genus of the pair $(\llbracket t_i \rrbracket, \llbracket t_j \rrbracket)$ where $j = (i + 1) \pmod{s}$, then $k := \sum_{i=1}^{s}(u_i + v_i)$ and $g := m\left(\sum_{i=1}^{s}[\widetilde{g}_i + \beta(t_i, t_j)]\right)$ are called the *orbit number* and *genus* of the tuple, respectively.

The proof of the following theorem and its corollary are now analogous to that of Theorem 5.10 and Corollaries 5.12 and 5.14, keeping in mind Remark 5.15.

THEOREM 5.18. *Let $\mathcal{C}$ be a cyclical multicurve on S_g. Then, conjugacy classes of $(0, k)$ permuting roots of $t_{\mathcal{C}}$ of degree n with s non-trivial surface orbits correspond to cyclical s-compatible tuples of degree n, genus g and orbit number k.*

As in Definition 5.11, every root of $t_{\mathcal{C}}$ is naturally associated to a tuple of integers arising from the decomposition of $\mathcal{C}$ and of $S_g(\mathcal{C})$. Once again, an admissible tuple does not necessarily imply the existence of a root as the tuple does not capture the finite order actions and their compatibilities as in Theorem 5.18. Let $\mathcal{C}$ be a cyclical multicurve on S_g. We say that a tuple

$$\psi = (0, (g_1, k_1, m), (g_2, k_2, m), \ldots, (g_{s(\psi)}, k_{s(\psi)}, m_{s(\psi)}))$$

is said to be *admissible with respect to $\mathcal{C}$* if there is a decomposition of $\mathcal{C}$ in the form $\mathcal{C} = \sqcup_{i=1}^{s(\psi)} \sqcup_{k=1}^{m} \mathcal{C}_{k,i}$ and $S_g(\mathcal{C})$ in the form $S_g(\mathcal{C}) = \sqcup_{i=1}^{s(\psi)} \mathbb{S}_{g_i}(m)$ such that $S_g = \cup_{i=1}^{s(\psi)} \cup_{k=1}^{m} \Sigma_{i,k}\widehat{(\mathcal{C} \setminus \mathcal{C}_{k,i})}$, where $\Sigma_{i,k} := S_{g_i}^k \#_{\mathcal{C}_{k,i}} S_{g_j}^k$ with $j \equiv (i + 1) \pmod{s(\psi)}$. Given an admissible tuple ψ as above, we once again associate the number

$$M(\psi) = m \min_{1 \leq i \leq s(\psi)} \left[4(g_i + g_j - 1)^2 + 10(g_i + g_j - 1) + \frac{25}{4}\right],$$

where $j \equiv (i + 1) \pmod{s(\psi)}$. This tuple is now used to compute a bound on the degree of a root as in Corollary 5.12.

COROLLARY 5.19. *Let $\mathcal{C}$ be a cyclical multicurve on S_g, and let h be a $(0, k')$-permuting root of $t_{\mathcal{C}}$ of degree n. Then, $n \leq \sup_\psi M(\psi)$, where the supremum is taken over all tuples ψ that are admissible with respect to $\mathcal{C}$.*

5.5. Case 5: $\mathcal{C}$ is a non-cyclical and the root has multiple surface orbits of cardinality 1. We now consider the case of an (r, k)-permuting root of $t_{\mathcal{C}}$ where $r, k > 0$. We begin by writing S_g as a connected sum of subsurfaces S_{g_i} across those bounding submulticurves that are preserved by h. The restriction of h to each S_{g_i} is then a $(0, k_i)$-permuting root of the Dehn twist about the submulticurve $D_i := \mathcal{C} \cap S_{g_i}$. This allows us to apply Theorem 5.10 to obtain finite order actions on $S_{g_i}(D_i)$ such that pairs of actions on adjacent subsurfaces are compatible (in the sense of Theorem 5.4).

Let $\mathcal{C}$ be a non-cyclical separating multicurve in S_g, and let h be an (r, k)-permuting root of $t_{\mathcal{C}}$.

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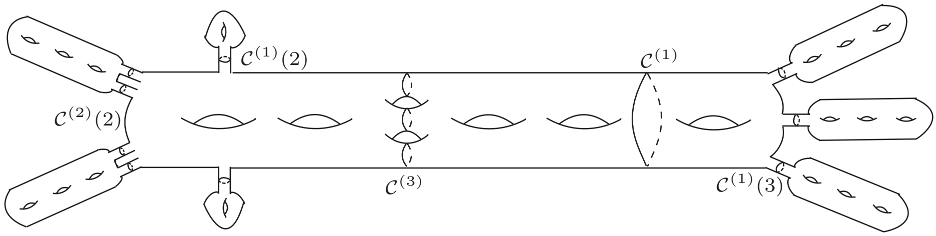

Figure 11. S_{24} with a separating multicurve $\mathcal{C}$.

