The socle of subshift algebras, with applications to subshift conjugacy

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We introduce the concept of 'irrational paths' for a given subshift and use it to characterize all minimal left ideals in the associated unital subshift algebra. Consequently, we characterize the socle as the sum of the ideals generated by irrational paths. Proceeding, we construct a graph such that the Leavitt path algebra of this graph is graded isomorphic to the socle. This realization allows us to show that the graded structure of the socle serves as an invariant for the conjugacy of Ott–Tomforde–Willis subshifts and for the isometric conjugacy of subshifts constructed with the product topology. Additionally, we establish that the socle of the unital subshift algebra is contained in the socle of the corresponding unital subshift C*-algebra.

1. Introduction

Leavitt path algebras are non-commutative algebras that have attracted significant attention, due to their versatility and deep connections with various branches of mathematics, including combinatorics, C*-algebras, and symbolic dynamics. Particularly noteworthy, the classification of Leavitt path algebras is closely related to the classification of graph C*-algebras and those of subshifts of finite type. This relation enables the utilization of algebraic techniques and the inherent structure of Leavitt path algebras to address problems in symbolic dynamics, see [17] and [18] for a recent overview.

Expanding from the setup above, C*-algebras associated with subshifts over finite alphabets have been formally defined in prior works. However, due to the intricacies involved in dealing with arbitrary subshifts (not limited to finite type), the definition has gone through refinements, culminating with the formulation given in [15]. Among other applications, conjugacy of subshifts can be described in terms of certain isomorphisms of the associated algebras, see [13].

In the general context of subshifts over arbitrary alphabets, both purely algebraic and C*-algebraic subshift algebras have recently been defined, see [9, 11]. These algebras generalize Leavitt path algebras of graphs and ultragraphs and can be seen as Leavitt path algebras of labelled graphs [8, 10]. As with Leavitt path algebras, these new algebras exhibit a strong correlation with the underlying dynamics of the associated subshifts, including the characterization of conjugacy between Ott–Tomforde and Willis subshifts via algebraic terms. Consequently, the understanding of their structure is important.

When addressing infinite alphabets, multiple definitions of a subshift exist. Among these, the approach outlined by Ott–Tomforde and Willis is extensively studied, see, for example, [19, 20, 23, 24, 27], particularly in relation to the non-commutative subshift algebras introduced in [9, 11], exhibiting a strong connection with these algebras. It is worth noting that the subshifts introduced by Ott–Tomforde and Willis, hereafter referred to as OTW-subshifts, conform to the typical definition of subshifts when the alphabet is finite.

Our contribution, in this article, to the description of the structure of subshift algebras lies in the socle and its grading. In the context of Leavitt path algebras, the socle is examined in [5, 6] and used in the classification program to generate an atlas of Leavitt path algebras of small graphs in [4]. The socle series is studied in [3], and additionally, the socle of Kumijian–Pask algebras is considered in [14]. In our investigation of the socle of a purely algebraic subshift algebra, we have identified a graph wherein the socle of the subshift algebra is graded isomorphic to the Leavitt path algebra of the said graph. This finding is novel even within the context of Leavitt path algebras, as previous studies have characterized the socle solely as an ideal, and enables us to discern properties of the socle and use it in applications. Indeed, it allows us to establish a combinatorial criterion that the underlying graphs of conjugate OTW-subshifts must satisfy. Building on [9, theorem 6.11], this implies that the graded socle serves as an invariant for isometric conjugacy of subshifts with the $\frac{1}{2^i}$-metric (which induces the product topology, see the discussion above [9, theorem 6.11] for the precise definition of such subshifts). We give more details of our work below.

We devote §2 to preliminaries, where we recall the notions of subshift, the unital algebra associated with a subshift, and a few results that we use in the article. We refer the reader to [11] for a more comprehensive introduction to subshift algebras.

In §3, we describe the minimal left ideals in an unital subshift algebra, see corollary 3.11. To do this, we identify certain special elements in a subshift, which we call irrational paths (definition 3.4). We show that any minimal left ideal is isomorphic to a left ideal generated by a one-point set given by an irrational path. Previously, left minimal ideals were linked with line paths, see [22]. The use of irrational paths (which contain the line paths) completes the description of the left minimal ideals.

With the description of the minimal left ideals obtained in §3, we proceed, in §4, to the study of the socle, which is the sum of all left minimal ideals of the algebra. We prove that the socle of an unital subshift algebra is not only isomorphic but equal to the sum of left ideals generated by the irrational paths, see theorem 4.2. This result is new also in the context of Leavitt path algebras, where it was previously shown that the socle contains the ideal generated by line points, see remark 4.4. As a consequence of our results, we show that the socle of a unital subshift algebra is always different from the algebra and is non-zero if and only if the set of irrational points is non-empty, see 4.5. We finish the section describing a smaller generating set for the socle. For this, we consider tail equivalence (definition 4.7) in the set of irrational paths and show that the socle is the direct sum of the left minimal ideals generated by each equivalence class, see corollary 4.9.

The construction of a graph associated with a subshift is the focus of §5. We show that the Leavitt path algebra of this graph is graded isomorphic to the subshift algebra, see corollary 5.7. Moreover, we observe that the aforementioned graph is always acyclic, row-finite, and sinkless, see remark 5.8. We use this to characterize, in terms of condition (Y), when the socle is strongly $\mathbb{Z}$-graded, see proposition 5.10. As an application, we show that if two OTW-subshifts are conjugate, then the graphs associated with the corresponding subshifts satisfy condition (Y), see theorem 5.14. We employ the latter criterion for two specific examples and conclude that neither the associated OTW-subshifts are conjugate nor there is an isometric conjugacy between the associated subshifts built using the $\frac{1}{2^i}$-metric (which induces the product topology), see corollary 5.18 for details.

We finish the article in §6, with a brief examination of the socle of the subshift C*-algebra. Specifically, for a given subshift, we show that the socle of the subshift algebra is contained in the socle of the subshift C*-algebra and establish that this inclusion may be proper. This is the same behaviour presented by Leavitt path algebras, see [6, theorem 3.6]. However, the example we provide to illustrate the possibility of strict containment differs from the example in [6], which relies on a graph with sinks (a setting not permitted for subshift algebras).

2. Preliminaries

In this section, we recall the definition of the unital subshift algebra and some relevant results regarding it. We start setting up notation and recalling the notion of a subshift.

Throughout the article, R stands for a commutative unital ring, $\mathbb{N} = \{0, 1, 2, \ldots\}$, and $\mathbb{N}^* = \{1, 2, \ldots\}$.

2.1. Symbolic dynamics

Let $\mathscr{A}$ be a non-empty set, referred to as an *alphabet*. The shift map σ is a map from $\mathscr{A}^{\mathbb{N}}$ to itself, defined as $\sigma(x) = y$, where $x = (x_n)$ $n \in \mathbb{N}$ and $y = (x_{n+1})_{n \in \mathbb{N}}$. We designate elements of $\mathscr{A}^* = \bigcup_{k=0}^{\infty} \mathscr{A}^k$ as *blocks* or *words*, with ω representing the empty word. Additionally, we define $\mathscr{A}^+ = \mathscr{A}^* \setminus \{\omega\}$. For any $\alpha \in \mathscr{A}^* \cup \mathscr{A}^{\mathbb{N}}$, $|\alpha|$ denotes its length. Given $1 \leq i, j \leq |\alpha|$, we define $\alpha_{i,j} := \alpha_i \cdots \alpha_j$ if $i \leq j$, and $\alpha_{i,j} = \omega$ if $i > j$. If $\beta \in \mathscr{A}^*$, then $\beta\alpha$ denotes concatenation. For a block $\alpha \in \mathscr{A}^k$,

α^∞ represents the infinite word $\alpha\alpha\dots$. A subset $\mathsf{X} \subseteq \mathscr{A}^{\mathbb{N}}$ is *invariant* for σ if $\sigma(\mathsf{X}) \subseteq \mathsf{X}$. For an invariant subset $\mathsf{X} \subseteq \mathscr{A}^{\mathbb{N}}$, we define $\mathcal{L}_n(\mathsf{X})$ as the set of all words of length n that appear in some sequence of X, that is,

$$\mathcal{L}_n(\mathsf{X}) := \{(a_0 \dots a_{n-1}) \in \mathscr{A}^n : \exists\, x \in \mathsf{X} \text{s.t. } (x_0 \dots x_{n-1}) = (a_0 \dots a_{n-1})\}.$$

It is evident that $\mathcal{L}_n(\mathscr{A}^{\mathbb{N}}) = \mathscr{A}^n$, and it is always the case that $\mathcal{L}_0(\mathsf{X}) = \omega$. The *language* of X, denoted as $\mathcal{L}_{\mathsf{X}}$, encompasses all finite words that manifest within some sequence of X. Formally,

$$\mathcal{L}_{\mathsf{X}} := \bigcup_{n=0}^{\infty} \mathcal{L}_n(\mathsf{X}).$$

For elements $c, d \in \mathcal{L}_{\mathsf{X}}$, we use the notation $c_{|c|} \neq d_{|d|}$ meaning that the last letter of c is different from the last letter of d.

Given $F \subseteq \mathscr{A}^*$, the *subshift* $\mathsf{X}_F \subseteq \mathscr{A}^{\mathbb{N}}$ is the set of all sequences x in $\mathscr{A}^{\mathbb{N}}$ such that no word of x belongs to F. When the context is clear, we denote X_F by X. The key sets that are used in the definition of the algebra associated with a subshift are defined below.

DEFINITION 2.1. *Let X be a subshift for an alphabet $\mathscr{A}$. Given $\alpha, \beta \in \mathcal{L}_{\mathsf{X}}$, define*

$$C(\alpha, \beta) := \{\beta x \in \mathsf{X} : \alpha x \in \mathsf{X}\}.$$

In particular, the set $C(\omega, \beta)$ is denoted by Z_β and called a cylinder set, and the set $C(\alpha, \omega)$ is denoted by F_α and called a follower set. Notice that $\mathsf{X} = C(\omega, \omega)$.

2.2. Unital algebras of subshifts

In this section, we recall the definition of the unital algebra associated with a general subshift X, as done in [11]. We start with the definition of the Boolean algebra associated with the sets of the form $C(\alpha, \beta)$.

DEFINITION 2.2. *Let X be a subshift. Define $\mathcal{U}$ to be the Boolean algebra of subsets of X generated by all $C(\alpha, \beta)$ for $\alpha, \beta \in \mathcal{L}_{\mathsf{X}}$, that is, $\mathcal{U}$ is the collections of sets obtained from finite unions, finite intersections, and complements of the sets $C(\alpha, \beta)$.*

We can now recall the definition of the unital algebra associated with a subshift.

DEFINITION 2.3. *Let X be a subshift. We define the unital subshift algebra $\widetilde{\mathcal{A}}_R(\mathsf{X})$ as the universal unital R-algebra with generators $\{p_A : A \in \mathcal{U}\}$ and $\{s_a, s_a^* : a \in \mathscr{A}\}$, subject to the relations:*

- *(i) $p_{\mathsf{X}} = 1$, $p_{A \cap B} = p_A p_B$, $p_{A \cup B} = p_A + p_B - p_{A \cap B}$, and $p_\emptyset = 0$ for every $A, B \in \mathcal{U}$;*
- *(ii) $s_a s_a^* s_a = s_a$ and $s_a^* s_a s_a^* = s_a^*$ for all $a \in \mathscr{A}$;*
- *(iii) $s_\beta s_\alpha^* s_\alpha s_\beta^* = p_{C(\alpha, \beta)}$ for all $\alpha, \beta \in \mathcal{L}_{\mathsf{X}}$, where $s_\omega := 1$, and for $\alpha = \alpha_1 \dots \alpha_n \in \mathcal{L}_{\mathsf{X}}$, $s_\alpha := s_{\alpha_1} \cdots s_{\alpha_n}$ and $s_\alpha^* := s_{\alpha_n}^* \cdots s_{\alpha_1}^*$.*

It follows from the third relation of the previous definition that $s_\alpha s_\alpha^* = p_{Z_\alpha}$ and $s_\alpha^* s_\alpha = p_{F_\alpha}$ for each $\alpha \in \mathcal{L}_X$. Recall that the relative range of (A, α), where $\alpha \in \mathcal{L}_X$ and $A \in \mathcal{U}$, is given by

$$r(A, \alpha) = \{x \in X : \alpha x \in A\}.$$

Next, we recall some results that will be necessary in our work.

