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THE EXISTENCE OF CONTINUABLE SOLUTIONS OF A SECOND ORDER DIFFERENTIAL EQUATION

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1. Introduction. A much-studied equation in recent years has been the second order nonlinear ordinary differential equation

$$(1)\qquad y''(t) + q(t)f(y(t)) = 0, \quad t \geqq t_0,$$

where q and f are continuous on the real line and, in addition, f is monotone increasing with $yf(y) > 0$ for $y \neq 0$. Although the original interest in (1) lay largely with the case that $q(t) \geqq 0$ for all large values of t, a number of papers have recently appeared in which this sign restriction is removed. It is then that questions of continuability of solutions become a serious matter, for Burton and Grimmer have shown [**2**] that if $q(t)$ is allowed to take on negative values and provided that f satisfies a certain growth condition, there will always exist solutions of (1) which have a bounded maximal interval of existence. It is of some interest, therefore, to obtain some conditions on a sign-varying function $q(t)$ that will guarantee that there exists at least one non-trivial solution of (1) which is continuable to $[t_0, \infty)$, particularly in the context of oscillation criteria for (1) which are invariably prefaced by some assumption of continuability of solutions. Apparently, the only result of a general nature appearing in the literature which bears on this problem, is a rather complicated condition due to Kiguradze [**6**] for the function $f(y) = y^{2n+1}$, n a natural number.

Throughout this paper, we shall use the term "continuable" for any non-trivial solution of (1) which is continuable to $[t_0, \infty)$.

When $q(t) > 0$ for all t, there is the useful result [**3**] that if q is of locally bounded variation, then all solutions of (1) are continuable in the case $f(y) = y^{2n+1}$. This result is easily extended to any function f for which $\int_0^{\pm\infty} f(u)du = \infty$. In [**3**], it was also shown that if the preceding condition on q is violated at even one point, then there may exist non-continuable solutions.

If we only have $q(t) \geqq 0$ for all t, then it is not at all clear whether or not the above condition on q is still sufficient even for there to exist one continuable solution; however it is still sufficient for the continuability of all solutions if, in addition, $q(t)$ is piecewise monotone in the neighbourhood of each of its zeros.

2. Statement of results. The purpose of this paper is to prove the following.

THEOREM. *Let f be locally lipschitz with $yf(y) > 0$ for $y \neq 0$ and such that*

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$\lim_{|y|\to\infty} f(y)/y = \infty$. *Let q be continuous with isolated zeros, such that if I is any interval on which $q(t) > 0$ and ϕ is any solution of* (1) *with initial point in I, then ϕ is continuable to the closure of I.*

Then (1) *has infinitely many continuable solutions, and furthermore, if q oscillates* (*changes sign on every interval* (t_1, ∞) *with* $t_1 \geqq t_0$), *then* (1) *has infinitely many* (*continuable*) *oscillatory solutions.*

COROLLARY. *Let f be locally lipschitz with $yf(y) > 0$ for $y \neq 0$ and* $\lim_{|y|\to\infty} f(y)/y = \infty$. *Let q be continuous with isolated zeros, and piecewise monotone on each bounded interval of R^1. Then* (1) *has infinitely many continuable solutions. Moreover, if q oscillates, then* (1) *has infinitely many oscillatory solutions.*

Before proving the theorem in § 3, we introduce some notation. Throughout, we shall assume that f and q satisfy the conditions required in the hypotheses of Theorem 1. If $(a, \gamma) \in \mathbf{R}^1 \times \mathbf{R}^2$, we shall denote by $y(a, \gamma; t)$ the solution of (1) with initial conditions $y(a) = \gamma_1$, $y'(a) = \gamma_2$, $(\gamma = (\gamma_1, \gamma_2))$. The ordered pair $(y(a, \gamma; t), y'(a, \gamma; t))$ will be denoted by $Z(a, \gamma; t)$. Where we may do so without fear of ambiguity, we shall use shorter forms to avoid onerous notation. Modulus signs will always denote the appropriate Euclidean norm. For a subset S of $[t_0, \infty)$, we shall use $\bar{S}$ and ∂S for its closure and boundary, respectively. We shall use $\tilde{f}(y)$ to denote the function $\max_{|v|\leqq y} |f(v)|$. Since we shall frequently be making inferences from the continuous dependence of solutions on initial conditions, we shall use the abbreviation (*CD*) (see, for example [**4**, page 94]).