(i) We shall denote the set of all bounding submulticurves of $\mathcal{C}$ that are preserved by h by $\text{Fix}_h(\mathcal{C}) = \{\mathcal{E}_1, \ldots, \mathcal{E}_{m(h)}\}$.

(ii) Writing $S_g = \overline{\#}_{i=1}^{m(h)}(S_{g_i} \#_{\mathcal{E}_i} S_{g_{i+1}})$ as in Notation 2.3, and $D_i := \mathcal{C} \cap S_{g_i}$, we have that for each i, $S_{g_i} \cap \mathcal{C}_{r,k}(h)$ is a $(0, k_i)$-partition of D_i, which has the form $S_{g_i} \cap \mathcal{C}_{r,k}(h) = \{\mathcal{C}_{i,j}^{(k_{i,j})}(m_{i,j}) : 1 \leq j \leq k_i\}$.

(iii) For $1 \leq i \leq r+1$, we write $S_{g_i} = \overline{\#}_{j=1}^{k_i}(S_{g_{i,1}} \#_{D_{i,j}} \mathbb{S}_{g_{i,2,j}}(m_{i,j}))$, where $D_{i,j} = \mathcal{C}_{i,j}^{(k_{i,j})}(m_{i,j})$ and $g_i = g_{i,1} + \sum_{j=1}^{k_i} m_{i,j}(g_{i,2,j} + k_{i,j} - 1)$.

In Figure 11, we have the surface S_{24} with the multicurve

$$\mathcal{C} = \mathcal{C}^{(2)}(2) \sqcup \mathcal{C}^{(1)}(2) \sqcup \mathcal{C}^{(3)} \sqcup \mathcal{C}^{(1)} \sqcup \mathcal{C}^{(1)}(3).$$

According to the notation introduced above, we have

$$S_{24} = \left(\mathbb{S}_3(2) \#_{\mathcal{C}^{(2)}(2)} S_2 \#_{\mathcal{C}^{(1)}(2)} \mathbb{S}_1(2)\right) \#_{\mathcal{C}^{(3)}} S_2 \#_{\mathcal{C}^{(1)}} \left(S_1 \#_{\mathcal{C}^{(1)}(3)} \mathbb{S}_3(3)\right).$$

Definition 5.20. Fix $m, n \in \mathbb{N}$, and for $1 \leq i \leq m+1$, let $\overline{t_i} = ([\![t_{i,1}]\!], \ldots, [\![t_{i,2,s_i}]\!])$ be an $(s_i + 1)$-compatible tuple as in Definition 5.9. Then, the tuple $(\overline{t_1}, \ldots, \overline{t_{m+1}})$ is said to form an *(m+1)-compatible multituple of degree n* if for each $1 \leq i \leq (m+1)$,

(i) the pair $([\![t_{i,1}]\!], [\![t_{i+1,1}]\!])$ forms an $(r_{i,1}, k_{i,1})$-compatible pair of degree n, and

(ii) $\mathbb{O}(t_{i,1}, t_{i+1,1}) \cap \left(\sqcup_{j=1}^{s_i} \mathbb{O}(t_{i,1}, t_{i,2,j})\right) = \emptyset = \mathbb{P}(t_{i,1}, t_{i+1,1}) \cap \left(\sqcup_{j=1}^{s_i} \mathbb{P}(t_{i,1}, t_{i,2,j})\right)$.

If $g(\overline{t_i})$ denotes the genus of $\overline{t_i}$, and $\alpha_i := \sum_{\mathbb{O} \in \mathbb{O}(t_{1,i}, t_{1,i+1})} |\, \mathbb{O} \,|$, then the number

$$g = \sum_{i=1}^{m+1} g(\overline{t_i}) + \sum_{i=1}^{m} (r_{1,i} + k_{1,i}\alpha_i - 1)$$

is called the *genus* of the multituple.

The following theorem follows from Theorem 5.4 and 5.10.

Theorem 5.21. *Let $\mathcal{C}$ be a non-cyclical separating multicurve in S_g. Then, the conjugacy class of a root h of $t_{\mathcal{C}}$ of degree n with m surface orbits of cardinality 1 corresponds to an $(m+1)$-compatible multituple of degree n and genus g.*

Let $\mathcal{C}$ be a non-cyclical separating multicurve in S_g. We say that a tuple $\psi = (\psi_1, \psi_2, \ldots, \psi_{m(\psi)})$ is *admissible with respect to $\mathcal{C}$* if there are disjoint submulticurves $\{\mathcal{E}_i\}_{i=1}^{m} \subset \mathcal{C}$, and a decomposition of S_g in the form $S_g = \overline{\#}_{i=1}^{m}(S_{g_i} \#_{\mathcal{E}_i} S_{g_{i+1}})$, where each ψ_i is an admissible tuple with respect to $\mathcal{C} \cap S_{g_i} \subset S_{g_i}$, $1 \leq i \leq m$ in the sense of Definition

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5.11. Using Theorem 5.21 and Corollary 5.12, we obtain the following bound on the degree of a root in this case.