LEMMA 2.4. [7, lemma 3.7] *Let* X *be a subshift,* $a, b \in \mathscr{A}$, *and* $\gamma, \alpha \in \mathcal{L}_X$.

(i) *If* $\beta := b\gamma \in \mathcal{L}_X$, *then* $s_\beta^* s_a = \delta_{b,a} s_\gamma^* p_{F_a}$.
(ii) *If* $A \in \mathcal{U}$, *then* $p_A s_\alpha = s_\alpha p_{r(A,\alpha)}$ *and* $s_\alpha^* p_A = p_{r(A,\alpha)} s_\alpha^*$.

By [11], $\widetilde{\mathcal{A}}_R(X)$ has a $\mathbb{Z}$-grading given by

$$\widetilde{\mathcal{A}}_R(X)_n = \mathrm{span}_R\{s_\alpha p_A s_\beta^* : \alpha, \beta \in \mathcal{L}_X, \ A \in \mathcal{U} \text{ and } |\alpha| - |\beta| = n\}, n \in \mathbb{Z}.$$

It is also graded by the free group on the alphabet. This grading arises from the isomorphism of the algebra with a certain partial skew group ring, as stated below.

THEOREM 2.5 [11, theorem 5.9] *Let* X *be a subshift. Then,* $\widetilde{\mathcal{A}}_R(X) \cong \widetilde{\mathcal{D}}_R(X) \rtimes_\tau \mathbb{F}$ *via an isomorphism* Φ *that sends* s_a *to* $1_a \delta_a$ *and* s_a^* *to* $1_{a^{-1}} \delta_{a^{-1}}$.

Another important result regarding $\widetilde{\mathcal{A}}_R(X)$ is the so-called Reduction Theorem (which was proved in [7]). We recall this result below, after providing the necessary definitions.

DEFINITION 2.6. [7, definitions 4.1 and 4.2] *Let* X *be a subshift,* $\alpha \in \mathcal{L}_X \setminus \{w\}$, *and* $\emptyset \neq A \in \mathcal{U}$. *The pair* (A, α) *is called a cycle if* $A \subseteq r(A, \alpha)$. *We say that a cycle* (A, α) *has an exit if* $A \neq \{\alpha^\infty\}$. *Otherwise, in case* $A = \{\alpha^\infty\}$, *we say that* (A, α) *is a cycle without exit.*

DEFINITION 2.7. [7, definition 4.4] *Let* X *be a subshift. We say that a cycle without exit* (A, α) *is minimal if there is no element* $\beta \in \mathcal{L}_X$, *with* $1 \leq |\beta| < |\alpha|$, *such that* $\alpha = \beta^k$ *for some* $k \geq 2$.

THEOREM 2.8 [7, Reduction Theorem] *Let* X *be a subshift and* $x \in \widetilde{\mathcal{A}}_R(X)$ *be a non-zero element. Then, there exists* $\mu, \nu \in \widetilde{\mathcal{A}}_R(X)$ *such that* $\mu x \nu \neq 0$ *and either*

(i) $\mu x \nu = \gamma p_D$ *with* $D \in \mathcal{U}$ *and* $\gamma \in R$ *or*

(ii) $\mu x \nu = \gamma_1 p_A + \sum_{i=2}^k \gamma_i s_\beta q_i p_A$, *where* (A, β) *is a minimal cycle without exit,* $q_i \in \mathbb{N} \setminus \{0\}$, *and* $0 \neq \gamma_i \in R$.

3. Minimal left ideals in $\widetilde{\mathcal{A}}_R(\mathbf{X})$

In this section, we characterize, up to isomorphism, the minimal left ideals of $\widetilde{\mathcal{A}}_R(X)$ (see corollary 3.11). To begin, we show a couple of results on the structure of left

ideals of $\widetilde{\mathcal{A}}_R(\mathsf{X})$. As in the previous section, unless otherwise stated, R stands for a commutative unital ring.

PROPOSITION 3.1. *Let X be a subshift. Then,*

    (i) *for each $x \in \widetilde{\mathcal{A}}_R(\mathsf{X})$, $\widetilde{\mathcal{A}}_R(\mathsf{X})x$ is a left ideal of $\widetilde{\mathcal{A}}_R(\mathsf{X})$.*

    (ii) *if $0 \neq I \subseteq \widetilde{\mathcal{A}}_R(\mathsf{X})$ is a left minimal ideal, then there exists some $x \in \widetilde{\mathcal{A}}_R(\mathsf{X})$ such that $I = \widetilde{\mathcal{A}}_R(\mathsf{X})x$.*

    (iii) *for all $A, B \in \mathcal{U}$ with $\emptyset \neq B \subsetneq A$, it holds that $0 \neq \widetilde{\mathcal{A}}_R(\mathsf{X})p_B \subsetneq \widetilde{\mathcal{A}}_R(\mathsf{X})p_A$.*

Proof. The first item is clear, so we begin with the proof of the second one.

Let $0 \neq I \subseteq \widetilde{\mathcal{A}}_R(\mathsf{X})$ be a left minimal ideal, and let $0 \neq x \in I$. Then, $\widetilde{\mathcal{A}}_R(\mathsf{X})x$ is a left ideal of $\widetilde{\mathcal{A}}_R(\mathsf{X})$, which is contained in I. Since $\widetilde{\mathcal{A}}_R(\mathsf{X})x \neq 0$ and I is minimal, we obtain that $I = \widetilde{\mathcal{A}}_R(\mathsf{X})x$.

Next, we prove the third item. Notice that $0 \neq \widetilde{\mathcal{A}}_R(\mathsf{X})p_B$, since $p_B \in \widetilde{\mathcal{A}}_R(\mathsf{X})p_B$. Furthermore, for any $z \in \widetilde{\mathcal{A}}_R(\mathsf{X})$, we have $zp_B = zp_Bp_A$ (since $B \subseteq A$), which implies that $\widetilde{\mathcal{A}}_R(\mathsf{X})p_B \subseteq \widetilde{\mathcal{A}}_R(\mathsf{X})p_A$. It remains to prove that the inclusion is proper. Suppose that $\widetilde{\mathcal{A}}_R(\mathsf{X})p_B = \widetilde{\mathcal{A}}_R(\mathsf{X})p_A$. This implies the existence of some $z \in \widetilde{\mathcal{A}}_R(\mathsf{X})$ such that $zp_B = p_A$. Define $C = B^c \cap A$, which is an element of $\mathcal{U}$. Observe that $p_Bp_C = 0$ since $B \cap C = \emptyset$, and $p_Ap_C = p_C \neq 0$ because $C \neq \emptyset$. Therefore, $0 = zp_Bp_C = p_Ap_C \neq 0$, a contradiction. We conclude that $\widetilde{\mathcal{A}}_R(\mathsf{X})p_B \neq \widetilde{\mathcal{A}}_R(\mathsf{X})p_A$. $\qquad\square$

PROPOSITION 3.2. *Let X be a subshift. Then, for each $A \in \mathcal{U}$ with $|A| > 1$, $\widetilde{\mathcal{A}}_R(\mathsf{X})p_A$ is a left ideal of $\widetilde{\mathcal{A}}_R(\mathsf{X})$, which is not minimal.*

Proof. Using the first item of proposition 3.1, we obtain that $\widetilde{\mathcal{A}}_R(\mathsf{X})p_A$ is a left ideal of $\widetilde{\mathcal{A}}_R(\mathsf{X})$. Given that $|A| > 1$, there exists $x, y \in A$ with $x \neq y$. Write $x = x_1x_2...$ and $y = y_1y_2...$, where $x_i, y_i \in \mathscr{A}$ for each i. Since $x \neq y$, there exists an index $j \in \mathbb{N}$ such that $x_j \neq y_j$. Let $B = Z_{x_{(1,j)}} \cap A$. Since $x \in B$ and $y \notin B$, we have that $\emptyset \neq B \subsetneq A$. Consequently, using the third item of proposition 3.1, we obtain that $\widetilde{\mathcal{A}}_R(\mathsf{X})p_A$ is not minimal. $\qquad\square$

Given the above, we are interested in the left ideals of the form $\widetilde{\mathcal{A}}_R(\mathsf{X})p_A$, where $|A| = 1$. Before proceeding, we review the definition of a line path below.

DEFINITION 3.3. [22, definition 4.1] *Let X be a subshift. We say that an element $p = a_0a_1a_2a_3... \in \mathsf{X}$ is a line path if $Z_{a_0} = \{p\}$, and for every $\beta \in \mathcal{L}_\mathsf{X}$ and $k \in \mathbb{N}$, we have that $\beta^\infty \neq a_ka_{k+1}a_{k+2}...$. We denote by $\mathcal{P}_\mathsf{X}$ the set of all the elements $A \in \mathcal{U}$ such that $A = \{p\}$ for some line path $p \in \mathsf{X}$.*

Line paths are associated with minimal left ideals. In fact, it is proved in [22, corollary 4.8] that, for each $A \in \mathcal{P}_\mathsf{X}$, $\widetilde{\mathcal{A}}_R(\mathsf{X})p_A$ is a minimal left ideal. However, there are other minimal ideals of the form $\widetilde{\mathcal{A}}_R(\mathsf{X})p_B$, with $B \notin \mathcal{P}_\mathsf{X}$. To show this, we need the following definition.

DEFINITION 3.4. *Let* X *be a subshift. Denote by* $\mathcal{Q}_X$ *the set of all the single point sets* $\{p\} \in \mathcal{U}$ *such that* $p \neq \alpha\beta^\infty$ *for each* $\alpha, \beta \in \mathcal{L}_X$. *We call an element of* $\mathcal{Q}_X$ *an irrational path.*

Notice that $\mathcal{P}_X \subseteq \mathcal{Q}_X$ but, in general, $\mathcal{P}_X \neq \mathcal{Q}_X$, as we show in the next example.

EXAMPLE 3.5. *Let* $\mathscr{A} = \{a, b, c\}$ *be an alphabet with three letters, and let* $x \in \mathscr{A}^\infty$ *be the element*

$$x = bcb^2cb^3cb^4c....$$

Define

$$X = \{a^\infty, b^\infty, ax, cx\} \cup \{\sigma^n(x) : n \in \mathbb{N}\},$$

which is a subshift.

This subshift has no line paths since the cardinalities of Z_a, Z_b, and Z_c are all greater than one. However, $\mathcal{Q}_X \neq \emptyset$ since $\mathcal{U} \ni C(c, a) = \{ax\}$ is an element of $\mathcal{Q}_X$.

Lemma 3.6 allows us to establish a dichotomy for the set of irrational paths in a subshift: either $\mathcal{Q}_X$ is empty or it has infinitely many elements.

LEMMA 3.6. *Let* X *be a subshift. If* $A = \{p\} \in \mathcal{Q}_X$, *then* $\{\sigma^n(p)\} \in \mathcal{Q}_X$ *for each* $n \in \mathbb{N}$, *and moreover,* $\sigma^n(p) \neq \sigma^m(p)$ *for each* $m \neq n$. *Consequently, if* $\mathcal{Q}_X$ *is non-empty, then it contains infinitely many elements.*

Proof. Let $A = \{p\} \in \mathcal{Q}_X$, where $p = p_0p_1p_2... \in X$, $n \in \mathbb{N}$, and define $a = p_0p_1..p_{n-1}$. Notice that $\{\sigma^n(p)\} = r(A, a) \in \mathcal{U}$. Suppose that $\{\sigma^n(p)\} \notin \mathcal{Q}_X$, that is, $\sigma^n(p) = \alpha\beta^\infty$ for some $\alpha, \beta \in \mathcal{L}_X$. Then, $p = a\alpha\beta^\infty$ and hence $\{p\} \notin \mathcal{Q}_X$, a contradiction. Therefore, $\{\sigma^n(p)\} \in \mathcal{Q}_X$.

