A DIRECT PROOF OF A THEOREM OF WEST ON SEQUENCES OF RIESZ OPERATORS

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We recall (cf. [2] Definitions 3.1 and 3.2, p. 322) that a bounded linear operator T on a Banach space $\mathfrak{X}$ into itself is said to be *asymptotically quasi-compact* if $\kappa(T^n)^{1/n} \to 0$ as $n \to \infty$, where $\kappa(U) = \inf \| U - C \|$ for every bounded linear operator U on $\mathfrak{X}$ into itself, the infimum being taken over all compact linear operators C on $\mathfrak{X}$ into itself. For a complex Banach space, this is equivalent (cf. [2], pp. 319, 321 and 326) to T being a Riesz operator.

THEOREM 1 (West). *If the sequence $\{T_n\}$ of asymptotically quasi-compact bounded linear operators on a Banach space $\mathfrak{X}$ into itself converges to the bounded linear operator T on $\mathfrak{X}$ into itself, and T commutes with each operator T_n of the sequence, then T is asymptotically quasi-compact.*

West's proof (cf. [3], Corollary 4.3, p. 135) depends heavily on Banach algebra theory. The following is a direct proof.

Let $\varepsilon > 0$ be given. We first choose p so that $\| T_p - T \| < \varepsilon/3$, and put $U_p = T - T_p$ (so that $\| U_p \| < \varepsilon/3$); we shall suppose that U_p is not the zero operator, for otherwise $T = T_p$ and the conclusion follows immediately. We next choose q so that $\kappa(T_p^n)^{1/n} < \varepsilon/3$ when $n > q$. Then, since T_p commutes with U_p, we have (provided that $n \geqq q$)

$$T^n = (T_p + U_p)^n = \sum_{r=0}^{q} \binom{n}{r} T_p^r U_p^{n-r} + \sum_{r=q+1}^{n} \binom{n}{r} T_p^r U_p^{n-r},$$

and so

$$\kappa(T^n) \leqq \sum_{r=0}^{q} \binom{n}{r} \| T_p \|^r \| U_p \|^{n-r} + \sum_{r=q+1}^{n} \binom{n}{r} \kappa(T_p^r) \| U_p \|^{n-r}$$

$$= \| U_p \|^n \sum_{r=0}^{q} \binom{n}{r} (\| T_p \| / \| U_p \|)^r + \sum_{r=q+1}^{n} \binom{n}{r} \kappa(T_p^r) \| U_p \|^{n-r}.$$

But $\sum_{r=0}^{q} \binom{n}{r} (\| T_p \| / \| U_p \|)^r$ is a polynomial in n, and so there is a positive constant k such that

$$\sum_{r=0}^{q} \binom{n}{r} (\| T_p \| / \| U_p \|)^r < k \,.\, 2^n$$

when n is sufficiently large. Hence

$$\kappa(T^n) < (\varepsilon/3)^n \,.\, k \,.\, 2^n + \sum_{r=q+1}^{n} \binom{n}{r} (\varepsilon/3)^r (\varepsilon/3)^{n-r}$$

$$\leqq (k+1)(2/3)^n \varepsilon^n$$

when n is sufficiently large. But $(k+1)(2/3)^n \to 0$ as $n \to \infty$. Hence

$$\kappa(T^n) < \varepsilon^n,$$

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and so

$$\kappa(T^n)^{1/n} < \varepsilon,$$

when n is sufficiently large. It follows that $\kappa(T^n)^{1/n} \to 0$ as $n \to \infty$, that is, T is asymptotically quasi-compact.

NOTE. In a later paper ([4]), West discusses the possibility of expressing a Riesz operator as the sum of a compact linear operator and a quasi-nilpotent bounded linear operator, but he does not mention that such an operator is necessarily a Riesz operator; indeed in a later joint paper ([1], p. 108), he and his co-author use an ad hoc argument to prove a particular case. In fact we have the following theorem.

THEOREM 2. *If C is a compact linear operator on a Banach space $\mathfrak{X}$ into itself, and Q is a quasi-nilpotent bounded linear operator on $\mathfrak{X}$ into itself, then $T = C+Q$ is asymptotically quasi-compact.*

Proof. If we expand $T^n = (C+Q)^n$ in the natural way, we obtain a sum of 2^n terms, each term a product of n factors, each factor either C or Q (since we are not assuming that C and Q commute, we cannot combine terms as in the binomial theorem). One term will be Q^n; all the other terms will have at least one factor C, and so will be compact, and their sum will therefore be compact. Hence

$$\kappa(T^n)^{1/n} \leqq \| Q^n \|^{1/n} \to 0 \quad \text{as } n \to \infty,$$

that is T is asymptotically quasi-compact.

Note added in proof. In fact, the completeness of $\mathfrak{X}$ is not essential in Theorems 1 and 2. They remain true for incomplete spaces provided that the definition of asymptotic quasi-compactness is extended to these spaces.

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