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AN INTEGRAL EQUATION FOR THE DISTRIBUTION OF THE FIRST EXIT TIME OF A REFLECTED BROWNIAN MOTION

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Abstract

Reflected Brownian motion is used in areas such as physiology, electrochemistry and nuclear magnetic resonance. We study the first-passage-time problem of this process which is relevant in applications; specifically, we find a Volterra integral equation for the distribution of the first time that a reflected Brownian motion reaches a nondecreasing barrier. Additionally, we note how a numerical procedure can be used to solve the integral equation.

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1. Introduction

Reflected Brownian motion has received considerable attention due to its capacity to model several real phenomena in physics, biology and chemistry; see Grebenkov [12], where several applications are mentioned. It is known that this model is very suitable for modelling the interaction between a particle diffusing in a medium and an interface, where the particle may suffer a "reflection" (see for example [12, 18]). First-passage-time problems are important in such applications (see for instance Levitz *et al.* [18]), which motivates us to study the distribution of the time when the process surpasses an increasing varying level/barrier, that is, the time when the particle interacting with the

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interface reaches a specific level in the medium (separated from the interface). Hence, we concentrate on finding the distribution of

$$T := \inf\{t : |B_t| = f(t)\},$$

where $B = \{B_t : t \geq 0\}$ is a standard Brownian motion (BM) and $f(t)$ is a nondecreasing function with $f(0) > 0$.

However, this issue can be seen as a particular case of the first time that the BM exits a moving range. Let $f(t)$ and $g(t)$ be two positive functions such that $f(0) > 0$ and $g(0) > 0$ (that is, the BM starts in between the barriers). The general probabilistic question is to find the distribution of

$$T^* := \inf\{t : B_t \notin (-g(t), f(t))\}. \tag{1.1}$$

There are works in the literature concerning first-exit-time problems from varying regions, with single and double barriers. Considering a single barrier, see for example Ricciardi *et al.* [24], Peskir [22], De-la-Peña and Hernández-del-Valle [10], Darling and Siegert [7].

Regarding two-sided barriers, as in (1.1), there has been also a lot of interest, mainly from a theoretical point of view. Lifshits and Shi [21], address the tail behaviour of the exit-time distribution from parabolic domains of a planar Brownian motion, see also [1, 19, 20]. Deblassie [9] studies the probability that a Brownian motion (with dimension higher than one) remains in what he calls *horn-shaped domains*. Other related works are [5, 8, 19].

Notice that reflected Brownian motion is a *one-dimensional Bessel process*; in Betensky [3] first-passage issues for Bessel processes are addressed.

Here we focus on characterizing the distribution of T.

The paper is organized as follows. In Section 2 we give preliminary results. In Section 3 we find an integral equation for the first exit time distribution of the reflected Brownian motion, and in Section 4 we state the integral equation for the density. Finally we use a numerical method to solve integral equations. We also give conclusions and mention some open problems.

2. Hitting times of a Brownian motion

To start studying exit-times distribution, we define common variables that we use. Process B will be Brownian motion (BM) throughout the paper. The following hitting times are of interest:

$$T^f := \inf\{t \geq 0 : B_t = f(t)\}, \quad T_{-g} := \inf\{t \geq 0 : B_t = -g(t)\},$$

where f and g are two functions such that $f(0) > 0$ and $g(0) > 0$.

The first fact we have to notice is that $T_{-f} \overset{d}{=} T^f$, by symmetry of the BM ($\overset{d}{=}$ stands for equality in distribution). We readily see that the exit time T, defined in (1.1), satisfies $T = T^f \wedge T_{-g}$. So, the exit time is the first time B hits any of the barriers.