Now, suppose that $\sigma^n(p) = \sigma^m(p)$ for some $m \neq n$, and suppose, without loss of generality, that $n < m$. Let $z = \sigma^n(p) = \sigma^m(p)$ and let $b = p_np_{n+1}...p_{m-1}$. Then, $z = \sigma^n(p) = b\sigma^m(p) = bz$, and consequently, $z = b^\infty$. Hence, $p = p_0p_1...p_{n-1}z = p_0p_1...p_{n-1}b^\infty$, which implies that $\{p\} \notin \mathcal{Q}_X$, a contradiction. Therefore, $\sigma^n(p) \neq \sigma^m(p)$. $\square$

Recall that a non-zero idempotent e in an algebra A is minimal if eAe is a division ring, see [12, definition 30.1]. Furthermore, if A is semiprime, then L is a minimal left ideal of A if and only if $L = Ae$, where e is a minimal idempotent in A, see [12, proposition 30.6]. We will use these concepts below to prove that left ideals associated with irrational paths are minimal. From now on, we assume that the base ring R is a field, so that $\widetilde{\mathcal{A}}_R(X)$ is semiprime (in fact, it is enough to ask that R has no zero divisors to obtain semiprimeness, see [7, corollary 5.6]).

PROPOSITION 3.7. *Let* X *be a subshift and* R *be a field. Then, for each* $A \in \mathcal{Q}_X$, $\widetilde{\mathcal{A}}_R(X)p_A$ *is a left minimal ideal of* $\widetilde{\mathcal{A}}_R(X)$.

Proof. Fix an $A = \{p\} \in \mathcal{Q}_X$, and let $\alpha, \beta \in \mathcal{L}_X$ and $B \in \mathcal{U}$ be such that $p_As_\alpha p_B s_\beta^* p_A \neq 0$. Then, α is an initial path of p, since otherwise $p_As_\alpha = 0$. Similarly, β is an initial path of p. Write $p = \alpha x$ and $p = \beta y$, where $x, y \in X$.

Then, since

$$0 \neq p_A s_\alpha p_B s_\beta^* p_A = s_\alpha p_{r(A,\alpha)} p_B p_{r(A,\beta)} s_\beta^*,$$

we have that $\emptyset \neq r(A, \alpha) \cap B \cap r(A, \beta)$ and hence $r(A, \alpha) = r(A, \alpha) \cap B \cap r(A, \beta) = r(A, \beta)$. Notice that $r(A, \alpha) = \{x\} = \{\sigma^{|\alpha|}(p)\}$ and $r(A, \beta) = \{y\} = \{\sigma^{|\beta|}(p)\}$, so that $\sigma^{|\alpha|}(p) = \sigma^{|\beta|}(p)$. From lemma 3.6, we get that $|\alpha| = |\beta|$, and consequently, $\alpha = \beta$. Hence,

$$p_A s_\alpha p_B s_\beta^* p_A = s_\alpha p_{r(A,\alpha)} p_B p_{r(A,\beta)} s_\beta^* = s_\alpha p_{r(A,\alpha)} p_{r(A,\alpha)} s_\alpha^* =$$

$$= p_A s_\alpha s_\alpha^* p_A = p_A p_{Z_\alpha} p_A = p_A.$$

Therefore, as elements of the form $s_\alpha p_B s_\beta^*$ generate $\widetilde{\mathcal{A}}_R(\mathsf{X})$ (see the comment about the $\mathbb{Z}$-grading of $\widetilde{\mathcal{A}}_R(\mathsf{X})$ below lemma 2.4), we obtain that $p_A \widetilde{\mathcal{A}}_R(\mathsf{X}) p_A = R p_A$. This means that p_A is a minimal idempotent, and so, by [12, proposition 30.6], we conclude that $\widetilde{\mathcal{A}}_R(\mathsf{X}) p_A$ is a minimal left ideal. □

Next, we completely characterize minimal left ideals associated with elements of $\mathcal{U}$.

PROPOSITION 3.8. *Let X be a subshift, R be a field, and $A \in \mathcal{U}$. Then, $\widetilde{\mathcal{A}}_R(\mathsf{X}) p_A$ is a left minimal ideal if and only if $A \in \mathcal{Q}_\mathsf{X}$.*

Proof. Let $A \in \mathcal{Q}_\mathsf{X}$. Then, by proposition 3.7, $\widetilde{\mathcal{A}}_R(\mathsf{X}) p_A$ is minimal.

To prove the converse, let $A \in \mathcal{U}$ and assume that $\widetilde{\mathcal{A}}_R(\mathsf{X}) p_A$ is minimal. If $|A| > 1$ then, by proposition 3.2, $\widetilde{\mathcal{A}}_R(\mathsf{X}) p_A$ is not minimal, a contradiction. Hence, $|A| = 1$, and we write $A = \{p\}$. Suppose that $A \notin \mathcal{Q}_\mathsf{X}$. Then, $p = \alpha \mu^\infty$, with $\mu, \alpha \in \mathcal{L}_\mathsf{X}$.

Suppose first that $|\alpha| = 0$, that is, $p = \mu^\infty$. Let $\beta \in \mathcal{L}_\mathsf{X}$ be the element of minimal length such that $\mu = \beta^k$ for some $k \in \mathbb{N}$. Then, $\mu^\infty = p = \beta^\infty$, and the pair (A, β) is a minimal cycle without exit (see definition 2.7). Applying [7, lemma 5.3], we obtain that $p_A \widetilde{\mathcal{A}}_R(\mathsf{X}) p_A$ and $R[x, x^{-1}]$ are isomorphic algebras, where $R[x, x^{-1}]$ denotes the Laurent polynomials ring. Since $R[x, x^{-1}]$ is not a division ring, neither is $p_A \widetilde{\mathcal{A}}_R(\mathsf{X}) p_A$. Hence, p_A is not a minimal idempotent and $\widetilde{\mathcal{A}}_R(\mathsf{X}) p_A$ is not a minimal ideal, which contradicts the hypothesis.

Next, suppose that $p = \alpha \mu^\infty$, with $|\alpha| > 0$. Define $B = \{\mu^\infty\}$ and notice that $r(A, \alpha) = \{\mu^\infty\}$. Then, $p_A s_\alpha = s_\alpha p_{r(A,\alpha)} = s_\alpha p_B$, and similarly, $s_\alpha^* p_A = p_B s_\alpha^*$. From the previous paragraph, we get that $\widetilde{\mathcal{A}}_R(\mathsf{X}) p_B$ is not minimal. Let J be a left ideal of $\widetilde{\mathcal{A}}_R(\mathsf{X})$ such that $0 \neq J \subsetneq \widetilde{\mathcal{A}}_R(\mathsf{X}) p_B$. Define $I = J s_\alpha^*$, which is a left ideal of $\widetilde{\mathcal{A}}_R(\mathsf{X})$. Notice that $I = J s_\alpha^* = J p_B s_\alpha^* = J s_\alpha^* p_A$, so that $I \subseteq \widetilde{\mathcal{A}}_R(\mathsf{X}) p_A$. Let $0 \neq x \in J$. Then, $x = x p_B = x p_B p_{F_\alpha} = x p_B s_\alpha^* s_\alpha$, and therefore, $0 \neq x p_B s_\alpha^*$. So, $I \neq 0$. Aiming to show that $I \neq \widetilde{\mathcal{A}}_R(\mathsf{X}) p_A$, suppose that $I = \widetilde{\mathcal{A}}_R(\mathsf{X}) p_A$. Then, $p_A \in I = J s_\alpha^*$, and there is $y \in J$ such that $p_A = y p_B s_\alpha^*$. Multiplying this equality on the right by s_α, we get that $p_A s_\alpha = y p_B s_\alpha^* s_\alpha = y p_B$, and since $p_A s_\alpha = s_\alpha p_B$, we conclude that $s_\alpha p_B = y p_B$. Now, multiplying this equality on the left by s_α^*, we get that $s_\alpha^* s_\alpha p_B = s_\alpha^* y p_B$, which is an element of J. As $s_\alpha^* s_\alpha p_B = p_B$, we obtain that $p_B \in J$, and consequently, $J = \widetilde{\mathcal{A}}_R(\mathsf{X}) p_B$, which is a contradiction. Therefore, $I \neq \widetilde{\mathcal{A}}_R(\mathsf{X}) p_A$ and $\widetilde{\mathcal{A}}_R(\mathsf{X}) p_A$ is not minimal. □

Our next goal is to completely characterize the left minimal ideals of $\widetilde{\mathcal{A}}_R(\mathsf{X})$. For this, we need the following lemma:

LEMMA 3.9. *Let $0 \neq x \in \widetilde{\mathcal{A}}_R(\mathsf{X})$ be such that $\widetilde{\mathcal{A}}_R(\mathsf{X})x$ is a left minimal ideal of $\widetilde{\mathcal{A}}_R(\mathsf{X})$. Then, for each $\mu', \nu' \in \widetilde{\mathcal{A}}_R(\mathsf{X})$ with $\mu' x \nu' \neq 0$, $\widetilde{\mathcal{A}}_R(\mathsf{X})x$ and $\widetilde{\mathcal{A}}_R(\mathsf{X})\mu' x \nu'$ are isomorphic as left $\widetilde{\mathcal{A}}_R(\mathsf{X})$-modules (and consequently, $\widetilde{\mathcal{A}}_R(\mathsf{X})\mu' x \nu'$ is also minimal).*

Proof. First, observe that $\widetilde{\mathcal{A}}_R(\mathsf{X})\mu' x = \widetilde{\mathcal{A}}_R(\mathsf{X})x$. Indeed, this follows from the minimality of $\widetilde{\mathcal{A}}_R(\mathsf{X})x$ and the fact that $0 \neq \widetilde{\mathcal{A}}_R(\mathsf{X})\mu' x$ is a left $\widetilde{\mathcal{A}}_R(\mathsf{X})$ ideal contained in $\widetilde{\mathcal{A}}_R(\mathsf{X})x$.

Define $\varphi : \widetilde{\mathcal{A}}_R(\mathsf{X})\mu' x \to \widetilde{\mathcal{A}}_R(\mathsf{X})\mu' x \nu'$ by $\varphi(a) = a\nu'$ for each $a \in \widetilde{\mathcal{A}}_R(\mathsf{X})\mu' x$. Notice that this map is a surjective left $\widetilde{\mathcal{A}}_R(\mathsf{X})$-module homomorphism. We show that φ is injective. Since $\varphi(\mu' x) = \mu' x \nu' \neq 0$, we have that $ker(\varphi) \neq \widetilde{\mathcal{A}}_R(\mathsf{X})\mu' x$. Moreover, $Ker(\varphi)$ is a left $\widetilde{\mathcal{A}}_R(\mathsf{X})$ ideal contained in $\widetilde{\mathcal{A}}_R(\mathsf{X})\mu' x = \widetilde{\mathcal{A}}_R(\mathsf{X})x$. By the minimality of $\widetilde{\mathcal{A}}_R(\mathsf{X})x$, we conclude that $ker(\varphi) = 0$. Therefore, φ is an $\widetilde{\mathcal{A}}_R(\mathsf{X})$-left module isomorphism. $\qquad\square$

PROPOSITION 3.10. *Let X be a subshift, R be a field, and let $x \in \widetilde{\mathcal{A}}_R(\mathsf{X})$ be such that $\widetilde{\mathcal{A}}_R(\mathsf{X})x$ is a minimal left ideal. Then, $\widetilde{\mathcal{A}}_R(\mathsf{X})x$ is isomorphic (as a left $\widetilde{\mathcal{A}}_R(\mathsf{X})$-module) to $\widetilde{\mathcal{A}}_R(\mathsf{X})p_D$ for some $D \in \mathcal{Q}_\mathsf{X}$.*

Proof. Applying the Reduction Theorem (theorem 2.8) for x, we obtain $\mu, \nu \in \widetilde{\mathcal{A}}_R(\mathsf{X})$ such that $\mu x \nu \neq 0$ and $\mu x \nu = \lambda p_D$, where $D \in \mathcal{U}$ and $\lambda \in R$, or $\mu x \nu = \gamma_1 p_A + \sum_{i=2}^{k} \gamma_i s_\beta q_i p_A$, where (A, β) is a minimal cycle without exit (so $A = \{\beta^\infty\}$), $q_i \in \mathbb{N} \setminus \{0\}$, and $0 \neq \gamma_i \in R$.

