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A DEFINITION OF SEPARATION AXIOM

BY
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0. **Introduction.** Several separation axioms, defined in terms of continuous functions, were examined by van Est and Freudenthal [3], in 1951. Since that time, a number of new topological properties which were called separation axioms were defined by Aull and Thron [1], and later by Robinson and Wu [2]. This paper gives a general definition of separation axiom, defined in terms of continuous functions, and shows that the standard separation axioms, and all but one of these new topological properties, fit this definition. Moreover, it is shown that the remaining property, defined in [2], can never fit the expected form of the definition. In addition, a new class of separation axioms lying strictly between T_0 and T_1 are defined and characterized, and examples of spaces satisfying these axioms are produced.

1. **The definition.** Suppose that $\mathscr{A}$ and $\mathscr{B}$ are classes of subsets of topological spaces. Given a class of $\mathscr{X}$ of topological spaces, with a distinguished pair of subsets A_X and B_X of each member $X \in \mathscr{X}$, we define $T(\mathscr{A}, \mathscr{B}, \mathscr{X})$ as the class of topological spaces Y such that for every disjoint pair of non-empty subsets A and B of Y with $A \in \mathscr{A}$ and $B \in \mathscr{B}$, there is a continuous map $f: Y \to X$ with $f[A]=A_X$ and $f(B)=B_X$, for some $X \in \mathscr{X}$. By a *separation axiom* we shall mean a class $T(\mathscr{A}, \mathscr{B}, \mathscr{X})$, or the intersection of such classes. Throughout, we shall identify all separation axioms with the class of topological spaces satisfying that axiom.

Let $\mathscr{S}$ be the class of all singletons of topological spaces, $\mathscr{C}$ the class of all closed sets, $\mathscr{D}$ the class of all doubletons, $\mathscr{F}$ the class of all finite sets, and $\mathscr{D}\mathscr{S}$ the class of all derived sets of singletons of topological spaces. This terminology will be preserved throughout.

The following simple observations follow quickly from the definition. If $f: X \to Y$ is a continuous map such that $f[A_X]=A_Y$ and $f[B_X]=B_Y$, then $T(\mathscr{A}, \mathscr{B}, X) \subseteq T(\mathscr{A}, \mathscr{B}, Y)$. The converse is not true, however, as will be shown following Theorem 2.4. Also, if the classes $\mathscr{A}$ and $\mathscr{B}$ are closed under the weakening of topologies, for example the classes $\mathscr{S}$, $\mathscr{D}$, and $\mathscr{F}$ and the class of all compact sets, then $T(\mathscr{A}, \mathscr{B}, \mathscr{X})$ is closed under the strengthening of topologies. Again, suppose $\mathscr{A}$ and $\mathscr{B}$ satisfy the following property. If the subsets A and B of a space Y belong to $\mathscr{A}$ and $\mathscr{B}$ respectively, and if Y is a subspace of a space X, then A and B belong to $\mathscr{A}$ and $\mathscr{B}$ when considered as subsets of X. For example, the classes

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$\mathcal{S}$, $\mathcal{D}$, $\mathcal{F}$, and the class of all compact sets satisfy this property. Under these circumstances the class $T(\mathcal{A}, \mathcal{B}, \mathcal{X})$ is hereditary; that is, every subspace of a member is a member. Finally, $T(\mathcal{S}, \mathcal{S}, \mathcal{X})$ is closed under arbitrary products, for all families $\mathcal{X}$ of topological spaces with distinguished pairs of subsets.

2. **The standard axioms and the axioms of Aull and Thron.** The first theorem shows that the commonly known separation axioms, and the axioms defined by Aull and Thron in [1] fit our definition. For reference, we shall give the definitions of the axioms in [1]. The closure of a set A will be denoted by $\mathrm{cl}(A)$, the closure of a singleton $\{x\}$ by $\mathrm{cl}(x)$, and the derived set of a point x by $\{x\}'$. A space X is T_D if for each $x \in X$, $\{x\}'$ is closed. A space is T_{DD} if it is T_D, and for all distinct points x and y, $\{x\}' \cap \{y\}' = \varnothing$. A space is T_{UD} if for each point x, $\{x\}'$ is a union of disjoint closed sets. A space is T_Y if for all distinct points x and y, $\mathrm{cl}(x) \cap \mathrm{cl}(y)$ contains at most one point. A space is T_{YS} if for all distinct points x and y, $\mathrm{cl}(x) \cap \mathrm{cl}(y)$ is one of $\{x\}$, $\{y\}$, or $\varnothing$. A space is T_F if for each point x and finite set F not containing x, there is an open set containing F but not x, or containing x and disjoint from F. A space is T_{FF} if for each disjoint pair of finite sets there is an open set containing one and disjoint from the other.