3. Proof of the theorem. We shall require two lemmas.

LEMMA 1. *Let $I = (a, b)$ and $U = (\alpha, \beta)$ be open intervals and suppose that $q(t) > 0$ for $t \in I$ and that $\Gamma = \{p \in \mathbf{R}^2 : p = \gamma(s) = (\gamma_1(s), \gamma_2(s)), s \in U\}$ is a continuous arc in $\mathbf{R}^2$ which has the properties:*

(i) $\lim_{s\to\partial U} |\gamma(s)| = \infty$,

(ii) *there are neighbourhoods of α and of β in which one of $\gamma_1(s)$, $\gamma_2(s)$ is non-vanishing.*

Define $N(s)$ and $\gamma^*(s)$ by

$N(s)$ = *number of zeros of $y(a, \gamma(s); t)$ in I, and*

$\gamma^*(s) = (y(a, \gamma(s); b), y'(a, \gamma(s); b)) = (\gamma_1^*(s), \gamma_2^*(s))$.

Then if $\arg \gamma^*(s)$ *is any continuous argument function on*

$$\Gamma^* = \{p \in \mathbf{R}^2 : p = \gamma^*(s), s \in U\},$$

we have

(a) $\lim_{s\to\partial U} N(s) = \infty$,

(b) $\lim_{s\to\partial U} |\gamma^*(s)| = \infty$, *and*

(c) $\lim_{s\to\partial U} |\arg \gamma^*(s)| = \infty$.

LEMMA 2. *Let $q(t) < 0$ for $t \in I = (a, b)$. Then for each $s \in \mathbf{R}^1$, we may define an open interval $V(s)$ such that*

(a) *$y(a, (s, m); t)$ is continuable to $\bar{I}$ for all $m \in V(s)$.*
(b) *$V(s)$ is a maximal interval with this property.*
(c) *$V(s)$ satisfies the following continuity property: for each real s_0, there exists an open $V^*(s_0)$ such that $V(s) \supset V^*(s_0)$ for s sufficiently close to s_0, and $V(s)$, $V^*(s)$ are non-empty.*

Proof of Lemma 1. Let $C = \max_{t \in \bar{I}} q(t)$. Choose any positive number δ with $\delta < \min(2, \frac{1}{2}(b-a))$. Define

$$c(\delta) = \min_{t \in [a+\frac{1}{2}\delta, b-\frac{1}{2}\delta]} q(t),$$

and define

(2) $\quad \bar{m}(\delta, B) = \delta C\tilde{f}(B) + 2B/\delta.$

Now choose a positive number $C_1 = C_1(\delta)$ large enough that

(3_1) $\quad a + 2\delta + 4/C_1 < b.$

Because of the hypothesis $\lim_{|y| \to \infty} f(y)/y = \infty$, we may find $B_1 > 0$ such that

(4_1) $\quad |f(y)| > C_1^2 \pi^2 |y|/c(\delta), \quad |y| \geqq \frac{1}{2}B_1.$

We may clearly assume that $B_1 \geqq C_1$. Denote $\bar{m}(\delta, B_1)$ by $\bar{m}_1$. Whenever $d \in [a, b]$, $p \in \mathbf{R}^2$, the solution $Z(d, p; t)$ is continuable to all of $[a, b]$. From (CD), it follows therefore, that given any $R > 0$, there exists $M(\delta, R) > 0$ such that for any $d \in [a, b - \frac{1}{2}\delta]$, $|Z(d + \frac{1}{2}\delta, p; d)| \leqq M(\delta, R)$, whenever $|p| \leqq R$. In the remainder of the proof of part (a) of this lemma, we shall suppress the initial conditions from the notation and shall use $y(t)$, $Z(t)$ throughout as abbreviations for $y(a, \gamma; t)$, $Z(a, \gamma; t)$, respectively.

We assert that if $|\gamma| > M(\delta, \sqrt{2}\,\bar{m}_1)$, then

(*) $\quad$ there exists $t_1' \in [a + \delta/2, a + 3\delta/2]$ such that $|y(t_1')| \geqq B_1$.

Suppose this to be false, so that

(5) $\quad |y(t)| < B_1, \quad t \in [a + \delta/2, a + 3\delta/2].$

From the definition of M, we see that $|Z(a + \delta/2) > \sqrt{2}\,\bar{m}_1$, whenever $|\gamma| > M(\delta, \sqrt{2}\,\bar{m}_1)$.