By lemma 3.9, $\widetilde{\mathcal{A}}_R(\mathsf{X})x$ and $\widetilde{\mathcal{A}}_R(\mathsf{X})\mu x \nu$ are isomorphic as left $\widetilde{\mathcal{A}}_R(\mathsf{X})$-modules. So, if $\mu x \nu = \lambda p_D$, then $\widetilde{\mathcal{A}}_R(\mathsf{X})x$ is isomorphic to $\widetilde{\mathcal{A}}_R(\mathsf{X})\lambda p_D$. As R is a field, $\widetilde{\mathcal{A}}_R(\mathsf{X})\lambda p_D = \widetilde{\mathcal{A}}_R(\mathsf{X})p_D$, and by proposition 3.8, we get that $D \in \mathcal{Q}_\mathsf{X}$.

Next, we show that the second possibility for $\mu x \nu$ does not happen. For this, suppose that $\mu x \nu = \gamma_1 p_A + \sum_{i=2}^{k} \gamma_i s_\beta q_i p_A$, and to simplify notation, let $z = \mu x \nu$.

From [7, lemma 5.3], there exists an R-isomorphism $\psi : p_A \widetilde{\mathcal{A}}_R(\mathsf{X})p_A \to R[x, x^{-1}]$. Let $I = p_A \widetilde{\mathcal{A}}_R(\mathsf{X})z$, which is a non-zero two-sided ideal of $p_A \widetilde{\mathcal{A}}_R(\mathsf{X})p_A$ (to see that it is an ideal on the right, use that $z = p_A z p_A$ since (A, β) is a minimal cycle without exit, and that $R[x, x^{-1}]$ is commutative).

Next, we show that I is a minimal ideal of $p_A \widetilde{\mathcal{A}}_R(\mathsf{X})p_A$. Let $0 \neq J$ be a $p_A \widetilde{\mathcal{A}}_R(\mathsf{X})p_A$ ideal such that $J \subseteq I$. Then, $0 \neq \widetilde{\mathcal{A}}_R(\mathsf{X})J$ is an $\widetilde{\mathcal{A}}_R(\mathsf{X})$ left ideal contained in $\widetilde{\mathcal{A}}_R(\mathsf{X})z$. Since $\widetilde{\mathcal{A}}_R(\mathsf{X})z$ is minimal (by lemma 3.9), we obtain that $\widetilde{\mathcal{A}}_R(\mathsf{X})J = \widetilde{\mathcal{A}}_R(\mathsf{X})z$. Consequently, $p_A \widetilde{\mathcal{A}}_R(\mathsf{X})J = p_A \widetilde{\mathcal{A}}_R(\mathsf{X})z = I$, and since $p_A \widetilde{\mathcal{A}}_R(\mathsf{X})J = p_A \widetilde{\mathcal{A}}_R(\mathsf{X})p_A J = J$, we conclude that $J = I$. Therefore, I is minimal.

The minimality of I implies that $\psi(I)$ is a minimal ideal of $R[x, x^{-1}]$, which is a contradiction, since $R[x, x^{-1}]$ has no minimal ideals. Therefore, the case $\mu x \nu = \gamma_1 p_A + \sum_{i=2}^{k} \gamma_i s_\beta q_i p_A$ is not possible, as desired. $\qquad\square$

COROLLARY 3.11. *Let* X *be a subshift,* R *be a field, and* I *be a left ideal in* $\widetilde{\mathcal{A}}_R(X)$. *Then,* I *is minimal if and only if it is isomorphic (as a left* $\widetilde{\mathcal{A}}_R(X)$-*submodule) to* $\widetilde{\mathcal{A}}_R(X)p_A$ *for some* $A \in \mathcal{Q}_X$.

Proof. If I is isomorphic as a left $\widetilde{\mathcal{A}}_R(X)$-ideal to $\widetilde{\mathcal{A}}_R(X)p_A$ for some $A \in \mathcal{Q}_X$, then it is minimal by proposition 3.8. For the other statement, suppose that I is a left $\widetilde{\mathcal{A}}_R(X)$ minimal ideal. By proposition 3.1, there exists $x \in \widetilde{\mathcal{A}}_R(X)$ such that $I = \widetilde{\mathcal{A}}_R(X)x$. The result now follows from proposition 3.10.   □

4. The socle of $\widetilde{\mathcal{A}}_R(X)$

Recall that the (left) socle of an algebra B, denoted by $Soc(B)$, is the sum of all the left minimal ideals in B, and that $Soc(B)$ is a two-sided ideal. If B is semiprime, then $Soc(B)$ coincides with the sum of all the right minimal ideals in B, as proved in [25, chapter 3, proposition 4]). As we we are assuming that R is a field, we have that $\widetilde{\mathcal{A}}_R(X)$ is semiprime.

Our goal in this section is to characterize the socle of $\widetilde{\mathcal{A}}_R(X)$ as the sum of the minimal left ideals associated with elements of $\mathcal{Q}_X$. By proposition 3.8, we have that

$$\sum_{D \in \mathcal{Q}_X} \widetilde{\mathcal{A}}_R(X)p_D \subseteq Soc(\widetilde{\mathcal{A}}_R(X)).$$

To prove the reverse inclusion, we first need a lemma.

LEMMA 4.1. *Let* X *be a subshift,* $x \in X$, *and* $D = \{x\} \in \mathcal{U}$.

   (i) *If* $\beta \in \mathcal{L}_X$ *is such that* $D \subseteq F_\beta$, *then* $\{\beta x\} \in \mathcal{U}$.
   (ii) *For each* $\alpha, \beta \in \mathcal{L}_X$ *and* $A \in \mathcal{U}$ *such that* $p_D s_\alpha p_A s_\beta^* \neq 0$, *there exists* $y \in X$ *such that* $x = \alpha y$, $\{y\}$ *and* $\{\beta y\} \in \mathcal{U}$, *and*

$$p_D s_\alpha p_A s_\beta^* = s_\alpha s_\beta^* p_{\{\beta y\}}.$$

   (iii) *For each* $\alpha, \beta \in \mathcal{L}_X$ *and* $A \in \mathcal{U}$, *if* $s_\alpha p_A s_\beta^* p_D \neq 0$, *then* $s_\alpha p_A s_\beta^* p_D = s_\alpha s_\beta^* p_D$.

Proof. We begin with the first item. Let Φ be the isomorphism of theorem 2.5. Notice that

$$\Phi(s_\beta p_D s_\beta^*) = 1_\beta \delta_\beta 1_D 1_{\beta^{-1}} \delta_{\beta^{-1}} = 1_{\{\beta x\}} \delta_0.$$

Recall from [11, §5] that $\widetilde{\mathcal{D}}_R(X)$ is generated by the characteristic functions of the sets $C(\alpha, \beta)$, with $\alpha, \beta \in \mathcal{L}_X$. Consequently, $1_{\{\beta x\}}$ is a finite sum of the form $1_{\beta x} = \sum_{i \in F} \lambda_i 1_{A_i}$, where $A_i \in \mathcal{U}$ and F is a finite set. Moreover, since $\mathcal{U}$ is a Boolean algebra, we can suppose that $A_i \cap A_j = \emptyset$ for each $i \neq j$ (see [10, lemma 3.5]). Since $\{\beta x\}$ is a set with only one element, and all the A_i are disjoint, we obtain that F is also a set with only one element, say $F = \{i\}$. Hence, $1_{\beta x} = \lambda_i 1_{A_i}$, which implies that $\lambda_i = 1$ and $\{\beta x\} = A_i \in \mathcal{U}$.

Next, we prove the second item. Let $\alpha, \beta \in \mathcal{L}_{\mathsf{X}}$ and $A \in \mathcal{U}$. Notice that if $x \notin Z_\alpha$, then $p_D s_\alpha = 0$ and hence $p_D s_\alpha p_A s_\beta^* = 0$. Therefore, if $p_D s_\alpha p_A s_\beta^* \neq 0$, then $x \in Z_\alpha$, say $x = \alpha y$ for some $y \in \mathsf{X}$.

Thus,

$$p_D s_\alpha p_A s_\beta^* = p_{\{\alpha y\}} s_\alpha p_A s_\beta^* = s_\alpha p_{\{y\}} p_A s_\beta^*.$$

From the last term in the equality above, we obtain that $p_{\{y\}} p_A = 0$ if $y \notin A$. Therefore, we have that $y \in A$. Hence,

$$s_\alpha p_{\{y\}} p_A s_\beta^* = s_\alpha p_{\{y\}} s_\beta^*.$$

In case that $y \notin F_\beta$, the above implies that $p_{\{y\}} s_\beta^* = 0$, and so, we conclude that $y \in F_\beta$. Observe that $\{y\} \in \mathcal{U}$ since $\{y\} = r(\{\alpha y\}, \alpha)$, and so, from the first item of this lemma, we get that $\{\beta y\} \in \mathcal{U}$. Thus, from the second item of [lemma 2.4](), we obtain that $p_{\{y\}} s_\beta^* = s_\beta^* p_{\{\beta y\}}$ and hence $s_\alpha p_{\{y\}} s_\beta^* = s_\alpha s_\beta^* p_{\{\beta y\}}$. Thus, $p_D s_\alpha p_A s_\beta^* = s_\alpha s_\beta^* p_{\{\beta y\}}$, as desired.

To prove the third item, let $\alpha, \beta \in \mathcal{L}_{\mathsf{X}}$ and $A \in \mathcal{U}$ be such that $s_\alpha p_A s_\beta^* p_D \neq 0$. Since $0 \neq s_\alpha p_A s_\beta^* p_D = s_\alpha p_A p_{r(D,\beta)} s_\beta^*$, we have that $r(D, \beta) \subseteq A$. Hence, $s_\alpha p_A p_{r(D,\beta)} s_\beta^* = s_\alpha p_{r(D,\beta)} s_\beta^* = s_\alpha s_\beta^* p_D$, where the last equality follows from the second item of [lemma 2.4](). $\qquad\square$

Next, we prove that the socle of a subshift algebra coincides with the sum of the minimal left ideals associated with elements of $\mathcal{Q}_{\mathsf{X}}$.

THEOREM 4.2 *Let X be a subshift and R be a field. Then,*

$$Soc(\widetilde{\mathcal{A}}_R(\mathsf{X})) = \sum_{A \in \mathcal{Q}_{\mathsf{X}}} \widetilde{\mathcal{A}}_R(\mathsf{X}) p_A.$$

Proof. Let $0 \neq I$ be a left minimal ideal of $\widetilde{\mathcal{A}}_R(\mathsf{X})$. By [lemma 3.1](), there exists $0 \neq x \in \widetilde{\mathcal{A}}_R(\mathsf{X})$ such that $I = \widetilde{\mathcal{A}}_R(\mathsf{X})x$. For this x, let μ, ν be as in the Reduction Theorem ([theorem 2.8]()) and use [lemma 3.9]() to conclude that $\widetilde{\mathcal{A}}_R(\mathsf{X})\mu x \nu$ is minimal. Then, by the proof of [proposition 3.10](), we get that $\mu x \nu = \lambda p_D$ for some $0 \neq \lambda \in R$ and $D \in \mathcal{U}$. Notice that since $\widetilde{\mathcal{A}}_R(\mathsf{X})x$ is minimal, we have that $\widetilde{\mathcal{A}}_R(\mathsf{X})\mu x = \widetilde{\mathcal{A}}_R(\mathsf{X})x$. Define $x' = \mu x$ and let $\varphi : \widetilde{\mathcal{A}}_R(\mathsf{X})x' \to \widetilde{\mathcal{A}}_R(\mathsf{X})p_D$ be the ($\widetilde{\mathcal{A}}_R(\mathsf{X})$ left) isomorphism of the proof of [lemma 3.9](), that is, $\varphi(zx') = zx'\nu = z\lambda p_D$. Observe that $\varphi(x') = \lambda p_D$, and since φ is a left $\widetilde{\mathcal{A}}_R(\mathsf{X})$-isomorphism, this implies that

$$x' = \varphi^{-1}(\varphi(x')) = \varphi^{-1}(\lambda p_D) = \varphi^{-1}(p_D \lambda p_D) = p_D \varphi^{-1}(\lambda p_D) = p_D x'.$$