Next, we must name the following topological spaces with distinguished pairs of subsets. Let a, b, c, and d be distinct points. Then P_0 is the two point space with base $\{\{a\}, \{a, b\}\}$ and distinguished subsets $A=\{a\}$ and $B=\{b\}$. Let P_0' be the space with base $\{\{b\}, \{a, b\}\}$ and with $A=\{a\}$ and $B=\{b\}$. Let P_1 have the base $\{\{a\}, \{b\}, \{a, b, c\}\}$ and $A=\{a\}$, $B=\{b\}$. Define P_2 as the set $\{a, b\}$ with the discrete topology, and with $A=\{a\}$, $B=\{b\}$. Next, P_3 has base $\{\{a, b\}, \{a, b, c\}\}$ with $A=\{a\}$, $B=\{b, c\}$. Finally, let P_4 have base $\{\{a, c\}, \{a, b, c, d\}\}$ with $A=\{a, b\}$, $B=\{c, d\}$.

For each initial ordinal α, let $P^{(\alpha)}$ be the distinct set of points $\{a_\beta : \beta<\alpha\}$, with $A=\{a_0\}$, $B=\{a_\beta : 0<\beta<\alpha\}$, and a base consisting of all complements of the singletons $\{a_\beta\}$, $0<\beta<\alpha$. Define U as the unit interval $[0, 1]$ with the usual topology, and $A=\{0\}$, $B=\{1\}$.

LEMMA 2.1. *Suppose that X is a topological space with at least three points. Then the following are equivalent:*

(i) $X \in T(\mathcal{S}, \mathcal{D}, P_3)$;

(ii) $X \in T_0$ *and for each pair of distinct points* $x, y \in X$, $\{x\}' \cap \{y\}' = \varnothing$;

and (iii) *for each pair of distinct points* $x, y \in X$, $\mathrm{cl}(x) \cap \mathrm{cl}(y)$ *is one of* $\{x\}$, $\{y\}$, *or* $\varnothing$.

Proof. (i)$\Rightarrow$(ii). Suppose that $X \in T(\mathcal{S}, \mathcal{D}, P_3)$ and x and y are distinct points in X. Pick another point z and then there is a continuous map $f: X \to P_3$ such that $f^{-1}[\{a, b\}]$ is an open set containing z and exactly one of x and y. Thus $X \in T_0$.

Also, $z \in \{x\}' \cap \{y\}'$ for some x, y and z in X implies there cannot be a continuous map $f: X \to P_3$ with $f(z)=a$, and $f[\{x, y\}]=\{b, c\}$, so $\{x\}' \cap \{y\}' = \varnothing$ for all distinct x and y in X. (ii)$\Rightarrow$(iii) Assuming condition (ii), if $z \in \mathrm{cl}(x) \cap \mathrm{cl}(y)$ for some distinct points x and y in X, then $z=x$ or $z=y$. But $X \in T_0$ implies $\{x, y\} \nsubseteq \mathrm{cl}(x) \cap \mathrm{cl}(y)$. (iii)$\Rightarrow$(i) Assume (iii) holds for X, and that $\{z\}$ and the pair $\{x, y\}$ of distinct points are disjoint. Then there is an open set U such that $z \in U$ and one of x or y belongs to U. Define $f: X \to P_3$ by $f(z)=a$, $f[U \backslash \{z\}]=b$, and $f[X \backslash U]=c$. Then f is continuous so $X \in T(\mathscr{S}, \mathscr{D}, P_3)$.