COROLLARY 5.22. *Let $\mathcal{C}$ be a non-cyclical separating multicurve in S_g, and let h be root of $t_{\mathcal{C}}$ of degree n. Then,*

$$n \leq \sup_{\psi} \min_{1\leq i\leq m(\psi)} \operatorname{lcm}([M(\psi_i)], [M(\psi_{i+1})]),$$

where $M(\psi_i)$ is as in Definition 5.11, $[x]$ denotes the greatest integer $\leq x$, and the supremum is taken over all multituples ψ that are admissible with respect to $\mathcal{C}$.

If $\mathcal{C}$ is a generic separating multicurve, then $\mathcal{C}$ can be expressed as a disjoint union of non-separating multicurves and separating multicurves that are either cyclical or bounding. Consequently, the general theory for a separating multicurve will encompass the theories developed earlier sections. For the sake of brevity and clarity of exposition, we shall refrain from developing a theory for this case. However, we will classify such roots in $\mathrm{Mod}(S_4)$, thereby indicating how such a theory would follow from the ideas developed in Sections 4 and 5.

6. Classification of roots in $\mathrm{Mod}(S_4)$**.** In this section, we classify roots of Dehn twists about multicurves in $\mathrm{Mod}(S_4)$. When classifying an $(m+1)$-compatible multituples $(\overline{t_1}, \ldots, \overline{t_{m+1}})$ that corresponds to a root, Condition (i) of Definition 5.20 and Condition (ii) of Definition 5.9 help in eliminating data sets that do not lead to roots. For the sake of brevity, we only list those data sets that do lead to roots. Furthermore, in each case, a careful examination of the data set $\mathcal{D}$ also gives $\mathbb{O}_{\mathcal{D}}$, and so we only display the former.

Finally, when $\overline{t_i}$ is a permuting (n_i, r_i, k_i)-action on $\mathbb{S}_{g_i}(m_i)$, we use Remark 5.7 and replace $\overline{t_i}$ by the corresponding action on S_{g_i}, which has a root $\widetilde{t_i}$ of degree m_i, whose equivalence class can be encoded by a data set D_i. Therefore, an $(m+1)$-compatible multituple $(\overline{t_1}, \ldots, \overline{t_{m+1}})$ is described by a tuple $(D_1, D_2, \ldots, D_{m+1})$ of data sets, which will be listed in a table. While enumerating the curves in a multicurve, as a general convention, separating curves will be denoted with the letter c, while non-separating curves will be denoted with the letter d.

$\mathcal{C}$ is a non-separating multicurve

(n, r, k)	$\mathcal{C}$	D_1
$(4, 0, 1)$	$\{d_1, d_2, d_3, d_4\}$	$(4, 0, 1; (1, 4), (1, 4))$
$(4, 0, 1)$	$\{d_1, d_2, d_3, d_4\}$	$(4, 0, 3; (3, 4), (3, 4))$
$(2, 0, 2)$	$\{d_1, d_2, d_3, d_4\}$	$(2, 0, 1; (1, 2), (1, 2))$
$(3, 0, 1)$	$\{d_1, d_2, d_3\}$	$(3, 1, 1;)$
$(3, 0, 1)$	$\{d_1, d_2, d_3\}$	$(3, 1, 2;)$
$(2, 0, 1)$	$\{d_1, d_2\}$	$(2, 1, 0; (1, 2), (1, 2))$

Note that this shows that a non-separating multicurve of size 3 on S_4 does not have a root of degree 6. Hence, the upper bound obtained in part (i) of Corollary 4.4 is not realizable in general.

$\mathcal{C}$ is a separating multicurve

Table 1. $\mathcal{C} = \{c_1, c_2, c_3, c_4\}$, $S_4 = S_0 \#_{\mathcal{C}} S_1(4)$

(n, r, k)	D_1	D_2
$(4, 0, 1)$	$(4, 0, 1; (1, 4), (1, 4))$	$(1, 1;)$
$(4, 0, 1)$	$(4, 0, 3; (3, 4), (3, 4))$	$(1, 1;)$

Table 2. $\mathcal{C} = \{c_1, c_2, c_3, c_4\}$, $S_4 = S_1(2) \#_{\{c_1,c_2\}} S_0 \#_{\{c_3,c_4\}} S_1(2)$

(n, r, k)	D_1	D_2	D_3
$(2, 0, 2)$	$(1, 1, 0;)$	$(2, 0, 1; (1, 2), (1, 2))$	$(1, 1, 0;)$
$(6, 0, 2)$	$(6, 0, 0; (1, 6), (1, 2), (1, 3))$	$(2, 0, 1; (1, 2), (1, 2))$	$(6, 0, 0; (1, 6), (1, 2), (1, 3))$