The following cases are known (refer to [11, 14–16, 23]). If $f(t) = a$ for all $t \geq 0$, the density $\varphi_a(s)$ of T^f is given by

$$\varphi_a(s) = \frac{a}{\sqrt{2\pi s^3}} \exp\left\{-\frac{a^2}{2s}\right\}, \qquad s \geq 0.$$

If $f(t) = ct + a$, with $a > 0$, the density becomes

$$\varphi_f(s) = \frac{a}{\sqrt{2\pi s^3}} \exp\left\{-\frac{(cs+a)^2}{2s}\right\}, \qquad t \geq 0. \tag{2.1}$$

For the case of constant barriers $f(t) = g(t) = a$, it is known that the Laplace transform of T [23, Proposition 2.3.7] is

$$E\big(e^{-\theta T}\big) = \frac{1}{\cosh(a\sqrt{2\theta})}.$$

Thus, the density is given by the inverse function. In this paper, we shall work with a more general form.

PROPOSITION 2.1 ([4, Page 172]). *Consider a Brownian motion B starting at x. The density of T, the first time B exits $[-a, b]$ with $x \in [-a, b]$, is given by*

$$P(T \in dt) = ss_{(b-x,b+a)}(t) + ss_{(x+a,b+a)}(t), \tag{2.2}$$

where

$$ss_{(u,v)}(t) = \sum_{k=-\infty}^{\infty} \frac{v - u + 2kv}{\sqrt{2\pi t^3}} e^{-(v-u+2kv)^2/(2t)}.$$

Furthermore, the expected value of T is $E(T) = (x+a)(b-x)$ [23, Exercise 2.3.11].

We denote by $\varphi_x^{(-a,b)}(t)$ the density function $P(T \in dt)$ in (2.2).

3. Integral equation

When the functions f and g in (1.1) are nondecreasing, T is said to be the first time that B exits the "horn-shape" $\{(-g(t), f(t)) : t \geq 0\}$. In this section, we shall derive an integral equation to compute the distribution of the first time a BM leaves a region determined by reflective barriers, that is, when $g = f$. First we need some notation.

3.1. Notation Let f and g be two positive functions.

- We denote by T^f the first time a BM hits f, and T_{-g} when it hits $-g$. The notation is the same when f and g are constants, namely T^b or T_{-a}.
- We denote by $T_{-g}^f(x)$ the first time a BM starting at x hits f or $-g$, where $x \in [-g(0), f(0)]$. So $T_{-a}^b(x)$ denotes the hitting time when the barriers are constants. When there is no ambiguity we write $T_{-g}^f(0)$ as T.

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- The distribution of $T_{-g}^{f}(x)$ is denoted by $\Phi_x^{(-g,f)}(t) := P(T_{-g}^{f}(x) < t)$, or $\Phi(t)$ when $x = 0$.

Notice that if the barriers are constants, the distribution is known, and the density is given by $\varphi_x^{(-a,b)}(u)$ in (2.2). For example, if we fix t and s, then $\varphi_x^{(-g(t),f(s))}(u)$ denotes the density of the first time a BM hits either of the fixed points $-g(t)$ or $f(s)$ (the functions evaluated at specific nodes t and s).

3.2. Methodology As part of the technique, we shall take approximations of the barrier. Hence, given a function f, we consider a partition of the time domain $[0, t]$, namely $\Pi_n := \{0 = t_{0,n} < t_{1,n} < \cdots < t_{n,n} = t\}$. Then the approximating barrier is

$$f_n(t) := \sum_{i=1}^{n} f(t_{i-1,n}) \mathbb{I}_{[t_{i-1,n}, t_{i,n})}(t), \tag{3.1}$$

where $\mathbb{I}$ is the indicator function. We take the partitions such that $\Pi_m \subset \Pi_n$ for $m < n$, and $\max_{0 \le i,j \le n} |t_{i,n} - t_{j,n}| \to 0$ as $n \to \infty$.

We have the following proposition.

PROPOSITION 3.1. *For any pair of nondecreasing functions f and g on $\mathbb{R}^+$ with $f(0) > 0$ and $g(0) > 0$,*

$$P\Big(T_{-g_n}^{f_n} < t\Big) \to P\Big(T_{-g}^{f} < t\Big), \quad \forall t > 0, \text{ as } n \to \infty,$$

where f_n and g_n are the approximations as in (3.1) on the same partitions $\{\Pi_n, n = 1, 2, \ldots\}$.