LEMMA 2.2. *Suppose that X is a topological space with at least three points. Then $X \in T(\mathscr{S}, \mathscr{D}, \{P_0, P_0'\})$ iff for each pair x, y of distinct points in X, $x \in \{y\}'$ implies $\{x\}'=\varnothing$.*

Proof. Suppose that for all distinct points $x, y \in X$, $x \in \{y\}'$ implies $\{x\}'=\varnothing$. Given a disjoint singleton $\{x\}$ and doubleton $\{y, z\}$, if $x \notin \mathrm{cl}(y)$ and $x \notin \mathrm{cl}(z)$ then there is a continuous map $f: X \to P_0$ such that $f(x)=a$ and $f[\{y, z\}]=\{b\}$. Otherwise, suppose $x \in \mathrm{cl}(y)$. Then $z \notin \mathrm{cl}(x)$ and $y \notin \mathrm{cl}(x)$ so there is a continuous map $f: X \to P_0'$ such that $f(x)=a$ and $f[\{y, z\}]=\{b\}$. Thus $X \in (\mathscr{S}, \mathscr{D}, \{P_0, P_0'\})$.

If $X \in T(\mathscr{S}, \mathscr{D}, \{P_0, P_0'\})$ then since X has at least three points, X is T_0. Suppose $x \in \{y\}'$ and $z \in \{x\}'$. Clearly $z \neq y$. But if $f: X \to P_0$ with $f(x)=a$ and $f[\{y, z\}]=b$ is continuous then $x \notin \{y\}'$, and if $f: X \to P_0'$ with $f(x)=a$ and $f[\{y, z\}]=b$ is continuous then $z \notin \{x\}'$.

LEMMA 2.3. *A space X belongs to $T(\mathscr{D}, \mathscr{D}, P_4)$ iff for each pair $\{x, y\}$ of distinct points in X, $\{x\}' \cap \{y\}'$ contains at most one point.*

Proof. Note that every space with less than four points trivially satisfies the lemma. Suppose $\{x\}' \cap \{y\}'$ contains at most one point, for each distinct pair $x, y \in X$. Let $\{x, y\}$ and $\{u, v\}$ be disjoint pairs of points in X. If there is an open set U containing exactly one point from $\{x, y\}$, and exactly one point from $\{u, v\}$, then there is a continuous map $f: X \to P_4$ with $f[U]=\{a, c\}$ and $f[X \backslash U]=\{b, d\}$, and $X \in T(\mathscr{D}, \mathscr{D}, P_4)$. Since our arguments are symmetric in x and y, in u and v, and in $\{x, y\}$ and $\{u, v\}$, it is sufficient to consider three cases. In each case the statement that a pair of points belongs to U shall mean that there is an open set U containing that pair and disjoint from the other pair. (A) Suppose $\mathrm{cl}(x) \cap \mathrm{cl}(y) \neq \varnothing$ and $\mathrm{cl}(u) \cap \mathrm{cl}(v) \neq \varnothing$. For example, suppose $y \in \mathrm{cl}(x)$ and $u \in \mathrm{cl}(v)$. Then $x \notin \mathrm{cl}(y)$, $v \notin \mathrm{cl}(y)$, $x \notin \mathrm{cl}(u)$ and $v \notin \mathrm{cl}(u)$ so $x, v \in U$. (B) Suppose $\mathrm{cl}(x) \cap \mathrm{cl}(y) \neq \varnothing$ but $\mathrm{cl}(u) \cap \mathrm{cl}(v)=\varnothing$. Then, for instance, $y \in \mathrm{cl}(x)$ and thus $x \notin \mathrm{cl}(y)$, $x \notin \mathrm{cl}(u)$, $v \notin \mathrm{cl}(y)$ and $v \notin \mathrm{cl}(u)$ so $x, v \in U$. (C) Suppose $\mathrm{cl}(x) \cap \mathrm{cl}(y)=\varnothing$ and $\mathrm{cl}(u) \cap \mathrm{cl}(v)=\varnothing$. If $x \notin \mathrm{cl}(u)$ and $v \notin \mathrm{cl}(y)$ then $x, v \in U$. Assume $x \in \mathrm{cl}(u)$. We will take two subcases. If (a) $y \in \mathrm{cl}(u)$ then not both $x \in \mathrm{cl}(v)$ and $y \in \mathrm{cl}(v)$ or else $\{x, y\} \subseteq \mathrm{cl}(u) \cap \mathrm{cl}(v)$. Therefore, either $x, u \in U$ or $y, u \in U$. If (b) $y \in \mathrm{cl}(u)$ then since $x \in \mathrm{cl}(u)$, $v \notin \mathrm{cl}(x)$ and $y, v \in U$. Finally, if $v \in \mathrm{cl}(y)$ then a pair of subcases similar to (a) and (b) produces the desired result. Thus $X \in T(\mathscr{D}, \mathscr{D}, P_4)$.