Let $Z(a + \delta/2) = (\zeta_1, \zeta_2)$ and consider first the case that $|\zeta_2| \geqq \bar{m}_1$. If we have $\zeta_1 \geqq 0$, $\zeta_2 \geqq \bar{m}_1$, we integrate (1) and use (5) to obtain the estimate $y'(t) \geqq \zeta_2 - \delta C\tilde{f}(B_1)$, $a + \delta/2 \leqq t \leqq a + 3\delta/2$. Therefore, integrating again and using (2),

$$y(a + \delta) \geqq \zeta_1 + (\zeta_2 - \delta C\tilde{f}(B_1))\delta/2 \geqq \zeta_1 + (\bar{m}_1 - \delta C\tilde{f}(B_1))\delta/2 \geqq B_1,$$

contradicting (5). If $\zeta_1 \geqq 0$, $\zeta_2 \leqq -\bar{m}_1$, we use the fact that on subintervals of $[a, b]$ on which $y(t)$ is non-negative, it is a concave function, together with (5), to see that $y(t)$ must vanish for some value of $t = t_1$ with $a + \delta/2 < t_1 < a + \delta/2 + B_1/\bar{m}_1 < a + \delta$, using (2) to obtain the last of these inequalities.

Furthermore, we have $y'(t_1) \leqq y'(a + \delta/2) \leqq -\bar{m}_1$, and integrating (1) we obtain $y'(t) \leqq -\bar{m}_1 + \delta C\tilde{f}(B_1)$, $t_1 \leqq t \leqq a + 3\delta/2$, and $y(t_1 + \delta/2) \leqq (-\bar{m}_1 + \delta C\tilde{f}(B_1))\delta/2 \leqq -B_1$. Since $a + \delta/2 \leqq t_1 + \delta/2 \leqq a + 3\delta/2$, we again have a contradiction of (5).

The cases in which $\zeta_1 < 0$, $|\zeta_2| \geqq \bar{m}_1$ may be similarly dealt with, and if $|\zeta_2| < \bar{m}_1$, we must have $|y(a + \delta/2)| = |\zeta_1| \geqq \bar{m}_1 > 2B_1/\delta > B_1$, contradicting the assumption that (*) is false. Thus we have verified (*). We shall now, without any loss of generality, assume that $y(t_1') \geqq B_1$.

By virtue of (4_1), it follows by way of the Sturm comparison theorem that there exists $t_1'' \in (t_1', t_1' + 2/C_1)$ such that

$$y(t_1'') < \tfrac{1}{2}B_1, \quad y(t) > 0, \quad t_1' < t < t_1''.$$

Using the mean value theorem and the concave behaviour of $y(t)$, we infer that $y'(t_1'') < -\frac{1}{4}B_1C_1 \leqq -\frac{1}{4}C_1^2$ and that there exists $t_1''' \in (t_1'', t_1'' + 2/C_1)$ for which $y(t) > 0$, $t_1'' < t < t_1'''$, $y(t_1''') = 0$ and $y'(t_1''') < -\frac{1}{4}C_1^2$. We also note that $a + \delta/2 < t_1''' < a + 3\delta/2 + 4/C_1 < b - \delta/2$, by (3_1). Arguing similarly, we see that if δ is any positive number with $\delta < \min(2, 2(b - a)/7)$ and if we choose positive numbers $C_1 \geqq C_2$ so large that

$$(3_2) \quad C_2^2/4 > M(\delta, \sqrt{2}\,\bar{m}(B_1)), \quad a + 7\delta/2 + 4(C_1^{-1} + C_2^{-1}) < b$$

where $B_1 \geqq C_1$, $B_2 \geqq C_2$ are such that

$$(4_2) \quad f(y) \geqq C_i^2\pi^2|y|/c(\delta), \quad |y| \geqq B_i/2, \quad i = 1, 2$$

then whenever $|\gamma| \geqq M(\delta, \sqrt{2}\,\bar{m}(B_2))$, $y(a, \gamma; t)$ will have zeros at t_1, t_2, where $a + \delta/2 < t_1 < t_2 < a + 3\delta + 4(C_1^{-1} + C_2^{-1}) < b - \delta/2$, and $|y'(a, \gamma; t_2)| > C_1^2/4$.