PROOF. By construction $f_n(s) \le f(s)$ and $g_n(s) \le g(s)$ for all s. Thus, the sequence $P(T_{-g_n}^{f_n} < t)$, $n = 1, 2, \ldots$ is decreasing, and is bounded from below by $P(T < t)$. Since we have $\{T_{-g_n}^{f_n} < t\} \to \{T_{-g}^{f} < t\}$ as $n \to \infty$, convergence holds. □

Exploiting well known properties of the BM, in the following result we obtain an integral equation of Volterra type for the exit-time distribution. We use arguments similar to those in [10] or [13].

THEOREM 3.2. *Let f be a nondecreasing function such that $f(0) > 0$. Then, the distribution Φ of the first exit time $T_{-f}^{f}(0)$ obeys the integral equation*

$$\begin{aligned} \Phi(t) = &\int_0^t \varphi_0^{(-f(u),f(u))}(u)\, du \\ &- \int_0^t \left(\int_0^s \varphi_0^{(-f(s)-f(u),f(s)-f(u))}(s-u)\, \Phi(du) \right) ds, \end{aligned} \tag{3.2}$$

where the function $\varphi_x^{(a,b)}(u)$ is given by (2.2).

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PROOF. Consider an approximation $\{f_n, n = 1, 2, \dots\}$ for f as in (3.1). We note

$$\Phi_n(t) = P\Big(T^{f_n}_{-f_n} < t\Big) = \sum_{i=1}^{n} P\Big(t_{i-1,n} \le T^{f_n}_{-f_n} < t_{i,n}\Big). \tag{3.3}$$

Here, $\Phi_n(\cdot) := \Phi_0^{(-f_n, f_n)}(\cdot)$, and for simplicity we write T^f_{-g} instead of $T^f_{-g}(0)$. By Proposition 3.1 we have that $\Phi_n \to \Phi$ as $n \to \infty$.

The proof is divided into three parts.

PART 1. We analyse the right-hand side of identity (3.3). For notational convenience, set $A_{i,n} = \big(t_{i-1,n} \le T^{f(t_{i-1,n})}_{-f(t_{i-1,n})} < t_{i,n}\big)$. Then for each term we observe that

$$P\Big(t_{i-1,n} \le T^{f_n}_{-f_n} < t_{i,n}\Big) = P(A_{i,n}) - P\Big(A_{i,n}, T^{f_n}_{-f_n} < t_{i-1,n}\Big). \tag{3.4}$$

PART 2. The first part of the right-hand side of (3.4) can be obtained from (2.2) by integrating on the interval $[t_{i-1,n}, t_{i,n})$. Hence, it reads as

$$P(A_{i,n}) = \int_{t_{i-1,n}}^{t_{i,n}} \varphi_0^{(-f(t_{i-1,n}), f(t_{i-1,n}))}(u)\, du.$$

By the mean value theorem for integrals (see [2]) the previous equation becomes

$$P(A_{i,n}) = \varphi_0^{(-f(t_{i-1,n}), f(t_{i-1,n}))}(t^*_{i,n})(t_{i,n} - t_{i-1,n})$$

for some value $t^*_{i,n} \in [t_{i-1,n}, t_{i,n})$. Thus, in (3.3) we actually have a Riemann sum which converges to the desired quantity:

$$\lim_{n\to\infty} \sum_{i=1}^{n} \phi_0^{(-f(t_{i-1,n}), f(t_{i-1,n}))}(t^*_{i,n})(t_{i,n} - t_{i-1,n}) = \int_0^t \varphi_0^{(-f(u), f(u))}(u)\, du.$$

Hence, the second element of (3.2) is obtained.