On the other hand, if x, y, u, and v are distinct elements of a space X and

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$\{u, v\} \subseteq \text{cl}(x) \cap \text{cl}(y)$ then obviously there can be no continuous map $f: X \to P_4$ with $f[\{x, y\}] = \{a, b\}$ and $f[\{u, v\}] = \{c, d\}$.

Using the characterizations in Lemmas 2.1, 2.2, and 2.3, we readily obtain the following theorem.

THEOREM 2.4. *The separation axioms listed below can be represented as* $T(\mathscr{A}, \mathscr{B}, \mathscr{X})$, *where the classes* $\mathscr{A}$, $\mathscr{B}$, *and* $\mathscr{X}$ *are given in the table.*

Axiom	$\mathscr{A}$	$\mathscr{B}$	$\mathscr{X}$
T_0	$\mathscr{S}$	$\mathscr{S}$	P_0, P_0'
T_1	$\mathscr{S}$	$\mathscr{S}$	P_0
T_2	$\mathscr{S}$	$\mathscr{S}$	P_1
regular	$\mathscr{S}$	$\mathscr{C}$	P_1
completely regular	$\mathscr{S}$	$\mathscr{C}$	U
normal	$\mathscr{C}$	$\mathscr{C}$	P_1
totally disconnected	$\mathscr{S}$	$\mathscr{S}$	P_2
T_F	$\mathscr{S}$	$\mathscr{F}$	P_0, P_0'
T_{FF}	$\mathscr{F}$	$\mathscr{F}$	P_0, P_0'
T_D	$\mathscr{S}$	$\mathscr{D}\mathscr{S}$	P_0
T_{UD}	$\mathscr{S}$	$\mathscr{D}\mathscr{S}$	$\{P^{(\alpha)}: \alpha$ is an initial ordinal$\}$

Also, $T_{DD} = T(\mathscr{S}, \mathscr{D}, P_3) \cap T_D$, $T_{YS} = T(\mathscr{S}, \mathscr{D}, P_3) \cap T_0$, *and* $T_Y = T(\mathscr{S}, \mathscr{D}, \{P_0, P_0'\}) \cap T(\mathscr{D}, \mathscr{D}, P_4) \cap T_0$.

It is obvious that the characterizations of the separation axioms offered in Theorem 2.4 are not unique. For example, let P_5 be the set $\{a, b, c, d\}$ of distinct points with a base for the topology consisting of $\{\{a, c\}, \{b, c\}, \{a, b, c, d\}\}$, and $A = \{a\}$, $B = \{b\}$. Then $T(\mathscr{S}, \mathscr{S}, P_5) = T(\mathscr{S}, \mathscr{S}, P_0) = T_1$. Indeed, if X is T_1, then for each pair $x, y \in X$ with $x \neq y$, there are open sets U and V with $x \in U$, $y \notin U$, $y \in V$, and $x \notin V$. Define $f: X \to P_5$ by $f[U \setminus V] = \{a\}$, $f[V \setminus U] = \{b\}$, $f[U \cap V] = \{c\}$, and $f[X \setminus (U \cup V)] = \{d\}$, and then f will be continuous. This example also shows that $T(\mathscr{A}, \mathscr{B}, X) \subseteq (\mathscr{A}, \mathscr{B}, Y)$ does not imply that there is a continuous function $f: X \to Y$ with $f[A_X] = A_Y$ and $f[B_X] = B_Y$, because there is no such function from P_0 to P_5.