Table 3. $\mathcal{C} = \{c_1, c_2, c_3\}$, $S_4 = S_1 \#_{\mathcal{C}} S_1(3)$

(n, r, k)	D_1	D_2
$(3, 0, 1)$	$(3, 1, 0;)$	$(1, 1, 0;)$
$(6, 0, 1)$	$(6, 0, 0; (1, 6), (1, 2), (1, 3))$	$(1, 1, 0;)$
$(6, 0, 1)$	$(6, 0, 0; (5, 6), (1, 2), (2, 3))$	$(1, 1, 0;)$
$(6, 0, 1)$	$(3, 1, 1;)$	$(6, 0, 0; (1, 6), (1, 2), (1, 3))$
$(6, 0, 1)$	$(3, 1, 1;)$	$(6, 0, 0; (5, 6), (1, 2), (2, 3))$

Table 4. $\mathcal{C} = \{d_1, d_2, d_3\}$ is cyclical

(n, r, k)	D_1
$(3, 0, 1)$	$(1, 1, 0;)$
$(3, 0, 1)$	$(3, 0, 0, (2, 2); (2, 3))$

Table 5. $\mathcal{C} = \{c_1, c_2, c_3\}$, $S_4 = S_1 \#_{c_1} S_1 \#_{\{c_2,c_3\}} S_1(2)$

(n, r, k)	D_1	D_2	D_3
$(4, 1, 1)$	$(1, 1, 0, (1);)$	$(4, 0, 0, (1); (1, 2), (1, 4))$	$(1, 0, 1;)$
$(6, 1, 1)$	$(1, 1, 0, (1);)$	$(6, 0, 0, (1); (1, 2), (1, 3))$	$(1, 0, 1;)$
$(6, 1, 1)$	$(3, 0, 0, (2); (2, 3), (2, 3))$	$(2, 0, 0, (1); (1, 2), (1, 2), (1, 2))$	$(6, 0, 0; (1, 6), (1, 2), (1, 3))$

Table 6. $\mathcal{C} = \{c_1, c_2, c_3\}$, $S_4 = S_1 \#_{c_1} S_1 \#_{c_2} S_1 \#_{c_3} S_1$

(n, r, k)	D_1	D_2	D_3	D_4
$(3, 3, 0)$	$(3, 0, 0, (2); (2, 3), (2, 3))$	$(3, 0, 0, (2, 2); (2, 3))$	$(3, 0, 0, (2, 2); (2, 3))$	$(3, 0, 0, (2); (2, 3), (2, 3))$
$(12, 3, 0)$	$(3, 0, 0, (1); (1, 3), (1, 3))$	$(4, 0, 0, (3, 3); (1, 2))$	$(3, 0, 0, (1, 1); (1, 3))$	$(4, 0, 0, (3); (1, 2), (3, 4))$
$(12, 3, 0)$	$(4, 0, 0, (3); (1, 2), (3, 4))$	$(3, 0, 0, (1, 1); (1, 3))$	$(4, 0, 0, (3, 2); (1, 2))$	$(3, 0, 0, (1); (1, 3), (1, 3))$

Table 7. $\mathcal{C} = \{c_1, c_2\}$, $S_4 = S_2 \#_{\mathcal{C}} S_1(2)$

(n, r, k)	D_1	D_2
$(6, 0, 1)$	$(2, 0, 0; (1, 2), (1, 2), (1, 2), (1, 2), (1, 2), (1, 2))$	$(6, 0, 0; (1, 6), (1, 2), (1, 3))$
$(6, 0, 1)$	$(2, 1, 0; (1, 2), (1, 2))$	$(6, 0, 0; (1, 6), (1, 2), (1, 3))$
$(6, 0, 1)$	$(6, 0, 0; (5, 6), (1, 3), (5, 6))$	$(1, 1, 0;)$

Table 8. $\mathcal{C} = \{c_1, c_2\}$, $S_4 = S_1 \#_{c_1} S_1 \#_{c_2} S_2$.

(n, r, k)	D_1	D_2	D_3
$(6, 2, 0)$	$(6, 0, 0, (5); (1, 2), (2, 3))$	$(3, 0, 0, (1, 1); (1, 3))$	$(6, 0, 0, (5); (1, 3), (5, 6))$

Table 9. $\mathcal{C} = \mathcal{C}^{(3)}$, $S_4 = S_1 \#_{\mathcal{C}} S_1$. The last three roots are side-reversing