PART 3. For the last term in (3.4), we have the analysis

$$\begin{aligned} P\Big(A_{i,n}, T^{f_n}_{-f_n} < t_{i-1,n}\Big) &= \int_0^{t_{i-1,n}} P\Big(A_{i,n} \,\Big|\, T^{f_n}_{-f_n} = ut\Big)\, \Phi_n(du) \\ &= \sum_{k=1}^{i-1} \int_{t_{k-1,n}}^{t_{k,n}} P\Big(A_{i,n} \,\Big|\, T^{f_n}_{-f_n} = u\Big)\, \Phi_n(du). \end{aligned}$$

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This in turn equals

$$\sum_{k=1}^{i-1}\int_{t_{k-1,n}}^{t_{k,n}} P\Big(A_{i,n}\,\Big|\,T^{f_n}<T_{-f_n},\,T_{-f_n}^{f_n}=u\Big)$$
$$\times P\Big(T^{f_n}<T_{-f_n}\,\Big|\,T_{-f_n}^{f_n}=u\Big)\,\Phi_n(du)$$
$$+\sum_{k=1}^{i-1}\int_{t_{k-1,n}}^{t_{k,n}} P\Big(A_{i,n}\,\Big|\,T^{f_n}\geq T_{-f_n},\,T_{-f_n}^{f_n}=u\Big)$$
$$\times P\Big(T^{f_n}\geq T_{-f_n}\,\Big|\,T_{-f_n}^{f_n}=u\Big)\,\Phi_n(du). \tag{3.5}$$

Now, we analyse the quantities involved in the sums. We know that

$$P\Big(T^{f_n}<T_{-f_n}\,\Big|\,T_{-f_n}^{f_n}=u\Big)+P\Big(T^{f_n}\geq T_{-f_n}\,\Big|\,T_{-f_n}^{f_n}=u\Big)=1.$$

By the symmetry of the BM,

$$P\Big(T^{f_n}<T_{-f_n}\,\Big|\,T_{-f_n}^{f_n}=u\Big)=P\Big(T^{f_n}\geq T_{-f_n}\,\Big|\,T_{-f_n}^{f_n}=u\Big)=\frac{1}{2}. \tag{3.6}$$

On the other hand, for each integral in the sums, let $u\in[t_{k-1,n},\,t_{k,n})$ with $k\leq i-1$. From the regenerative properties of the BM,

$$P\Big(A_{i,n}\,\Big|\,T^{f_n}\geq T_{-f_n},\,T_{-f_n}^{f_n}=u\Big)$$
$$=P\Big(t_{i-1,n}-u\leq T_{-f(t_{i-1,n})-f(t_{k-1,n})}^{f(t_{i-1,n})-f(t_{k-1,n})}<t_{i,n}-u\Big)$$
$$=\int_{t_{i-1,n}-u}^{t_{i,n}-u}\varphi_0^{(-f(t_{i-1,n}-f(t_{k-1,n}),\,f(t_{i-1,n})-f(t_{k-1,n}))}(w)\,dw. \tag{3.7}$$

For this last step, the assumption of nondecreasing barriers is important. After a change of variable and application of the mean value theorem for each $k=1,2,\ldots,i-1$ the probability (3.7) becomes

$$\varphi_0^{(-f(t_{i-1,n})-f(t_{k-1,n}),\,f(t_{i-1,n})-f(t_{k-1,n}))}(t_{k,i,n}^*-u)(t_{k,n}-t_{k-1,n}), \tag{3.8}$$

for some $t_{k,i,n}^*\in[t_{i-1,n},\,t_{i,n})$ and $k<i-1$ and $i=1,2,\ldots,n$.

Furthermore, for the last sums at (3.5), owing to the symmetry of the BM,

$$P\Big(A_{i,n}\,\Big|\,T^{f_n}<T_{-f_n},\,T_{-f_n}^{f_n}=u\Big)=P\Big(A_{i,n}\,\Big|\,T^{f_n}\geq T_{-f_n},\,T_{-f_n}^{f_n}=u\Big).$$

So, using also (3.6) and (3.8), probability (3.5) ends up as

$$2\sum_{k=1}^{i-1}\int_{t_{k-1,n}}^{t_{k,n}}\varphi_0^{(-f(t_{i-1,n})-f(t_{k-1,n}),\,f(t_{i-1,n})-f(t_{k-1,n}))}(t_{k,i,n}^*-u)(t_{k,n}-t_{k-1,n})\,\frac{1}{2}\Phi_n(du).$$