3. **The properties $T^{(m)}$, strong T_D, and strong T_0.** These properties are defined by Robinson and Wu in [2]. A space X is $T^{(m)}$, m an infinite cardinal, if for each $x \in X$, $\{x\}'$ is a union of at most m closed sets. A space X is strong T_D if for each $x \in X$, $\{x\}'$ is either empty, or the union of a finite family of non-empty closed sets whose common intersection is empty. And, X is strong T_0 if for each $x \in X$, $\{x\}'$ is either empty, or the union of a family of non-empty closed sets, such that the intersection of this family is empty, and at least one of its elements is compact.

Let m be an infinite cardinal, and λ the initial ordinal of cardinality m. Define R_α, $\alpha<\lambda$, to be the two point space $\{a, b\}$ with base $\{\{a\}, \{a, b\}\}$, and R^m the product $\prod \{R_\alpha : \alpha<\lambda\}$, with the product topology. Let π_α be the projection of R^m to R_α, and let x_1 be the point in R^m for which $\pi_\alpha(x_1)=a$, $\alpha<\lambda$. Define $\mathscr{R}^m$ to be the set of all subspaces of R^m which contain the point x_1, and for each $X \in \mathscr{R}^m$, set $A_X=\{x_1\}$ and $B_X=X\backslash\{x_1\}$.

THEOREM 3.1. *The separation axiom* $T^{(m)}$ *can be represented as* $T(\mathscr{S}, \mathscr{D}\mathscr{S}, \mathscr{R}^m)$, *for each infinite cardinal m.*

Proof. Easily $T^{(m)} \subseteq T(\mathscr{S}, \mathscr{D}\mathscr{S}, \mathscr{R}^m)$ because B_X is a union of at most m closed sets of the form $X \cap \pi_\alpha^{-1}(b)$, $\alpha<\lambda$, $X \in \mathscr{R}^m$.

Conversely, if $X \in T^{(m)}$ and $x \in X$, then if $\{x\}' \neq \phi$, $\{x\}'$ is the union of closed sets $\{M_\alpha : \alpha<\lambda\}$ (not necessarily distinct). Define $f: X \to R^m$ by putting $\pi_\alpha(f(y))$ to be b if $y \in M_\alpha$, $\alpha<\lambda$, and a otherwise. Since $f^{-1}[\pi_\alpha^{-1}(b)]=M_\alpha$ for each $\alpha<\lambda$, f is continuous and X must belong to $T(\mathscr{S}, \mathscr{D}\mathscr{S}, \mathscr{R}^m)$'

The next theorem will prove useful in characterizing the property strong T_D, and will be used in Section 4.

Given a partially ordered set P, and a point $x \in P$, let $(x]=\{y \in P : y \leq x\}$. We shall say that P has the *PO topology* if $\{P\backslash(x] : x \in P\}$ generates the topology. Let $\mathscr{P}_0$ be the class of all partially ordered sets with at least two elements and with a greatest element, each endowed with the *PO* topology. That is, if $P \in \mathscr{P}_0$, then there is an element $u \in P$ such that $x \leq u$ for all $x \in P$, and $P\backslash\{u\}$ is not empty. Let $A_P=\{u\}$ and $B_P=P\backslash\{u\}$ be the distinguished pair of subsets for each $P \in \mathscr{P}_0$.

THEOREM 3.2. *Let* $A \in \mathscr{P}_0$, *let u be the greatest element or A, and let* $B=A\backslash\{u\}_{\mathscr{D}}$. *Then a space X belongs to* $T(\mathscr{S}, \mathscr{D}\mathscr{S}, A)$ *iff for each* $x \in X$ *either* $\{x\}'=\phi$ *or else there is a family* $\mathscr{M}$ *of non-empty closed subsets of X such that* (i) $\{x\}'=\bigcup\mathscr{M}$, (ii) *for each* $y \in \bigcup\mathscr{M}$, $\bigcap\{M \in \mathscr{M} : y \in M\} \in \mathscr{M}$, (iii) *for each* $M \in \mathscr{M}$, $M \neq \bigcup\{N \in \mathscr{M} : N \subseteq M$ *and* $N \neq M\}$, *and* (iv) *there is an order isomorphism from* $\mathscr{M}$, *partially ordered by inclusion, to B, with the partial order induced from A.*