Write x' as a finite sum of the form $x' = \sum \lambda_i s_{\alpha_i} p_{A_i} s_{\beta_i}^*$, where $A_i \in \mathcal{U}$, $\alpha_i, \beta_i \in \mathcal{L}_{\mathsf{X}}$, and $\lambda_i \in R$ for each i. From the second item of [lemma 4.1](), we obtain that $p_D x'$ has the form

$$p_D x' = \sum \lambda_i s_{\alpha_i} s_{\beta_i}^* p_{\{\beta_i y_i\}},$$

where $D = \{\alpha_i y_i\}$ for each i (notice that we can apply lemma 4.1 because, as $\widetilde{\mathcal{A}}_R(\mathsf{X})p_D$ is minimal, D is a one-point set). Since $\{\alpha_i y_i\} \in \mathcal{Q}_\mathsf{X}$ for each i, we have that $\{\beta_i y_i\} \in \mathcal{Q}_\mathsf{X}$ for each i, and hence,

$$x' = p_D x' = \sum \lambda_i s_{\alpha_i} s_{\beta_i}^* p_{\{\beta_i y_i\}} \in \sum_{A \in \mathcal{Q}_\mathsf{X}} \widetilde{\mathcal{A}}_R(\mathsf{X})p_A.$$

Consequently, we obtain that

$$I = \widetilde{\mathcal{A}}_R(\mathsf{X})x = \widetilde{\mathcal{A}}_R(\mathsf{X})x' \subseteq \sum_{A \in \mathcal{Q}_\mathsf{X}} \widetilde{\mathcal{A}}_R(\mathsf{X})p_A$$

and so

$$Soc(\widetilde{\mathcal{A}}_R(\mathsf{X})) \subseteq \sum_{A \in \mathcal{Q}_\mathsf{X}} \widetilde{\mathcal{A}}_R(\mathsf{X})p_A.$$

As we mentioned at the beginning of the section, the other inclusion follows from proposition 3.8, as it gives us that $\widetilde{\mathcal{A}}_R(\mathsf{X})p_A$ is a left minimal ideal of $\widetilde{\mathcal{A}}_R(\mathsf{X})$ for each $A \in \mathcal{Q}_\mathsf{X}$.

So,

$$Soc(\widetilde{\mathcal{A}}_R(\mathsf{X})) = \sum_{A \in \mathcal{Q}_\mathsf{X}} \widetilde{\mathcal{A}}_R(\mathsf{X})p_A,$$

as desired. □

COROLLARY 4.3. *Let X be a subshift and R be a field. Then, $Soc(\widetilde{\mathcal{A}}_R(\mathsf{X})) = J$, where J is the two-sided ideal of $\widetilde{\mathcal{A}}_R(\mathsf{X})$ generated by the set $\{p_A : A \in \mathcal{Q}_\mathsf{X}\}$. Moreover, the socle is a $\mathbb{Z}$-graded ideal.*

Proof. The first part of the result follows from theorem 4.2 and the fact that $Soc(\widetilde{\mathcal{A}}_R(\mathsf{X}))$ is a two-sided ideal. The grading statement follows from [1, remark 2.1.2]. □

REMARK 4.4. Let E be a graph. By [6, proposition 5.1], the socle of the Leavitt path algebra $L_K(E)$ contains (but is not necessarily equal to) the sum of the minimal left ideals associated with the line points. In theorem 4.2, we have an equality, since we are summing over a larger set (notice that a line path in E induces an irrational path in the subshift associated with E, but the converse is not necessarily true, as can be seen in Example 3.5).

COROLLARY 4.5. *Let X be a subshift and R be a field. Then,*

   (i) *$\widetilde{\mathcal{A}}_R(\mathsf{X})$ has non-zero socle if and only if $\mathcal{Q}_\mathsf{X}$ is non-empty.*
   (ii) *$Soc(\widetilde{\mathcal{A}}_R(\mathsf{X})) \neq \widetilde{\mathcal{A}}_R(\mathsf{X})$.*

Proof. The first statement is a direct consequence of corollary 4.3, so we prove the second item. The result is clearly true if $\mathcal{Q}_\mathsf{X} = \emptyset$, and hence, we may assume that $\mathcal{Q}_\mathsf{X} \neq \emptyset$.

Seeking for a contradiction, suppose that $p_X \in Soc(\widetilde{\mathcal{A}}_R(X))$. From theorem 4.2, we obtain that p_X is a sum of the form

$$p_X = \sum_{j=1}^{m} z_j p_{D_j},$$

where $z_j \in \widetilde{\mathcal{A}}_R(X)$ and $D_j \in \mathcal{Q}_X$ for each j. By lemma 3.6, $\mathcal{Q}_X$ contains infinitely many elements. So, there exists $D \in \mathcal{Q}_X$ such that $D \neq D_j$ for each $j \in \{1, ..., m\}$, which implies that $p_{D_j} p_D = 0$ for each j. Therefore,

$$p_D = p_X p_D = \sum_{j=1}^{m} z_j p_{D_j} p_D = 0,$$

which is a contradiction, since $p_D \neq 0$. We conclude that $p_X \notin Soc(\widetilde{\mathcal{A}}_R(X))$, and consequently, $\widetilde{\mathcal{A}}_R(X) \neq Soc(\widetilde{\mathcal{A}}_R(X))$. □

REMARK 4.6. For finite graphs, the socle of a Leavitt path algebra is non-zero if and only if the graph has sinks, see [5]. So, for a finite graph without sinks E, the socle of $L_K(E)$ is zero. We can obtain the last statement using the result above, considering the subshift X associated with E, recalling that $\widetilde{\mathcal{A}}_R(X)$ is isomorphic to $L_K(E)$, and noticing that $\mathcal{Q}_X = \emptyset$.

In the remainder of this section, we focus on finding a smaller generating set for the socle of a subshift algebra. For this, we need the following definition.

DEFINITION 4.7. *Let* X *be a subshift and* $x, y \in X$. *We say that the elements* x *and* y *are equivalent and write* $x \sim y$ *if there exists* $m, n \in \mathbb{N}$ *such that* $\sigma^m(x) = \sigma^n(y)$.

In the proposition below, we denote by $\langle x \rangle$ the two-sided ideal generated by x in $\widetilde{\mathcal{A}}_R(X)$.

PROPOSITION 4.8. *Let* X *be a subshift and let* $A, B \in \mathcal{U}$ *be two single point sets, say,* $A = \{p\}$ *and* $B = \{q\}$, *where* $p, q \in X$. *If* $p \sim q$, *then* $\langle p_A \rangle = \langle p_B \rangle$, *and if* $p \not\sim q$, *then* $\langle p_A \rangle \langle p_B \rangle = 0$.

Proof. We begin the proof showing that $p \sim q$ implies $\langle p_A \rangle = \langle p_B \rangle$.

Suppose first that $p = \sigma^n(q)$ for some $n \in \mathbb{N}$. Then, $q = \alpha p$, where $\alpha \in \mathcal{L}_X$. Hence,

$$s_\alpha^* p_B s_\alpha = s_\alpha^* s_\alpha p_{r(B,\alpha)} = p_{F_\alpha} p_{r(B,\alpha)} = p_{F_\alpha \cap r(B,\alpha)} = p_A,$$

which implies that $\langle p_A \rangle \subseteq \langle p_B \rangle$. Multiplying the above equality on the left by s_α and on the right by s_α^*, we get that

$$s_\alpha p_A s_\alpha^* = s_\alpha s_\alpha^* p_B s_\alpha s_\alpha^* = p_{Z_\alpha} p_B p_{Z_\alpha} = p_B,$$

and consequently, $\langle p_B \rangle \subseteq \langle p_A \rangle$. Therefore, we have proved that $\langle p_A \rangle = \langle p_B \rangle$.

To prove the general case, suppose that $p \sim q$. Then, there exists $m, n \in \mathbb{N}$ such that $\sigma^m(p) = \sigma^n(q)$, and hence, there exists $a, b \in \mathcal{L}_X$ and $z \in X$ such that $p = az$ and $q = bz$. Using that $r(A, \alpha) = \{z\} = r(B, \beta)$ and what we proved in the previous paragraph, we obtain that $\langle p_A \rangle = \langle p_{\{z\}} \rangle = \langle p_B \rangle$.

Now, suppose that p and q are not equivalent. We will show that $\langle p_A \rangle \langle p_B \rangle = 0$.

From lemma 4.1, $\langle p_A \rangle$ is the linear span of elements of the form $s_\alpha s_\beta^* p_{\{\mu\}}$, where $\mu \sim p$ and $\alpha, \beta \in \mathcal{L}_X$, and also $\langle p_B \rangle$ is the linear span of elements of the form $s_a s_b^* p_{\{\nu\}}$, where $\nu \sim q$ and $a, b \in \mathcal{L}_X$. Therefore, it is enough to show that $s_\alpha s_\beta^* p_{\{\mu\}} s_a s_b^* p_{\{\nu\}} = 0$ for $\alpha, \beta, a, b, \mu,$ and ν as above.

For $a, b, \alpha, \beta, \mu, \nu$ as above, notice that, by the second item of lemma 4.1, either $p_{\{\mu\}} s_a s_b^* = 0$ or there exists $z \sim \mu$ such that $p_{\{\mu\}} s_a s_b^* = s_a s_b^* p_{\{z\}}$. As $p \not\sim q$, we have that $z \not\sim \nu$ and hence $p_{\{z\}} p_{\{\nu\}} = 0$. Therefore,

$$s_\alpha s_\beta^* p_{\{\mu\}} s_a s_b^* p_{\{\nu\}} = s_\alpha s_\beta^* s_a s_b^* p_{\{z\}} p_{\{\nu\}} = 0$$

, and consequently, $\langle p_A \rangle \langle p_B \rangle = 0$. □

Let $\mathcal{Q}_X / \sim$ be the quotient space of $\mathcal{Q}_X$ by $\sim$, whose elements we denote by $[q]$ (here, and when necessary below, we identify a one-point set in $\mathcal{Q}_X$ with its element). To each equivalence class $[q]$ in $\mathcal{Q}_X / \sim$, we associated a projection, denoted by $p_{[q]}$, which is defined as $p_{[q]} := p_{\{q'\}}$, where $q' \in [q]$. By the proposition above, $p_{[q]}$ is well-defined up to the generation of the same two-sided ideal. Joining this discussion with corollary 4.3 and proposition 4.8, we get the following description of the socle of a subshift algebra.

COROLLARY 4.9. *Let X be a subshift and R be a field. Then,*

$$Soc(\widetilde{\mathcal{A}}_R(X)) = \bigoplus_{[q] \in \mathcal{Q}_X / \sim} \langle p_{[q]} \rangle.$$

5. The socle of a subshift algebra as a Leavitt path algebra

In this section, starting from a subshift X, we build a graph such that the associated Leavitt path algebra is graded isomorphic to the socle of the subshift algebra. Furthermore, we provide applications for such construction. The definition of the graph depends on some subsets of X, which we define below.

Fix an element $p \in X$ such that $\{p\} \in \mathcal{Q}_X$, and define $I_0 = \mathbb{N}$. Moreover, let

$$J_0 = \{\sigma^n(p) : n \in I_0\},$$

$$J_1 = \sigma^{-1}(J_0) \setminus J_0,$$

and inductively define, for each $n \in \mathbb{N}$,

$$J_n = \sigma^{-1}(J_{n-1}) \setminus (J_0 \cup J_1 \cup \ldots \cup J_{n-1}).$$

Notice that $J_n \cap J_m = \emptyset$ for each $n \neq m$. Moreover, it is possible that $J_n = \emptyset$ for some n, in which case this process is finite.

For each $i \in I_0$, define $z_i = \sigma^i(p)$, so that $J_0 = \{z_i : i \in I_0\}$. Inductively, for each $n \geq 1$, chose an index set I_n that indexes the elements of J_n and such that $I_n \cap I_m = \emptyset$ for each $n \neq m$. Write $J_n = \{z_i : i \in I_n\}$.