(n, r, k)	D_1	D_2
(3, 0, 1)	(3, 0, 0; (1, 3), (1, 3), (1, 3))	(3, 0, 0; (1, 3), (1, 3), (1, 3))
(3, 0, 1)	(3, 0, 0; (1, 3), (1, 3), (1, 3))	(3, 0, 0; (2, 3), (2, 3), (2, 3))
(3, 0, 1)	(3, 0, 0; (2, 3), (2, 3), (2, 3))	(3, 0, 0; (2, 3), (2, 3), (2, 3))
(3, 0, 1)	(3, 0, 0; (2, 3), (2, 3), (2, 3))	(3, 0, 0; (1, 3), (1, 3), (1, 3))
(6, 0, 1)	(3, 0, 0; (1, 3), (1, 3), (1, 3))	(6, 0, 0; (1, 6), (1, 2), (1, 3))
(6, 0, 1)	(3, 0, 0; (1, 3), (1, 3), (1, 3))	(6, 0, 0; (5, 6), (1, 2), (2, 3))
(6, 0, 1)	(3, 0, 0; (2, 3), (2, 3), (2, 3))	(6, 0, 0; (1, 6), (1, 2), (1, 3))
(6, 0, 1)	(3, 0, 0; (2, 3), (2, 3), (2, 3))	(6, 0, 0; (5, 6), (1, 2), (2, 3))
(4, 1, 1)	(2, 0, 0, (1); (1, 2), (1, 2), (1, 2))	(4, 0, 0, (3); (1, 2), (3, 4))
(3, 3, 0)	(3, 0, 0, (2, 2, 2);)	(3, 0, 0, (2, 2, 2);)
(6, 3, 0)	(3, 0, 0, (2, 2, 2);)	(2, 0, 0, (1, 1, 1); (1, 2))
(3, 0, 1)	(1, 0, 0, 1;)	(1, 0, 0, 1;)
(3, 0, 1)	(1, 0, 0, 2;)	(1, 0, 0, 2;)
(6, 3, 0)	(3, 0, 0, (2, 2, 2);)	(3, 0, 0, (2, 2, 2);)

Table 10. $\mathcal{C} = \mathcal{C}^{(2)}(2)$, $S_4 = S_1 \#_{\mathcal{C}^{(2)}} S_0 \#_{\mathcal{C}^{(2)}} S_1$

(n, r, k)	D_1	D_2	D_3
(2, 0, 2)	(2, 0, 0; (1, 2), (1, 2), (1, 2), (1, 2))	(2, 0, 0; (1, 2), (1, 2))	(2, 0, 0; (1, 2), (1, 2), (1, 2), (1, 2))
(4, 0, 2)	(4, 0, 0; (1, 4), (1, 2), (1, 4))	(2, 0, 0; (1, 2), (1, 2))	(4, 0, 0; (1, 4), (1, 2), (1, 4))
(4, 0, 2)	(4, 0, 0; (1, 4), (1, 2), (1, 4))	(2, 0, 0; (1, 2), (1, 2))	(4, 0, 0; (3, 4), (1, 2), (3, 4))
(4, 0, 2)	(4, 0, 0; (3, 4), (1, 2), (3, 4))	(2, 0, 0; (1, 2), (1, 2))	(4, 0, 0; (3, 4), (1, 2), (3, 4))
(6, 0, 2)	(6, 0, 0; (1, 6), (1, 2), (1, 3))	(2, 0, 0; (1, 2), (1, 2))	(6, 0, 0; (1, 6), (1, 2), (1, 3))
(6, 0, 2)	(6, 0, 0; (1, 6), (1, 2), (1, 3))	(2, 0, 0; (1, 2), (1, 2))	(6, 0, 0; (5, 6), (1, 2), (2, 3))
(6, 0, 2)	(6, 0, 0; (5, 6), (1, 2), (2, 3))	(2, 0, 0; (1, 2), (1, 2))	(6, 0, 0; (5, 6), (1, 2), (2, 3))

Table 11. $\mathcal{C} = \mathcal{C}^{(2)}(2)$, $S_4 = S_0 \#_{\mathcal{C}} S_1(2)$

(n, r, k)	D_1	D_2
(2, 0, 2)	(2, 0, 0; (1, 2), (1, 2))	(1, 1, 0;)
(6, 0, 2)	(2, 0, 0; (1, 2), (1, 2))	(6, 0, 0; (1, 6), (1, 2), (1, 3))

Table 12. $\mathcal{C} = \{c_1\} \sqcup \mathcal{C}^{(2)}$, $S_4 = S_1 \#_{c_1} S_1 \#_{\mathcal{C}^{(2)}} S_1$

(n, r, k)	D_1	D_2	D_3
(6, 1, 1)	(3, 0, 0, (2); (2, 3), (2, 3))	(2, 0, 0, (1); (1, 2), (1, 2), (1, 2))	(6, 0; (1, 6), (1, 2), (1, 3))
(6, 1, 1)	(1, 1, 0, (1);)	(6, 0, 0, (1); (1, 2), (1, 3))	(2, 0; (1, 2), (1, 2), (1, 2), (1, 2))
(3, 3, 0)	(3, 0, 0, (2); (2, 3), (2, 3))	(3, 0, (2, 2, 2);)	(3, 0, (2, 2); (2, 3))
(12, 3, 0)	(4, 0, 0, (3); (1, 2), (3, 4))	(3, 0, (1, 1, 1);)	(4, 0, (3, 3); (1, 2))

Table 13. $\mathcal{C} = \mathcal{C}^{(1)}(4) \sqcup \{d_1, \ldots, d_4\}$,
$S_4 = S_0 \#_{\mathcal{C}^{(1)}(4)} S_1(4)$ with $d_i \in \widehat{S_1^i}$