Proof. Suppose that $X \in T(\mathscr{S}, \mathscr{D}\mathscr{S}, A)$ and that $x \in X$. If $\{x\}' \neq \varnothing$, there is a continuous function $f: X \to A$ such that $f(x)=u$ and $f[\{x\}']=B$. For each $a \in B$, let $M_a=f^{-1}[(a]] \cap \text{cl}(x)$, and let $\mathscr{M}=\{M_a : a \in B\}$. Then M_a is a non-empty closed set for each $a \in B$, and $\bigcup\mathscr{M}=\{x\}'$. Since $f[\{x\}']=B$, $f[M_a]=(a]$, for each $a \in B$, so the map $M_a \to a$ is an order isomorphism. This in turn implies that $\mathscr{M}$ satisfies (iii) because no set $(a]$, $a \in B$, is a union of sets $(b]$ with $b<a$. Finally, for $y \in \{x\}'$, if $f(y)=a$ then $\bigcap\{M_b \in \mathscr{M} : y \in M_b\}=M_a$ so $\mathscr{M}$ satisfies ii).

To prove the converse, suppose x is a point in some space X, $\{x\}' \neq \varnothing$, and $\mathscr{M}$ is a family of non-empty closed subsets of X satisfying (i)–(iv). Let $\theta: \mathscr{M} \to B$ be the order isomorphism, and define $f: X \to A$ as follows. Let $f[X\backslash\{x\}']=u$. For each $y \in \{x\}'$, let $M_y=\bigcap\{M \in \mathscr{M} : y \in M\}$, and since $M_y \in \mathscr{M}$, let $f(y)=\theta(M_y)$. For each $a \in B$ there is some set $M \in \mathscr{M}$ such that $\theta(M)=a$. Note that $y \in M$ iff $M_y \subseteq M$, and this happens iff $f(y) \leq a$ so $f^{-1}[(a]]=M$ and f is continuous. But also if $\theta(M)=a$, $M=\bigcup\{M_y : y \in M\}$ and by (iii), $M=M_y$ for some $y \in M$ so $f[\{x\}']=B$.

Let $\mathscr{Q}$ be the set of $X \in \mathscr{P}_0$ such that X is finite and X has no least element a, $a \leq x$ for all $x \in X$.

COROLLARY 3.3. *A space X is strong* T_D *iff* $X \in T(\mathscr{S}, \mathscr{D}\mathscr{S}, \mathscr{Q})$.

Proof. Suppose X is strong T_D and $x \in X$ with $\{x\}' \neq \varnothing$. Then $\{x\}'$ is a union of a finite family $\mathscr{M}_1$ of non-empty closed subsets of X. Let $\mathscr{M}_2$ be $\mathscr{M}_1$ together with all non-empty intersections of sets in $\mathscr{M}_1$. Define $M_1, \ldots, M_m$ to be those elements of $\mathscr{M}_2$ which have no proper intersections with the other elements of $\mathscr{M}_2$. Note that for each $N \in \mathscr{M}_2$, $N \setminus \bigcup\{M_i : i \leq m\}$ also has no proper intersections with elements from $\mathscr{M}_2$. Enumerate $\mathscr{M}_2=\{M_1, \ldots, M_m, N_1, \ldots, N_n\}$ and define inductively M_{m+k} to be the first N_j such that $N_j \subseteq \bigcup\{M_i : i<m+k\}$. An inductive argument shows that the resulting family $\mathscr{M}$ satisfies (i), (ii), and (iii) of Theorem 3.2. Since $\mathscr{M}_1 \neq \varnothing$, the set $M_1, \ldots, M_m$ must contain at least two elements. Adjoin to $\mathscr{M}$, ordered by inclusion, a largest element u, and the resulting partially ordered set with the PO topology must belong to $\mathscr{Q}$.

The converse is immediate from Theorem 3.2, because if $\mathscr{M}$ is a family of closed sets with no least element, (ii) implies its intersection must be empty.

EXAMPLE 3.4. The property strong T_0 cannot be represented as $T(\mathscr{S}, \mathscr{D}\mathscr{S}, \mathscr{X})$ for any family $\mathscr{X}$ of topological spaces with distinguished pairs of subsets.