Define $I = \bigcup_{n=0}^{\infty} I_n$ and $J = \bigcup_{n=0}^{\infty} J_n$, which are both disjoint unions. The set I is the index set of the elements of J, so the map $I \ni i \mapsto z_i \in J$ is a bijective map.

REMARK 5.1. Each element $z_i \in J$ has a unique representation of the form $z_i = az_j$, where $a \in \mathscr{A}$ and $z_j \in J$. This is clear when $i \in I_0$, since in this case, $z_i = \sigma^i(p) = a\sigma^{i+1}(p)$, where a is the first letter of z_i. When $z_i \in J_n$ for some $n \geq 1$, we have that $z_i \in \sigma^{-1}(J_{n-1})$ and hence $\sigma(z_i) = z_j$ for a (unique) $z_j \in J_{n-1}$. Therefore, $z_i = az_j$ for some $a \in \mathscr{A}$.

Now, we define a graph E_p associated to the fixed element $p \in \mathsf{X}$ (with $\{p\} \in \mathcal{Q}_{\mathsf{X}}$). Both the vertex and edge sets are indexed by the set I defined above, and we write $E_p^0 = \{v_i : i \in I\}$ and $E_p^1 = \{e_i : i \in I\}$. The source map is defined by $s(e_i) = v_i$ for each $i \in I$. To define the range map, for each $i \in I$, write $z_i = az_j$ as in remark 5.1, and let $r(e_i) = v_j$. Equivalently, the range map is defined as $r(e_i) = v_j$, where $\sigma(z_i) = z_j$. Therefore, we get that $r(e_i) = v_j = s(z_j) = s(\sigma(z_i))$.

REMARK 5.2. In the graph E_p, there are no closed paths, that is, strings $\alpha = e_{k_1} \ldots e_{k_n}$ such that $r(e_{k_l}) = s(e_{k_{l+1}})$, $l = 1 \ldots n-1$, and $r(e_{k_n}) = s(e_{k_1})$. Indeed, for each $n \geq 1$ and $i \in I_n$, we have that $s(e_i) = v_i$ and $r(e_i) = v_j$, where $j \in I_{n-1}$. So, if α is a path in the graph E_p such that $s(\alpha) = v_i$, with $i \in I_n$ and $n \geq 1$, then the range of α lies in some I_j with $j < n$. Hence α is not a closed path. Suppose now that α is a path such that $s(\alpha) = v_i$ for some $i \in I_0$. Notice that for $i \in I_0$, the source and range maps are defined as $s(e_i) = v_i$ and $r(e_i) = v_{i+1}$. Hence, α is not a closed path. Consequently, there are no closed paths in E_p.

Our next goal is to prove that the ideal generated by an irrational element $\{p\}$ is isomorphic to the Leavitt path algebra of the graph E_p defined above. Before we do this, we prove the following lemma:

LEMMA 5.3. *Let X be a subshift and $Y \subseteq \mathsf{X}$ be such that $\{y\} \in \mathcal{U}$ for each $y \in Y$. Then,*

(i) for each $n \in \mathbb{N}$ and $z \in \sigma^n(Y)$, it holds that $\{z\} \in \mathcal{U}$ and
(ii) for each $n \in \mathbb{N}$ and $z \in \sigma^{-n}(Y)$, it holds that $\{z\} \in \mathcal{U}$.

Proof. Let $n \in \mathbb{N}$ and $z \in \sigma^n(Y)$. Then, $z = \sigma^n(y)$ for some $y \in Y$ and hence $y = \alpha z$, for some $\alpha \in \mathcal{L}_{\mathsf{X}}$. Since $\{y\} \in \mathcal{U}$, we have that $\{z\} = r(\{y\}, \alpha) \in \mathcal{U}$.

To prove the second item, let $n \in \mathbb{N}$ and $z \in \sigma^{-n}(Y)$. In this case, $\sigma^n(z) = y$ for some $y \in Y$, and therefore, $z = \beta y$ for some $\beta \in \mathcal{L}_{\mathsf{X}}$. Since $\{y\} \in \mathcal{U}$, from the first item of lemma 4.1, we get that $\{z\} \in \mathcal{U}$. $\square$

As a consequence of the above lemma, we obtain that every single element set formed by an element in the equivalence class of an irrational path is in $\mathcal{U}$, as stated below.

PROPOSITION 5.4. *Let $\{p\} \in \mathcal{Q}_X$. Then, $[p] = J$, where J is the set constructed above. Moreover, for each $z \in [p]$, the set $\{z\}$ belongs to the Boolean algebra $\mathcal{U}$.*

Proof. The first statement follows by the definition of J. For the second statement, notice that, by the first item of lemma 5.3, we have that $\{z\} \in \mathcal{U}$ for each $z \in J_0$. Also, by applying successively the second item of lemma 5.3, we get that $\{z\} \in \mathcal{U}$ for each $z \in J_n$, $n \geq 1$. □

The following lemma will be useful in the study of condition (Y) within the graph associated with the socle of a subshift algebra.

LEMMA 5.5. *let X be a subshift, $p \in X$ be such that $\{p\} \in \mathcal{Q}_X$, and E_p be the associated graph. Denote by E_p^1 the edge set of E_p and by E_p^∞ the set of all the infinite paths in E_p. There exists bijections $\psi : [p] \to E_p^\infty$ and $\varphi : [p] \to E_p^1$ such that*

(i) *for each $z \in [p]$, it holds that $\psi(z) = \varphi(z)\psi(\sigma(z))$.*
(ii) *for each $z \in [p]$ and each $n \in \mathbb{N}$, it holds that*

$$\psi(z) = \varphi(z)\varphi(\sigma(z))\varphi(\sigma^2(z))...\varphi(\sigma^n(z))\psi(\sigma^{n+1}(z)).$$

(iii) *for each $z \in [p]$, it holds that*

$$\psi(z) = \varphi(z)\varphi(\sigma(z))\varphi(\sigma^2(z))\varphi(\sigma^3(z))...,$$

that is, the edge in position n of the infinite path $\psi(z)$ is $\varphi(\sigma^n(z))$ for each $n \in \mathbb{N}$.

Proof. We begin defining the maps ψ and φ. Let I and J be as at the beginning of §5. Recall that I is the index set of J, so that $J = \{z_i : i \in I\}$ and $E_p^1 = \{e_i : i \in I\}$. By proposition 5.4, we have that $[p] = J$ and hence $[p] = \{z_i : i \in I\}$. So, the map $\varphi : [p] \to E_p^1$ defined by $\varphi(z_i) = e_i$ is a bijection. Now we define ψ. From the definition of the source and range maps of E_p (see remark 5.1), we get that $s(\varphi(z_i)) = v_i$ for each $z_i \in [p]$ and $r(\varphi(z_i)) = s(\varphi(\sigma(z_i)))$. Moreover, from the definition of E_p, we have that each vertex v_i of E_p emits a unique edge, which is e_i, and so v_i is the source of a unique infinite path in E_p (the path beginning with e_i). Hence, we get a bijective map $\psi : [p] \to E_p^\infty$, defined by $\psi(z_i) = \xi_i$, where ξ_i is the unique infinite path in E_p with $s(\xi_i) = v_i$ (or, equivalently, ξ_i is the unique infinite path in E_p beginning with e_i).

Now we prove the first item. Let $z_i \in [p]$, so that $\varphi(z_i) = e_i$. Denote $\xi = \psi(z_i)$, that is, ξ is the unique infinite path in E_p beginning in the edge e_i. Write $z_i = a_i z_j$, where $a_i \in \mathscr{A}$ and $z_j \in X$. Notice that from the definition of E_p, we have that $r(e_i) = v_j$ and v_j is the source of a unique edge, which is e_j. Therefore, the second edge of ξ is e_j. Write $\xi = e_i e_j \widetilde{\xi}$. From the definition of ψ, we get that $\psi(z_j)$ is

the unique infinite path of E_p beginning with e_j, and since $e_j\widetilde{\xi}$ is an infinite path beginning with e_j, we conclude that $\psi(z_j) = e_j\widetilde{\xi}$. Then,

$$\psi(z_i) = \xi = e_i e_j\widetilde{\xi} = \varphi(z_i)\psi(z_j) = \varphi(z_i)\psi(\sigma(z_i)),$$

which proves the first item.

The second and third items follow by applying successively the first one. $\qquad\square$

Notice that, by [1, remark 2.1.2], the ideal generated by a projection associated with an irrational element $\{p\}$ is graded. Next, we prove that this ideal is $\mathbb{Z}-$graded isomorphic to the Leavitt path algebra of the graph E_p. We refer the reader to [1] for the concepts regarding Leavitt path algebras.

PROPOSITION 5.6. *Let* X *be a subshift, let* $D = \{p\} \in \mathcal{Q}_\mathsf{X}$, *and let* E_p *be the associated graph as above. Then,* $\langle p_D \rangle$ *and the Leavitt path algebra* $L_R(E_p)$ *are* $\mathbb{Z}$*-graded isomorphic (with its natural* $\mathbb{Z}$*-gradings).*

Proof. To obtain a homomorphism $\varphi : L_R(E_p) \to \widetilde{\mathcal{A}}_R(\mathsf{X})$, we use the universal property of $L_R(E_p)$. So, it is enough to define φ on $E^0 \cup E^1$ in a way such that the images of $E^0 \cup E^1$ satisfy the relations defining $L_R(E_p)$.

We begin with the definition of φ on E^0. Let $v_i \in E^0$, where $i \in I$, and let z_i be the associated element in J. Define $\varphi(v_i) = p_{\{z_i\}}$. Notice that from proposition 5.4, we have $\{z_i\} \in \mathcal{U}$, and hence, $\varphi(v_i)$ is well-defined. Next, for each $e_i \in E^1$, let $z_i \in J$ be the element corresponding to i, and let $a_i \in \mathscr{A}$ be the first letter of z_i. Define $\varphi(e_i) = p_{\{z_i\}}s_{a_i}$, and $\varphi(e_i^*) = s_{a_i}^* p_{\{z_i\}}$.

Now we verify that $\{\varphi(v_i), \varphi(e_i), \varphi(e_i^*) : i \in I\}$ satisfy the relations that define $L_R(E_p)$, see [1, definition 1.2.3]. First, notice that $\varphi(v_i)$ is idempotent for each $v_i \in E^0$. Moreover, for $i \neq j$, we have that $\varphi(v_i)\varphi(v_j) = p_{\{z_i\}}p_{\{z_j\}} = 0$ since $z_i \neq z_j$. Next, for a fixed $i \in I$, observe that

$$\varphi(s(e_i))\varphi(e_i) = \varphi(v_i)\varphi(e_i) = p_{\{z_i\}}p_{\{z_i\}}s_{a_i} = p_{\{z_i\}}s_{a_i} = \varphi(e_i).$$

Moreover, for this i, let $z_i \in J$ be the element associated with i and write $z_i = a_i z_j$, where a_i is the first letter of z_i. Recall that $r(e_i) = v_j$. Then, $r(\{z_i\}, a_i) = \{z_j\}$, and hence,

$$\varphi(e_i)\varphi(r(e_i)) = \varphi(e_i)\varphi(v_j) = p_{\{z_i\}}s_{a_i}p_{\{z_j\}} = s_{a_i}p_{r(\{z_i\},a_i)}p_{\{z_j\}} = p_{\{z_i\}}s_{a_i} = \varphi(e_i).$$

So, we proved that

$$\varphi(s(e_i))\varphi(e_i) = \varphi(e_i) = \varphi(e_i)\varphi(r(e_i)).$$

Similarly, the reader can check that $\varphi(e_i^*) = \varphi(e_i^*)\varphi(s(e_i)) = \varphi(e_i^*) = \varphi(r(e_i))\varphi(e_i^*)$.