Now we proceed inductively. Given a natural number N, choose any positive number δ with

$$\delta < \min\left(2, \frac{2(b - a)}{3N + 1}\right).$$

Then pick an increasing sequence $C_n = C_n(\delta)$, $n = 1, \ldots, N$, such that

$$(3_N) \quad \begin{cases} \frac{1}{4}C_{n+1}^2 > M(\delta, \sqrt{2}\,\bar{m}(B_n)), & n = 1, \ldots, N - 1 \\ \left(\dfrac{3N + 1}{2}\right)\delta + 4\displaystyle\sum_{n=1}^{N} C_n^{-1} < b - a \end{cases}$$

where $B_n \geqq C_n$, $n = 1, \ldots, N$, such that

$$(4_N) \quad |f(y)| \geqq C_n^2\pi^2|y|/c(\delta), \quad |y| \geqq B_n/2, \quad n = 1, \ldots, N.$$

Whenever $|\gamma| \geqq M(\delta, \sqrt{2}\,\bar{m}(B_N))$, the solution $y(a, \gamma; t)$ of (1) has at least N zeros in (a, b). Part (a) of the lemma now follows on account of the hypothesis (i).

Part (b) follows from the fact that $|Z(b, p; a)| \leqq M(2(b - a), R)$, whenever $|p| \leqq R$, and on noting that if $\gamma^* = Z(a, \gamma; b)$, then $\gamma = Z(b, \gamma^*; a)$.

As a preliminary to proving part (c) of the lemma, we make a few observations concerning solutions of (1). Since the zero solution is unique, non-trivial solutions of (1) have only simple zeros; consequently, two solutions with the same number n of zeros on $[a, b]$ and with (strictly) the same sign at a, will have the same sign on the open subintervals between their jth and $(j + 1)$st zeros on $[a, b]$ and on the open sub-interval between their last zero on $[a, b]$ and b (this last statement being vacuous should either solution vanish at b). Further, it is easily verified that if a sequence y_i of such solutions converges uniformly to the solution y on $[a, b]$, with the additional condition that the values $y_i(a)$ be uniformly bounded away from zero, then y has either n or $n + 1$ zeros on $[a, b]$. Similar observations may be made concerning the zeros of derivatives of solutions of (1), since any such zeros in (a, b) will be simple.

Proceeding with the proof of (c), suppose for definiteness, that there is an interval (β', β) in which γ_1 is non-vanishing, and let s, s' belong to this interval such that $\gamma_1^*(s) = \gamma_1^*(s') = 0$, $N(s') = N(s) + 1$, while $N(r) = N(s)$ for any r between s and s' for which $\gamma_1^*(r) = 0$. We claim that $\arg \gamma^*(s') = \arg \gamma^*(s) - \pi$. Again, for definiteness, we shall assume that $s < s'$ and $\gamma_2^*(s) > 0$. Then $\arg \gamma^*(s) = 2k\pi + \pi/2$ for some integer k, and $y(a, \gamma(s); t) < 0$ in some left neighbourhood of b. For $s < r < s'$, we either have $N(r) = N(s)$, $\gamma_1^*(r) \geqq 0$ (by the preceding preliminary observations) and $2k\pi + \pi/2 \geqq \arg \gamma^*(r) > 2k\pi - \pi/2$, or we have $N(r) = N(s) - 1$. Since $\lim_{r \to s'} y(a, \gamma(r); t) = y(a, \gamma(s'); t)$ uniformly on $[a, b]$, and since $N(s') = N(s) + 1$, we must have $N(r) = N(s)$ for $r \in (s, s')$ sufficiently close to s'. But $\arg \gamma^*(s') = 2k'\pi - \pi/2$ for some integer k', and continuity then implies that $k = k'$, and our claim is verified. Next, suppose that $\beta' < s < s' < \beta$, $\gamma_1^*(s) = \gamma_1^*(s') = 0$, with $N(s') = N(s) + m$. Then $\arg \gamma^*(s') = \arg \gamma^*(s) - m\pi$. For appealing to the preliminary remarks, we may find a sequence $s = s_0 < s_1 < \ldots < s_k = s'$ such that $\gamma_1^*(s_{i-1}) = \gamma_1^*(s_i) = 0$, $|N(s_i) - N(s_{i-1})| = 1$, and $N(r) = N(s_i')$ for all $r \in (s_{i-1}, s_i)$ for which $\gamma_1^*(r) = 0$, where

$$s_i' = \begin{cases} s_{i-1}, & \text{if } N(s_i) = N(s_{i-1}) + 1 \\ s_i, & \text{if } N(s_{i-1}) = N(s_i) + 1. \end{cases}$$

Applying the preceding argument m times, we obtain the result $\arg \gamma^*(s') = \arg \gamma^*(s) - m\pi$. Using this, together with part (a) of the lemma, yields the result that $\lim_{s \to \beta} \arg \gamma^*(s) = -\infty$, in the event that γ_1 is non-vanishing in some neighbourhood of β. If, instead, γ_2 is non-vanishing near β, we may argue analogously in terms of the derivatives of solutions of (1).