Proof. Let A_0 be the set $\{u, a, b\}$ of distinct elements with a partial order induced by: $a<u$ and $b<u$. Let A_0 have the PO topology and clearly A_0 is strong T_0. Define A_1 to be the set of all sequences $\{x(n) : n<\omega\}$ of elements from A_0 such that for each $x \in A$, there is some $n<\omega$ such that $x(m)=u$ iff $m \geq n$. Partially order A_1 by giving it the lexicographical order, and let A_1 have the PO topology. The sets $(x]$, $x \in A_1$, form a base for the closed sets, and $\mathrm{cl}(x)=(x]$ for each $x \in A_1$. None of the sets $(x]$ are compact, so clearly A_1 is not strong T_0. Given any point $x \in A_1$, with $x(m)=u$ iff $m \geq n$, define $f: A_1 \to A_0$ by setting $f(y)$ to be u if $y \not< x$; otherwise, set $f(y)$ to be a if $y(n)=a$, and $f(y)$ to be b if $y(n)=b$. Let $x_a, x_b, \in A_1$ be defined by $x_a(m)=x(m)$ for $m \neq n$, $x_a(n)=a$, and similarly $x_b(m)=x(m)$ for $m \neq n$, and $x_b(n)=b$. Then $f^{-1}(a)=(x_a]$ and $f^{-1}(b)=(x_b]$, so f is continuous and $f[\{x\}']=\{a, b\}=\{u\}'$, $f(x)=u$. Thus $A_0 \in T(\mathscr{S}, \mathscr{D}\mathscr{S}, \mathscr{X})$ implies $A_1 \in T(\mathscr{S}, \mathscr{D}\mathscr{S}, \mathscr{X})$.

4. A class of separation axioms between T_0 and T_1.

It is clear from Theorem 3.2 that if $A \in \mathscr{P}_0$ then $T(\mathscr{S}, \mathscr{D}\mathscr{S}, A)$ is contained in T_0, because a space is T_0 if and only if the derived set of every point is a union of closed sets. It is not clear from the definition, however, that there are any non-trivial spaces in $T(\mathscr{S}, \mathscr{D}\mathscr{S}, A)$; that is, any spaces that are not in T_1. In this section we shall construct examples of such spaces.

Given a space $A \in \mathscr{P}_0$ with largest element u, let $U(A)$ be the set of all sequences

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$\{x(n):n<\omega\}$ for which there exists $n<\omega$ with $x(m)=u$ iff $m\geq n$. Given x and y in $U(A)$, set $x\leq y$ if there is some $n<\omega$ such that $x(m)=y(m)$ for all $m<n$, $x(n)\leq y(n)$, and $y(n+1)=u$. This defines a partial order on $U(A)$, so let $U(A)$ have the *PO* topology.

THEOREM 4.1. *For each $A\in\mathscr{P}_0$, $U(A)$ belongs to $T(\mathscr{S},\mathscr{D}\mathscr{S},A)$ but not to T_1.*

Proof. Let $x\in U(A)$ and suppose that n is the least integer for which $x(n)=u$. Define $f:U(A)\to A$ as follows. If $y<x$ and $y(n)=u$ let $f(y)=y(n-1)$; if $y<x$ and $y(n)\neq u$ let $f(y)=y(n)$; and if $y\not< x$ let $f(y)=u$. Since $\{x\}'=\{y\in U(A):y<x\}$, it is clear that $f(x)=u$ and $f[\{x\}']=A\backslash\{u\}$. But for each $a\in A\backslash\{u\}, f^{-1}[(a)]=(y]\cup(z]$ where $y(m)=x(m)$ when $m<n-1$, $y(n-1)=a$, and $y(m)=u$ when $n\leq m$; and where $z(m)=x(m)$ when $m<n$, $z(n)=a$, and $z(m)=u$ when $n<m$. Thus f is continuous and $U(A)\in T(\mathscr{S},\mathscr{D}\mathscr{S},A)$. Also, since $A\backslash\{u\}$, is not empty, $U(A)\backslash\{w\}$ is not empty, where $w(n)=u$ for all $n<\omega$, and the only closed set containing w is $U(A)$, so $U(A)$ is not T_1.

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