Next, for $i, j \in I$ with $i \neq j$, observe that

$$\varphi(e_i^*)\varphi(e_j) = s_{a_i}^* p_{\{z_i\}} p_{\{z_j\}} s_{a_j} = 0$$

since $p_{\{z_i\}} p_{\{z_j\}} = 0$. Moreover, to conclude that $\varphi(e_i^*)\varphi(e_j) = \delta_{i,j}\varphi(r(e_i))$, write $z_i = a_i z_j$ as in proposition 5.4 and notice that

$$\varphi(e_i^*)\varphi(e_i) = s_{a_i}^* p_{\{z_i\}} s_{a_i} = s_{a_i}^* s_{a_i} p_{r(\{z_i\}, a_i)} = p_{F_{a_i}} p_{\{z_j\}} = p_{\{z_j\}} = \varphi(r(e_i)).$$

Finally, for each $i \in I$ (write $z_i = a_i z_j$ as in proposition 5.4), notice that $s^{-1}(v_i) = e_i$, and so

$$\sum_{e \in s^{-1}(v_i)} \varphi(s_e)\varphi(s_e^*) = \varphi(e_i)\varphi(e_i^*) = p_{\{z_i\}} s_{a_i} s_{a_i}^* p_{\{z_i\}} = p_{\{z_i\}} p_{Z_{a_i}} p_{\{z_i\}} = p_{\{z_i\}} = p_{v_i}.$$

We have checked that the image of φ satisfies all the relations defining $L_R(E_p)$. By the universal property of $L_R(E_p)$, we have that φ extends to a homomorphism, which we also call $\varphi : L_R(E_p) \mapsto \mathcal{A}_R(\mathsf{X})$. By the definition of φ on the generators, we obtain that it is a graded homomorphism.

To finish our proof, it remains to show that φ is injective and that $\varphi(L_R(E_p)) = \langle p_D \rangle$.

By remark 5.2, E_p has no closed paths, and hence, it satisfies (vacuously) condition (L). Since $\varphi(v_i) = p_{\{z_i\}} \neq 0$ for each $v_i \in E^0$, we obtain from Cuntz–Krieger Uniqueness Theorem, see [1, theorem 2.2.15], that φ is injective.

It remains to show that $\varphi(L_R(E_p)) = \langle p_D \rangle$. From proposition 5.4, each $z_i \in J$ is an element of $[p]$, and then from proposition 4.8, we get that $p_{\{z_i\}} \in \langle p_D \rangle$. Therefore, $\varphi(e_i), \varphi(e_i^*)$, and $\varphi(v_i)$ are all elements of $\langle p_D \rangle$ for all $i \in I$, and consequently, $\varphi(L_R(E_p)) \subseteq \langle p_D \rangle$.

To prove that $\langle p_D \rangle \subseteq \varphi(L_R(E_p))$, we first show the following claim.

Claim: Let $\alpha z \in [p]$, where $\alpha \in \mathcal{L}_\mathsf{X}$ and $z \in \mathsf{X}$. Then, $p_{\{\alpha z\}} s_\alpha$ and $s_\alpha^* p_{\{\alpha z\}}$ both belong to $\varphi(L_R(E_p))$.

We show that $p_{\{\alpha z\}} s_\alpha \in \varphi(L_R(E_p))$ and leave the proof that $s_\alpha^* p_{\{\alpha z\}} \in \varphi(L_R(E_p))$, which is analogous to the reader.

Since $\alpha z \in [p] = J$, we have that $\alpha z = z_{i_1}$ for some $i_1 \in I$. From the definition of φ, we obtain that $\varphi(e_{i_1}) = p_{\{z_{i_1}\}} s_{\alpha_1}$, where α_1 is the first letter of α. Notice that $\sigma(\alpha z) \in J$, so that $\sigma(\alpha z) = z_{i_2}$ for some $i_2 \in I$. From the definition of φ, we get that $\varphi(e_{i_2}) = p_{\{z_{i_2}\}} s_{\alpha_2}$. Proceeding inductively, we obtain indexes $i_k \in I$, with $k \in \{1, 2, ..., |\alpha|\}$, such that $\varphi(e_{i_k}) = p_{\{z_{i_k}\}} s_{\alpha_{i_k}}$.

From the second item of lemma 2.3, we get that $s_{\alpha_k} p_{\{z_{i_{k+1}}\}} = p_{\{z_{i_k}\}} s_{\alpha_k}$ for each $k \in \{1, ..., |\alpha| - 1\}$. Hence, making the proper computations, we obtain that

$$\varphi(e_{i_1})\varphi(e_{i_2})...\varphi(e_{i_{|\alpha|}}) = p_{\{\alpha z\}} s_{\alpha_1}...s_{\alpha_{|\alpha|}} = p_{\{\alpha z\}} s_\alpha.$$

Therefore, $p_{\{\alpha z\}} s_\alpha \in \varphi(L_R(E_p))$ and the claim is proved.

Now we show that $\varphi(L_R(E_p)) = \langle p_D \rangle$.

Let $\alpha, \beta \in \mathcal{L}_{\mathsf{X}}$ and $A \in \mathcal{U}$ be such that $p_D s_\alpha p_A s_\beta^* \neq 0$. From the second item of lemma 4.1, there exists $y \in \mathsf{X}$ such that $p = \alpha y$, $\{\beta y\} \in \mathcal{U}$, and moreover, $p_D s_\alpha p_A s_\beta^* = s_\alpha s_\beta^* p_{\{\beta y\}}$. Applying the second item of lemma 2.3, we get that $p_{\{\alpha y\}} s_\alpha = s_\alpha p_{\{y\}}$ and $s_\beta^* p_{\{\beta y\}} = p_{\{y\}} s_\beta^*$, and therefore,

$$p_{\{\alpha y\}} s_\alpha s_\beta^* p_{\{\beta y\}} = s_\alpha s_\beta^* p_{\{\beta y\}}.$$

It follows from the claim proved above that the element $p_{\{\alpha y\}} s_\alpha s_\beta^* p_{\{\beta y\}}$ belongs to $\varphi(L_R(E_p))$, and so $p_D s_\alpha p_A s_\beta^* \in \varphi(L_R(E_p))$.

Similarly one shows that $s_\alpha p_A s_\beta^* p_D \in \varphi(L_R(E_p))$ for all $\alpha, \beta \in \mathcal{L}_{\mathsf{X}}$ and $A \in \mathcal{U}$. Consequently, we obtain that $\langle p_D \rangle \subseteq \varphi(L_R(E_p))$, and this finishes the proof of the proposition. $\qquad\square$

COROLLARY 5.7. *Let X be a subshift. Then, there exists a graph E such that $L_R(E)$ and $Soc(\widetilde{\mathcal{A}}_R(\mathsf{X}))$ are $\mathbb{Z}$-graded isomorphic (with its natural $\mathbb{Z}$-gradings).*

Proof. For each $[p] \in \mathcal{Q}_{\mathsf{X}}/\sim$, let E_p represent the corresponding graph (ensuring that E_p and E_q are disconnected for distinct equivalence classes $[p]$ and $[q]$). Define E as the disjoint union of all such graphs E_p and observe that

$$L_R(E) = \bigoplus_{[p] \in \mathcal{Q}_{\mathsf{X}}/\sim} L_R(E_p).$$

For each $[p] \in \mathcal{Q}_{\mathsf{X}}/\sim$, proposition 5.6 guarantees the existence of an isomorphism $\varphi_p : L_R(E_p) \to \langle [p] \rangle$. From corollary 4.9, we get that $Soc(\widetilde{\mathcal{A}}_R(\mathsf{X})) = \bigoplus_{[p] \in \mathcal{Q}_{\mathsf{X}}/\sim} \langle [p] \rangle$. So, $\varphi : L_R(E) \to Soc(\widetilde{\mathcal{A}}_R(\mathsf{X}))$ defined by $\varphi = \bigoplus_{[p] \in \mathcal{Q}_{\mathsf{X}}/\sim} \varphi_p$ is an isomorphism. $\qquad\square$

REMARK 5.8. Notice that the graph associated with a subshift described in corollary 5.7 is always acyclic, row-finite, and sinkless. Since the graph is acyclic, corollary 5.7 and [2] imply that the socle is locally K-matricial, that is, it is the direct union of subalgebras, each isomorphic to a finite direct sum of finite matrix rings over the field R.

Moreover, for each $p \in \mathcal{Q}_{\mathsf{X}}$, the Leavitt path algebra $L_R(E_p)$ is simple, since E_p satisfies condition (L) and there are no proper saturated and hereditary subsets of E_p^0. From the proof of the previous corollary, we get that $L_R(E)$ is semisimple, and since $Soc(\widetilde{\mathcal{A}}_R(\mathsf{X}))$ and $L_R(E)$ are isomorphic, we conclude that $Soc(\widetilde{\mathcal{A}}_R(\mathsf{X}))$ is semisimple.

Using the description of the socle of a subshift algebra as the Leavitt path algebra of the associated graph given above, we provide next a criterion to determine when the socle is strongly graded (see [26] for a study of strongly $\mathbb{Z}$-graded Leavitt path algebras). This criterion depends on condition (Y) so, for the reader's convenience, we first recall this definition.

DEFINITION 5.9. see [16, 21]   *A graph E satisfies condition (Y) if for each infinite path $e_1 e_2 e_3 \ldots$ there exists a finite path α and $k \in \mathbb{N}$ such that $s(e_{k+1}) = r(\alpha)$ and $|\alpha| = k + 1$.*

PROPOSITION 5.10. *Let* X *be a subshift. The socle of* $\widetilde{\mathcal{A}}_R(\mathsf{X})$ *is strongly* $\mathbb{Z}$-*graded (with its natural* $\mathbb{Z}-$*grading) if and only if the associated graph* E *given in corollary 5.7 satisfies condition (Y).*

Proof. This follows from corollary 5.7, remark 5.8, and [16, theorem 4.2] (or [21, theorem 3.9]). □

REMARK 5.11. Let X be a subshift and let E be the graph as in corollary 5.7, that is, E is the disjoint union of all the graphs E_p, where $[p] \in \mathcal{Q}_{\mathsf{X}}/\sim$. Then, E satisfies condition (Y) if and only if each E_p satisfies condition (Y).

Next, we characterize the condition (Y) of E in terms of the shift map σ of X. Let $W \subseteq \mathsf{X}$ be the set $W = \{p \in \mathsf{X} : \{p\} \in \mathcal{Q}_{\mathsf{X}}\}$. From the first item of lemma 5.3, we get that $\sigma(W) \subseteq W$, and from the second one, we get that each $z \in \sigma^{-1}(x)$ is also an element of W for each $x \in W$.

PROPOSITION 5.12. *Let* X *be a subshift and let* E *be the associated graph as in corollary 5.7. Then,* E *satisfies condition (Y) if and only if for each* $q \in W$, *there exists an* $n \in \mathbb{N}$ *such that* $\sigma^{-(n+1)}(\sigma^n(q)) \neq \emptyset$.

Proof. First, suppose that for each $q \in W$, there exists an $n \in \mathbb{N}$ such that $\sigma^{-(n+1)}(\sigma^n(q)) \neq \emptyset$. By remark 5.11, it is enough to prove that, for each $[p] \in \mathcal{Q}_{\mathsf{X}}/\sim$, the graph E_p satisfies condition (Y). Fix a $[p] \in \mathcal{Q}_{\mathsf{X}}/\sim$. Let ξ be an infinite path in the graph E_p, ψ be the map defined in lemma 5.5, and $z \in [p]$ be such that $\psi(z) = \xi$. Furthermore, let $m \in \mathbb{N}$ be such that $\sigma^{-(m+1)}(\sigma^m(z)) \neq \emptyset$, and chose $y \in \sigma^{-(m+1)}(\sigma^m(z))$. Then, $\sigma^{m+1}(y) = \sigma^m(z)$. Denote this last element by x. So, we get that $y = ax$ and $z = bx$, where $a, b \in \mathcal{L}_{\mathsf{X}}$ are such that $|a| = m + 1$ and $|b| = m$. Now let $\eta = \psi(y)$. By the second item of lemma 5.5, we get that

$$\xi = \psi(z) = \varphi(z)\varphi(\sigma(z))...\varphi(\sigma^{m-1}(z))\psi(\sigma^m(z)),$$

that is, ξ has the form $\xi = \alpha\psi(\sigma^m(z))$, where α is the finite path

$$\alpha = \varphi(z)\varphi(\sigma(z))\varphi(\sigma^2(z))...\varphi(\sigma^{m-1}(z)).$$

Similarly, η has the form $\eta = \beta\psi(\sigma^{m+1}(y))$, where

$$\beta = \varphi(y)\varphi(\sigma(y))\varphi(\sigma^2(y))...\varphi(\sigma^m(y)).$$

Since $\sigma^{m+1}(y) = \sigma^m(z)$, we obtain that $\eta = \beta\psi(\sigma^m(z))$. So, we have proved that $\xi = \alpha\psi(\sigma^m(z))$ and that there exists an infinite path η of the form $\eta = \beta\psi(\sigma^m(z))$ with $|\beta| = |\alpha| + 1$. This proves that E_p satisfies condition (Y).