Substitution in equations (3.4) and (3.3) yields

$$\sum_{i=1}^{n}\sum_{k=1}^{i-1}\int_{t_{k-1,n}}^{t_{k,n}} \varphi_0^{(-f(t_{i-1,n})-f(t_{k-1,n}),\, f(t_{i-1,n})-f(t_{k-1,n}))} \times (t^*_{k,i,n}-u)(t_{k,n}-t_{k-1,n})\,\Phi_n(du). \tag{3.9}$$

Upon applying again the mean value theorem, there are $t^*_{i,k,n}\in[t_{k-1,n},t_{k,n})$ for each $i<n$, such that sum (3.9) equals

$$\sum_{i=1}^{n}\sum_{k=1}^{i-1} \varphi_0^{(-f(t_{i-1,n})-f(t_{k-1,n}),\, f(t_{i-1,n})-f(t_{k-1,n}))}(t^*_{k,i,n}-t^*_{i,k,n}) \times (t_{k,n}-t_{k-1,n})(\Phi_n(t_{i,n})-\Phi_n(t_{i-1,n})). \tag{3.10}$$

Equation (3.10) represents a Riemann–Stieltjes sum of a continuous function. Thus, when $n\to\infty$ we obtain the limit

$$\int_0^t \left(\int_0^s \varphi_0^{(-f(s)-f(u),\, f(s)-f(u))}(s-u)\,\Phi(du)\right) ds,$$

which finally yields the last part of (3.2). This concludes the proof. □

4. Equations for the density

In the previous section, Theorem 3.2 stated an integral equation for the distribution of T, the first exit time. There are results on first passage times requiring additional conditions in order to derive an integral equation for the density. In [10, 22, 24] the barrier needs to be differentiable. In our case, we can readily see in (3.2) that Φ is differentiable, and thus we are able to obtain an integral equation for the density.

COROLLARY 4.1. *Under the conditions of Theorem 3.2, we have that the density of T, which we denote by φ, satisfies the integral equation*

$$\varphi(t)=\varphi_0^{(-f(t),\,f(t))}(t)-\int_0^t \varphi_0^{(-f(t)-f(u),\, f(t)-f(u))}(t-u)\varphi(u)\,du, \tag{4.1}$$

where the function $\varphi_x^{(a,b)}(u)$ is given by (2.2).

REMARK 1. As mentioned above, generally one assumes differentiable barriers to ensure the existence of a density. However, since the barriers we use are nondecreasing, one also has that the distribution function is differentiable.

REMARK 2. Since the function $\varphi_x^{(-a,b)}(u)$ is continuous in all its arguments, (4.1) has unique solution. This is a classical result in the theory of integral equations (see [6, Theorem 5, Page 183]).

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4.1. Another expression for the exit time density From the proof of Theorem 3.2, we can see that it is possible to modify the integral equation slightly. Recall the notation $T^{f}_{-g}(x)$; in (3.7),

$$P\Big(A_{i,n}\;\Big|\;T^{f_n} \geq T_{-f_n},\, T^{f_n}_{-f_n} = u\Big) = P\Big(t_{i-1,n} - u \leq T^{f(t_{i-1,n})}_{-f(t_{i-1,n})}(f(t_{k-1,n})) < t_{i,n} - u\Big)$$

for $u \in (t_{k-1,n}, t_{k,n}]$. Finally this becomes

$$\int_{t_{i-1,n}-u}^{t_{i,n}-u} \varphi^{(-f(t_{i-1,n}),f(t_{i-1,n}))}_{f(t_{k-1,n})}(w)\,dw.$$

The idea is to consider a BM starting at $f(t_{k-1,n})$, rather than at 0, as was originally done in (3.7).