This completes the proof of (c) and hence the lemma.

Corollary. *Let s, s' be in a neighbourhood of β in which either γ_1 or γ_2 is non-vanishing and let $N(s') = N(s) + m$. Then $\arg \gamma^*(s') \leqq \arg \gamma^*(s) - (m - 1)\pi$. A corresponding result holds in any appropriate neighbourhood of α.*

Proof. We consider only the case that γ_1 is non-vanishing near β and $s < s'$; the other cases are similar. Let $n = N(s)$. We can find s_1, s_2 with $s \leqq s_1 \leqq s_2 \leqq s'$, $\gamma_1(s_1) = \gamma_1(s_2) = 0$ and $N(s_1) = n + 1$, $N(s_2) = n + m$. Then the proof of part (c) of Lemma 1 implies that

$$\arg \gamma^*(s) \geqq \arg \gamma^*(s_1) = \arg \gamma^*(s_2) + (m - 1)\pi \geqq \arg \gamma^*(s').$$

Remarks. 1) Statements similar to Proposition (a) have appeared in [**1**] and [**5**].

2) The existence of a continuous argument function on Γ^*, at least for values of s sufficiently close to ∂U, is guaranteed by (b).

Proof of Lemma 2. First we show that for each s there exists $m^* = m^*(s)$ for which $y(a, (s, m^*); t)$ is continuable to $\bar{I}$. For it is clear that given $s > 0$, say, $y(a, (s, m); t)$ will become negative somewhere in $\bar{I}$, provided that m is sufficiently negative, whereas it will never assume negative values if m is positive. Now define m^* to be inf $\{m: y(a, (s, m); t)$ does not assume negative values in $I\}$.

If $y(a, (s, m^*); t)$ were not continuable to $\bar{I}$, this solution would have a maximal interval of existence $[a, b^*)$ with $b^* \leqq b$ and $\lim_{t \to b^*} y(a, (s, m^*); t) = \infty$. From (CD) and the definition of m^*, we find that $y(t) = y(a, (s, m^*); t) \geqq 0$ on $[a, b^*)$. Because of the uniqueness of the zero solution and the behaviour of $y(t)$ in a neighbourhood of b^*, we see that $y(t)$ must in fact be bounded away from zero in $[a, b^*)$. (CD) indicates that this is a contradiction of the definition of m^* and so $y(a, (s, m^*); t)$ will be continuable to $\bar{I}$; again using (CD), it is easy to deduce the existence of $V(s)$ containing $m^*(s)$ and satisfying (a), (b) and (c). If $s < 0$, we define m^* to be sup $\{m: y(a, (s, m); t)$ does not assume positive values in $I\}$, and argue similarly.

Remark. From (c), we see that $\bigcup_{s \in \mathbf{R}^1} \{s\} \times V(s)$ is an open subset of $\mathbf{R}^2$, and we may construct a continuous arc $\Lambda = \{p \in \mathbf{R}^2: p = (s, \lambda(s)), s \in \mathbf{R}^1\}$ with $\lambda(s) \in V(s)$.

Proof of the theorem. If $q(t) \geqq 0$ for every t, there is nothing to prove as regards existence, on account of the basic hypothesis concerning q. If $q(t) < 0$ for every t, arguments similar to those employed in the construction of m^* in the proof of Lemma 2 point to the existence of $m = m(s)$ such that $y(a, (s, m); t)$ is continuable. Therefore we concentrate our attention on the case that there exist intervals $I_j = (t_j, t_{j+1})$ with $(-1)^j q(t) < 0$ for $t \in I_j$, $j = 0, 1, \ldots$.