(n, r, k)	D_1	D_2
(4, 0, 2)	(4, 0, 0; (1, 4), (1, 4)	(1, 0, 0;)

Table 14. $\mathcal{C} = \mathcal{C}^{(1)}(4) \sqcup \{d_1, d_2, d_1', d_2'\}$,
$S_4 = S_1(2) \#_{\mathcal{C}^{(1)}(2)} S_0 \#_{\mathcal{C}^{(1)}(2)} S_1(2)$ with $d_i, d_i' \in \widehat{S_1^i}$

(n, r, k)	D_1	D_2	D_3
(2, 0, 4)	(1, 0, 0;)	(2, 0, 0; (1, 2), (1, 2))	(1, 0, 0;)

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Table 15. $\mathcal{C} = \mathcal{C}^{(1)}(4) \sqcup \{d_1, d_2\}$, $S_4 = S_1(2)\#_{\mathcal{C}^{(1)}(2)} S_0 \#_{\mathcal{C}^{(1)}(2)} S_1(2)$ with $d_i \in \widehat{S}_1^i$

(n, r, k)	D_1	D_2	D_3
$(2, 0, 3)$	$(1, 1, 0;)$	$(2, 0, 0; (1, 2), (1, 2))$	$(1, 0;)$

Table 16. $\mathcal{C} = \mathcal{C}^{(1)}(2) \sqcup \{c_1, c_2, d_1, d_2\}$, $S_4 = S_1(2)\#_{\mathcal{C}^{(1)}(2)} \left(S_1\#_{c_1} S_0 \#_{c_2} S_1\right)$ with $d_i \in \widehat{S}_1^i$

(n, r, k)	D_1	D_2	D_3	D_4
$(2, 2, 2)$	$(1, 0;)$	$(1, 1, 1;)$	$(2, 0, 1; (1, 2))$	$(1, 1, 1;)$

Table 17. $\mathcal{C} = \mathcal{C}^{(1)}(3) \sqcup \{d_1, d_2, d_3\}$, $S_4 = S_1\#_{\mathcal{C}^{(1)}(3)} S_1(3)$ with $d_i \in \widehat{S}_1^i$

(n, r, k)	D_1	D_2
$(3, 0, 2)$	$(3, 1, 1;)$	$(1, 0, 0;)$

Table 18. $\mathcal{C} = \mathcal{C}^{(1)}(2) \sqcup \{c_1, d_1, d_2\}$, $S_4 = S_1\#_{c_1} S_1 \#_{\mathcal{C}^{(1)}(2)} S_1(2)$ with $d_i \in \widehat{S}_1^i$

(n, r, k)	D_1	D_2	D_3
$(2, 1, 2)$	$(2, 0, 1; (1, 2), (1, 2), (1, 2))$	$(1, 1, 1;)$	$(4, 0; (1, 4), (1, 4))$

Table 19. $\mathcal{C} = \mathcal{C}^{(1)}(2) \sqcup \{d_1, d_2\}$, $S_4 = S_2\#_{\mathcal{C}^{(1)}(2)} S_1(2)$ with $d_i \in \widehat{S}_1^i$

(n, r, k)	D_1	D_2
$(2, 0, 2)$	$(2, 0; (1, 2), (1, 2), (1, 2), (1, 2), (1, 2), (1, 2))$	$(4, 0; (1, 4), (1, 4))$
$(2, 0, 2)$	$(2, 1; (1, 2), (1, 2))$	$(4, 0; (1, 4), (1, 4))$

Table 20. $\mathcal{C} = \mathcal{C}^{(1)}(2) \sqcup \{d_1, d_2\}$, $S_4 = S_2\#_{\mathcal{C}^{(1)}(2)} S_1(2)$ with $d_i \in \widehat{S}_2$.

(n, r, k)	D_1	D_2
$(2, 0, 2)$	$(2, 0; (1, 2), (1, 2))$	$(4, 0; (1, 4), (1, 2), (1, 4))$
$(2, 0, 2)$	$(2, 0; (1, 2), (1, 2))$	$(4, 0; (3, 4), (1, 2), (3, 4))$

Table 21. $\mathcal{C} = \{c_1, d_1, d_2\}$, $S_4 = S_1\#_{c_1} S_3$ with $\{d_1, d_2\} \subset \widehat{S}_3$

(n, r, k)	D_1	D_2
$(2, 1, 1)$	$(1, 1, 1;)$	$(2, 0, 1; (1, 2), (1, 2), (1, 2))$

Table 22. $\mathcal{C} = \mathcal{C}^{(2)} \sqcup \{d_1, d_2\}$, $S_4 = S_2\#_{\mathcal{C}^{(2)}} S_1$ with $\{d_1, d_2\} \subset \widehat{S}_2$

(n, r, k)	D_1	D_2
$(2, 0, 2)$	$(2, 0; (1, 2), (1, 2))$	$(2, 0; (1, 2), (1, 2), (1, 2), (1, 2))$

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Table 23. $\mathcal{C} = \mathcal{C}^{(2)}$, $S_4 = S_2 \#_{\mathcal{C}} S_1$