This change gives a new expression for the integral equation, namely

$$\varphi(t) = \varphi^{(-f(t),f(t))}_{0}(t) - \int_0^t \varphi^{(-f(t),f(t))}_{f(u)}(t-u)\varphi(u)\,du.$$

4.2. Numerical solutions We now exploit a numerical procedure to solve (4.1). A numerical algorithm based on a recursive formulae is quite straightforward to implement; we summarize it briefly (the interested reader can refer to [17]).

We want to solve the Volterra integral equation

$$F(t) = G(t) + \int_0^t K(t,s)F(s)\,ds, \tag{4.2}$$

where G and K are known functions, the latter usually being called the *kernel*. Equation (4.2) is an equation of the second kind because function G is nonzero, which is important for the method to work.

Suppose that we want to obtain an approximation of F in the interval $[0, r]$, and we divide it into N equally spaced subintervals of size h. We have $N+1$ nodes $\{t_0, t_1, \ldots, t_N\}$ such that $t_{n+1} - t_n = h$, $n = 0, 1, \ldots, N-1$, with $t_0 = 0$. We denote by $\{F_1, \ldots, F_N\}$ the approximation of f at the nodes $\{t_1, \ldots, t_N\}$. The following recursive formula can be used:

$$F_{n+1} = \frac{G(t_{n+1}) + \sum_{k=1}^{n} F_k \int_{t_{k-1}}^{t_k} K(t_{n+1}, s)\,ds}{1 - \int_{t_n}^{t_{n+1}} K(t_{n+1}, s)\,ds}, \qquad n = 0, \ldots, N-1. \tag{4.3}$$

Notice that

$$F_1 = \frac{G(h)}{1 - \int_0^h K(h,s)\,ds}.$$

EXAMPLE 3. Using Scilab 4.1.2, we compute the density function of $T^{1}_{-1}(0)$ and $T^{f}_{-f}(0)$, where $f(t) = 1 + 0.0001t$. The two densities should be close to each other. In Figure 1 a sampling from the density of $T^{1}_{-1}(0)$ (determined by (2.2)) is shown as a line and similarly, the density of $T^{f}_{-f}(0)$ (determined by (4.3)) is shown with points.

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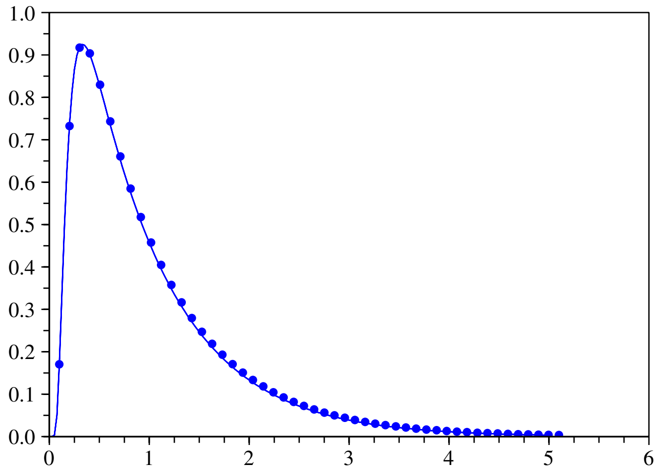

FIGURE 1. Densities for the exit times.

5. Conclusion

We studied the first-exit-time distribution of a reflected Brownian motion and found a Volterra integral equation for the density. The main result is derived from approximating the barriers by step functions and carrying out a careful analysis of the paths.

The solution of the integral equations does not seem a trivial task; however, it was shown to be feasible using numerical methods.

We briefly mention other possible directions to take.

- Extend the result to the case of nonsymmetric barriers. The main technical difficulty is the nonsymmetrical probabilities in (3.6).
- The problem of solving the integral equation explicitly remains open.
- There is an interesting relation between the maximum and the reflected Brownian motion. Let M be the maximum of the Brownian motion, that is, $M_t = \max(B_s, s \le t)$. It is known the two processes,

$$\{M_t - B_t : t \ge 0\} \quad \text{and} \quad \{|B_t| : t \ge 0\},$$

have the same law (see, for example, [14, Page 210]). We may want to use this to find the first passage distribution of M.

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