In [**2**], it was shown that (1) will have solutions with initial point in I_{2j}, but not continuable to $\bar{I}_{2j}$ if and only if either

$$\text{(I)} \quad \int_0^\infty (1 + F(u))^{-\frac{1}{2}} du < \infty$$

or

$$\text{(II)} \quad \int_0^{-\infty} (1 + F(u))^{-\frac{1}{2}} du > -\infty,$$

where

$$F(y) = \int_0^y f(u) du.$$

We shall assume that both (I) and (II) hold, since there is nothing to prove if neither holds, and the modifications required if just one of the conditions holds are trivial.

When (I) and (II) do both hold, it is easy to deduce via [**2**] that the sets $V(s)$ defined by Lemma 2 are bounded. We shall use $V_{2j}(s)$ to denote $V(s)$ when applying Lemma 2 with $I = I_{2j}$ and Λ_{2j} will denote the corresponding curve Λ (see remark following the proof of Lemma 2). S_j will denote $[t_0, t_{j+1}]$.

Let $M_{2j}(s) = \sup_{m \in V_{2j}(s)} |m|$. Fix $y_0 \in R^1$, and abbreviate $y(t_0, (y_0, m); t)$ to $y(m; t)$. Then $y(m; t)$ is continuable to $\bar{I}_0$ (and therefore to S_1) for $m \in V_0(y_0)$. Let $U_0 = U_{I_1}$ be $V_0(y_0)$ and define $\gamma_0(m) = \gamma_{I_1}(m)$ by

$$\gamma_0(m) = (y(m; t_1), y'(m; t_1)), \quad m \in U_0.$$

It may be verified, using the maximality of U_0, that

$$\Gamma_0 = \{p \in \mathbf{R}^2: p = \gamma_0(m), m \in U_0\}$$

satisfies the conditions of Lemma 1. (In fact, $|\gamma_0(m)| \rightarrow \infty$, arg $\gamma_0(m) \rightarrow \pi/2 \pmod \pi$ as $m \rightarrow \partial U_0$.) Therefore we may choose $m_{11} < m_{12}$ with $[m_{11}, m_{12}] \subset U_0$ and such that

(i) $|\arg \gamma_0^*(m_{12}) - \arg \gamma_0^*(m_{11})| = 2\pi$,

(ii) $|\gamma_0^*(m)| \geqq 2M_2(0), \quad m \in [m_{11}, m_{12}]$

where $\gamma_0^*(m) = \gamma_{I_1}^*(m)$, $m \in U_0$, in the notation of Lemma 1. Define Γ_0^* to be $\{p \in \mathbf{R}^2: p = \gamma_0^*(m), m \in U_0\}$. On account of (i) and (ii), it may be seen that the restriction of the curve Γ_0^* to the parameter interval $[m_{11}, m_{12}]$ and the curve Λ_2 will have a point of intersection. Using (c) of Lemma 2 as well, we shall be able to find $m_{11}' < m_{12}'$ with $[m_{11}', m_{12}'] \subset (m_{11}, m_{12})$ such that $y(m; t)$ is continuable to $\bar{I}_2$ (and thus to S_3) for $m \in (m_{11}', m_{12}')$ and (m_{11}', m_{12}') is a maximal such interval.

We now put $I_1 = U_{I_3} = (m_{11}', m_{12}')$, $\gamma_1 = \gamma_{I_3}(m) = (y(m; t_3), y'(m; t_3))$, $m \in U_1$, and $\Gamma_1 = \Gamma_{I_3} = \{p \in \mathbf{R}^2: p = \gamma_1(m), m \in U_1\}$, and verify that Γ_1 satisfies the conditions of Lemma 1.

We may proceed inductively to obtain sequences m_{n1}', m_{n2}' such that $m_{n1}' < m_{n+1,1}' < m_{n+1,2}' < m_{n2}'$ and $y(m; t)$ is continuable to S_{2n+1} for $m \in (m_{n1}', m_{n2}') = W_n$, say. We have $W = \bigcap_{n=1}^{\infty} W_n \neq \emptyset$. Now for $m \in W$, $y(m; t)$ is a continuable solution of (1).

The general case $((-1)^j q(t) \leqq 0$ on $I_j)$ needs only a minor modification of the above argument. The question of existence is thus dealt with.

Finally, given any sequence l_n of real numbers, we may use Lemma 1 to modify our construction to ensure that for $m \in W_n$, $y(m; t)$ has at least l_n zeros in S_{2n+1}, and this proves the final assertion of the theorem.

The corollary follows from the theorem and the discussion in § 1.

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