(n, r, k)	D_1	D_2
(4, 0, 1)	(2, 0, 0; (1, 2), (1, 2), (1, 2), (1, 2), (1, 2), (1, 2))	(4, 0, 0; (1, 4), (1, 2), (1, 4))
(4, 0, 1)	(2, 0, 0; (1, 2), (1, 2), (1, 2), (1, 2), (1, 2), (1, 2))	(4, 0, 0; (3, 4), (1, 2), (3, 4))
(4, 0, 1)	(2, 1, 0; (1, 2), (1, 2))	(4, 0, 0; (1, 4), (1, 2), (1, 4))
(4, 0, 1)	(2, 1, 0; (1, 2), (1, 2))	(4, 0, 0; (3, 4), (1, 2), (3, 4))
(4, 0, 1)	(4, 0, 0; (1, 2), (1, 2), (3, 4))	(2, 0, 0; (1, 2), (1, 2), (1, 2), (1, 2))
(4, 0, 1)	(4, 0, 0; (1, 2), (1, 2), (3, 4))	(2, 1, 0;)
(4, 0, 1)	(4, 0, 0; (1, 2), (1, 2), (2, 4))	(2, 0, 0; (1, 2), (1, 2), (1, 2), (1, 2))
(4, 0, 1)	(4, 0, 0; (1, 2), (1, 2), (2, 4))	(2, 1, 0;)
(6, 0, 1)	(2, 0, 0; (1, 2), (1, 2), (1, 2), (1, 2), (1, 2), (1, 2))	(6, 0, 0; (1, 6), (1, 2), (1, 3))
(6, 0, 1)	(2, 1, 0; (1, 2), (1, 2))	(6, 0, 0; (1, 6), (1, 2), (1, 3))
(6, 0, 1)	(6, 0, 0; (5, 6), (1, 3), (5, 6))	(2, 0, 0; (1, 2), (1, 2), (1, 2), (1, 2))
(6, 0, 1)	(6, 0, 0; (5, 6), (1, 3), (5, 6))	(2, 1, 1;)
(2, 2, 0)	(1, 2, 0, (1, 1);)	(2, 0, , 0(1, 1); (1, 2), (1, 2))
(2, 2, 0)	(2, 0, 0, (1, 1); (1, 2), (1, 2), (1, 2), (1, 2))	(1, 1, 0, 1, 1);)
(2, 2, 0)	(2, 1, 0, (1, 1);)	(1, 1, 0, (1, 1);)
(3, 2, 0)	(1, 2, 0, (1, 1);)	(3, 0, 0, (1, 1); (1, 3))
(3, 2, 0)	(3, 0, 0, (1, 1); (2, 3), (2, 3))	(1, 1, 0, (1, 1);)
(3, 2, 0)	(3, 0, 0, (2, 2); (1, 3), (1, 3))	(3, 0, 0, (2, 2); (1, 3), (1, 3))
(4, 2, 0)	(1, 2, 0, (1, 1);)	(4, 0, 0, (1, 1); (1, 2))
(5, 2, 0)	(5, 0, 0, (1, 1); (3, 5))	(1, 1, 0, (1, 1);)
(6, 2, 0)	(6, 0, 0, (1, 1); (2, 3))	(1, 1, 0, (1, 1);)
(6, 2, 0)	(2, 0, 0, (1, 1); (1, 2), (1, 2), (1, 2), (1, 2))	(3, 0, 0, (2, 2); (2, 3))
(6, 2, 0)	(2, 0, 0, (1, 1); (1, 2), (1, 2), (1, 2), (1, 2))	(3, 0, 0, (2, 2); (2, 3))
(6, 2, 0)	(2, 1, 0, (1, 1);)	(3, 0, 0, (2, 2); (2, 3))
(6, 2, 0)	(3, 0, 0, (2, 2); (1, 3), (1, 3))	(2, 0, 0, (1, 1); (1, 2), (1, 2))
(6, 2, 0)	(6, 0, 0, (5, 5); (1, 3)	(3, 0, 0, (1, 1); (1, 3))
(12, 2, 0)	(3, 0, 0, (1, 1); (2, 3), (2, 3))	(4, 0, 0, (3, 3); (1, 2))
(12, 2, 0)	(6, 0, 0, (5, 5); (1, 3))	(4, 0, 0, (1, 1); (1, 2))
(15, 2, 0)	(5, 0, 0, (3, 3); (4, 5))	(3, 0, 0, (2, 2); (2, 3))
(20, 2, 0)	(5, 0, 0, (4, 4); (2, 5))	(4, 0, 0, (1, 1); (1, 2))

7. Concluding remarks.

7.1. Roots and the Torelli group. Let $\Psi : \mathrm{Mod}(S_g) \to \mathrm{Sp}(2g, \mathbb{Z})$ be the symplectic representation of $\mathrm{Mod}(S_g)$ arising out of its action on $H_1(S_g, \mathbb{Z})$. For $m \in \mathbb{N}$, the natural surjection $\mathrm{Sp}(2g, \mathbb{Z}) \to \mathrm{Sp}(2g, \mathbb{Z}/m\mathbb{Z})$ induces a map $\Psi_m : \mathrm{Mod}(S_g) \to \mathrm{Sp}(2g, \mathbb{Z}/m\mathbb{Z})$. For $m \geq 3$ and $g \geq 1$, the kernel of this map, denoted by $\mathrm{Mod}(S_g)[m]$, is a torsion-free subgroup of finite index in $\mathrm{Mod}(S_g)$ [**2**, Theorem 6.9], called the *level m congruence subgroup* of $\mathrm{Mod}(S_g)$.

THEOREM 7.1. *Let h be the root of the Dehn twist $t_{\mathcal{C}}$ about a multicurve $\mathcal{C}$ in S_g. Then, $h \notin \mathrm{Mod}(S_g)[m]$, for $m \geq 3$.*

Proof. Let $\hat{\imath}(a, b)$ denote the algebraic intersection number between isotopy classes a, and b of simple closed curves in S_g. If c is a non-separating curve in S_g, there is a non-separating curve d such that $\hat{\imath}(c, d) = 1$, and $\{c, d\}$ can be extended to a geometric symplectic basis of $H_1(S_g; \mathbb{Z})$. Now, [**2**, Proposition 6.3] states that, for any $k \geq 0$, $\Psi(t_b^k)([a]) = [a] + k\hat{\imath}(a, b)[b]$, and so we have $\Psi(t_c)[d] = [c] + [d]$. Hence, if $\mathcal{C}$ is a multicurve contains at least one non-separating curve, then $t_{\mathcal{C}} \notin \mathrm{Mod}(S_g)[m]$, for all $m \geq 1$. So, we assume that every curve in $\mathcal{C}$ is a separating curve, and we fix $m \geq 3$. From the theory developed in earlier sections, we know that a root of $t_{\mathcal{C}}$ induces a

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non-trivial partition $S_g(\mathcal{C}) = \sqcup_{i=1}^{s} \mathbb{S}_{g_i}(m_i)$, and an action of the form $\sigma_{m_i} \circ t_i$ on each $\mathbb{S}_{g_i}(m_i)$, where t_i is a permuting $(n_i/m_i, r_i, k_i)$-action on a surface $S_{g_i}^1$.

Suppose that some $m_i > 1$, then we must have that $g_i > 0$ since all the curves in $\mathcal{C}$ are separating. If $t_i^{m_i} = \mathrm{id}_{S_{g_i}^1}$, then t_i is equivalent to $\mathrm{id}_{S_{gi}^1}$ (by Definition 5.6), and so h induces a non-trivial permutation of the $2g_im_i$ standard generators of $H_1(S_g; \mathbb{Z})$ contributed by $\mathbb{S}_{g_i}(m_i)$, and hence $\Psi_m(h)$ is non-trivial. If $t_i^{m_i} \neq \mathrm{id}_{S_{gi}^1}$, then $\Psi_m(t_i^{m_i})$ forms a finite order subblock of $\Psi_m(h^{m_i})$. Since $\mathrm{Mod}(S_g)[m]$ is torsion-free, it follows that $h^{m_i} \notin \mathrm{Mod}(S_g)[m]$, and so $h \notin \mathrm{Mod}(S_g)[m]$.

Suppose that every $m_j = 1$. Then, there must exist some component S_{g_k} of $S_g(\mathcal{C})$ with $g_k > 1$, where h induces a non-trivial permuting action. This action yields a non-trivial finite order subblock of $\Psi_m(h)$, and since $\mathrm{Mod}(S_g)[m]$ is torsion-free, we have that $h \notin \mathrm{Mod}(S_g)[m]$. □

Note that the above theorem does not hold for $m = 2$ as the first root listed in Table 10 provides a counterexample.

7.2. Roots of finite product of powers. We believe that the theory developed in this paper for classifying roots up to conjugacy for finite products of commuting Dehn twists can be naturally generalised to one that classifies roots of finite products of powers of commuting twists.

Currently, the compatibility condition requires that pairs of distinguished orbits (or fixed points) of permuting actions should have associated angles that add up to $2\pi/n \pmod{2\pi}$. When c is a single non-separating curve, the roots of t_c^{ℓ} for $1 \leq \ell < n$, were classified in [**10**] by using a variant of this condition, which required that the angles associated with compatible fixed points add up to $2\pi\ell/n \pmod{2\pi}$. This notion of compatibility of fixed points can be generalized to orbits, and this could lead to the classification of roots of homeomorphisms of the form $\prod_{i=1}^{m} t_{c_i}^{\ell_i}$, where $\{c_1, c_2, \ldots, c_m\}$ is a multicurve and each $\ell_i \in \mathbb{Z}$. In particular, this will account for bounding pair maps, which are maps of the form $t_c t_d^{-1}$ where c and d are homologous non-separating simple closed curves.

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