For the converse, suppose that E satisfies condition (Y), and let $q \in W$. Let ψ be as in lemma 5.5, and define $\xi = \psi(q)$, which is an infinite path in E_q. Given that E satisfies condition (Y), it follows that E_q also satisfies condition (Y). Consequently, there exists an infinite path η in E_q such that $\eta = \alpha\mu$ and $\xi = \beta\mu$, where μ is an infinite path in E_q and α, β are finite paths in E_q with $|\alpha| = |\beta| + 1$. Let $y \in [q]$ be

such that $\psi(y) = \eta$ and write $m = |\beta|$. By the second item of lemma 5.5, we get that

$$\beta\mu = \xi = \psi(q) = \varphi(q)\varphi(\sigma(q))\varphi(\sigma^2(q))...\varphi(\sigma^{m-1}(q))\psi(\sigma^m(q)),$$

from where we conclude that $\mu = \psi(\sigma^m(q))$. Similarly, from

$$\alpha\mu = \eta = \psi(y) = \varphi(y)\varphi(\sigma(y))\varphi(\sigma^2(y))...\varphi(\sigma^m(y))\psi(\sigma^{m+1}(y)),$$

we obtain that $\mu = \psi(\sigma^{m+1}(y))$. Therefore, $\psi(\sigma^{m+1}(y)) = \mu = \psi(\sigma^m(q))$, and since ψ is injective, we have that $\sigma^{m+1}(y) = \sigma^m(q)$. This last equality means that $y \in \sigma^{-(m+1)}(\sigma^m(q))$, and so we are done. $\qquad\square$

REMARK 5.13. If $\sigma_{|W} : W \to W$ is surjective then, by the proposition above, E satisfies condition (Y). However, surjectivity of σ is not necessary, as can be seen in the following example: consider the subshift X over the alphabet $\mathscr{A} = \{0, 1, 2\}$ determined by the set of forbidden words $F = \{00, 10, 20\}$. The associated graph satisfies condition (Y) but $\sigma_{|W}$ is not onto.

As an application of proposition 5.10, we show how to use the graded structure of the socle as an invariant of conjugacy between Ott–Tomforde–Willis subshifts.

THEOREM 5.14 *Let X_1 and X_2 be subshifts and E_i, $i = 1, 2$, be the associated graphs as in corollary 5.7. If E_1 satisfies condition (Y) and E_2 does not, then the associated OTW-subshifts are not conjugate.*

Proof. Suppose that the OTW-subshifts are conjugate. Then, by [11, theorem 7.6], there is a graded isomorphism between the associated subshift algebras, which induces a graded isomorphism between the socle of both algebras. By proposition 5.10, the socle of one subshift algebra is strongly graded, while the other is not. Since a graded isomorphism preserves the strong graded structure of an algebra, we obtain a contradiction. $\qquad\square$

REMARK 5.15. Under the hypothesis of the theorem above, using [9, theorem 6.11], we conclude that the associated subshifts built using the $\frac{1}{2^i}$ metric (which induces the product topology) are not isometrically conjugate. Moreover, for finite alphabets, OTW-subshifts coincide with the usual notion of subshifts (with the product topology). Hence, the result above also provides an invariant for the conjugacy of subshifts (with the product topology) over finite alphabets.

To illustrate an application of the theorem above, we provide two examples of subshifts. For one of them, condition (Y) is satisfied for the associated graph, while for the other, it is not. We begin with a subshift that induces a 2-step Ott–Tomforde–Willis subshift not conjugate to any 1-step Ott–Tomforde–Willis subshift. In this case, the graph does not satisfy condition (Y).

EXAMPLE 5.16. Let $\mathbb{N}^*$ be the alphabet and define P as the following subset of $(\mathbb{N}^*)^3$:

$$P := \{(1, j, 1), (j, 1, j) : j \in \mathbb{N}^* \setminus \{2\}\} \cup \{(j, j+1, j+2) : j \in \mathbb{N}^*\}.$$

Let $F = (\mathbb{N}^*)^3 \setminus P$ and $\mathsf{X}_F := \mathsf{X}$. By [19, §3], the Ott–Tomforde–Willis subshift associated with F is not conjugate to any 1-step Ott–Tomforde–Willis subshift. Notice that

$$\mathsf{X} := \{(1j)^\infty, (j1)^\infty : j \in \mathbb{N}^* \setminus \{2\}\} \cup \{(k+j)_{j \in \mathbb{N}^*} : k \in \mathbb{N}^*\}.$$

We show that $Soc(\widetilde{\mathcal{A}}_R(\mathsf{X}))$ is R-isomorphic to $M_\infty(R)$, the algebra of infinite matrices with finitely many non-zero entries.

The set of the irrational paths, in the subshift X, is $\mathcal{Q}_\mathsf{X} = \{\{(k+j)_{j \in \mathbb{N}}\} : k \in \mathbb{N}^*\}$. Let $p = 12345....$ Then, according to the equivalence relation given in definition 4.7, all the elements of $\mathcal{Q}_\mathsf{X}$ are equivalent to $A = \{p\}$. By corollary 4.9, we conclude that

$$Soc(\widetilde{\mathcal{A}}_R(\mathsf{X})) = \langle p_A \rangle.$$

For the element $p = 1234...$, the associated graph E_p is the graph below.
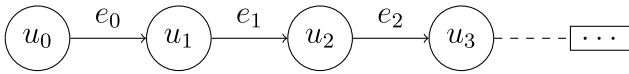
Given that $L_R(E_p)$ is isomorphic to $M_\infty(R)$ and, by proposition 5.6, $L_R(E_p)$ and $\langle p_A \rangle$ are isomorphic, it follows that $Soc(\widetilde{\mathcal{A}}_R(\mathsf{X}))$ and $M_\infty(R)$ are isomorphic.

The next example exhibits a subshift such that the graph associated with the socle satisfies condition (Y).

EXAMPLE 5.17. Let

$$\mathsf{X} := \{(k+j)_{j \in \mathbb{N}^*} : k \in \mathbb{Z}\}.$$

The set of the irrational paths is $\mathcal{Q}_\mathsf{X} = \{(k+j)_{j \in \mathbb{N}^*} : k \in \mathbb{Z}\}$. Let $p = 12345....$ As before, all the elements of $\mathcal{Q}_\mathsf{X}$ are equivalent to $A = \{p\}$, and hence, by corollary 4.9, we have that $Soc(\widetilde{\mathcal{A}}_R(\mathsf{X})) = \langle p_A \rangle$.

To construct the graph associated with X, we chose $I_n = \{-n : n \in \mathbb{N}^*\}$ and $I = \mathbb{Z}$. The graph E_p is depicted below.
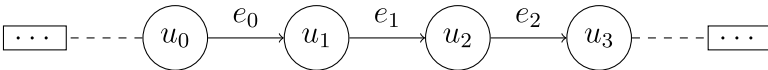
The Leavitt path algebra of the graph above is isomorphic to $M_\infty(R)$ and so is $Soc(\widetilde{\mathcal{A}}_R(\mathsf{X}))$. Moreover, by proposition 5.10, $Soc(\widetilde{\mathcal{A}}_R(\mathsf{X}))$ is strongly graded.

Since one of the graphs of the examples above satisfies condition (Y) and the other does not, we obtain that the corresponding OTW-subshifts are not conjugate, as stated below.

COROLLARY 5.18. *The Ott–Tomforde–Willis subshifts associated with the subshifts of Examples 5.16 and 5.17 are not conjugate. Moreover, there is no isometric conjugacy between the associated subshifts built using the $\frac{1}{2^i}$ metric (which induces the product topology).*

Proof. This follows directly from theorem 5.14 and remark 5.15. □

6. The relation between the algebraic and the C*-socle

We conclude the article with a concise examination of the socle of the C*-algebra associated with a subshift, aiming to describe the relation between the socles in both analytical and purely algebraic contexts. In the analytical context, the socle is also defined as the sum of all left minimal ideals (it is important to note that these ideals are not necessarily closed). Below, we recall the definition of the C*-algebra associated with a subshift over an arbitrary alphabet, as presented in [9].

DEFINITION 6.1. *Let X be a subshift. We define $\widetilde{\mathcal{O}}_{\mathsf{X}}$ as the universal unital C*-algebra generated by projections $\{p_A : A \in \mathcal{U}\}$ and partial isometries $\{s_a : a \in \mathscr{A}\}$ subject to the relations:*

(i) *$p_{\mathsf{X}} = 1$, $p_{A \cap B} = p_A p_B$, $p_{A \cup B} = p_A + p_B - p_{A \cap B}$, and $p_\emptyset = 0$ for every $A, B \in \mathcal{U}$;*

(ii) *$s_\beta s_\alpha^* s_\alpha s_\beta^* = p_{C(\alpha, \beta)}$ for all $\alpha, \beta \in \mathcal{L}_{\mathsf{X}}$, where $s_\omega := 1$, and for $\alpha = \alpha_1 \ldots \alpha_n \in \mathcal{L}_{\mathsf{X}}$, $s_\alpha := s_{\alpha_1} \cdots s_{\alpha_n}$ and $s_\alpha^* := s_{\alpha_n}^* \cdots s_{\alpha_1}^*$.*

REMARK 6.2. It is shown in [9] that the subshift algebra $\widetilde{\mathcal{A}}_R(\mathsf{X})$ is embedded densely in $\widetilde{\mathcal{O}}_{\mathsf{X}}$ via a homomorphism that sends generators of $\widetilde{\mathcal{A}}_R(\mathsf{X})$ to generators of $\widetilde{\mathcal{O}}_{\mathsf{X}}$.

Next, we show that the socle of $\widetilde{\mathcal{A}}_R(\mathsf{X})$ is contained in the socle of $\widetilde{\mathcal{O}}_{\mathsf{X}}$ and give an example where the inclusion is proper. We will use the concept of minimal idempotent recalled below lemma 3.6. Taking into consideration that $\widetilde{\mathcal{A}}_R(\mathsf{X})$ and $\widetilde{\mathcal{O}}_{\mathsf{X}}$ are semiprime (see [7, corollary 5.6] for primeness of $\widetilde{\mathcal{A}}_R(\mathsf{X})$), we have the following result.

LEMMA 6.3. *Let X be a subshift, R be a field, and $A \in \mathcal{Q}_{\mathsf{X}}$. Then, p_A is a minimal idempotent in $\widetilde{\mathcal{A}}_{\mathbb{C}}(\mathsf{X})$ and also in $\widetilde{\mathcal{O}}_{\mathsf{X}}$.*

Proof. Consider an element $A \in \mathcal{Q}_{\mathsf{X}}$. From the proof of proposition 3.7, it follows that $p_A \widetilde{\mathcal{A}}_{\mathbb{C}}(\mathsf{X}) p_A = \mathbb{C} p_A$, implying that p_A is a minimal idempotent in $\widetilde{\mathcal{A}}_{\mathbb{C}}(\mathsf{X})$. Furthermore, by remark 6.2, we deduce that the closure of $p_A \widetilde{\mathcal{A}}_{\mathbb{C}}(\mathsf{X}) p_A$ in $\widetilde{\mathcal{O}}_{\mathsf{X}}$, denoted by $\overline{p_A \widetilde{\mathcal{A}}_{\mathbb{C}}(\mathsf{X}) p_A}$, equals $p_A \widetilde{\mathcal{O}}_{\mathsf{X}} p_A$. However, as $p_A \widetilde{\mathcal{A}}_R(\mathsf{X}) p_A = \mathbb{C} p_A$, we conclude that

$$\mathbb{C} p_A = \overline{\mathbb{C} p_A} = \overline{p_A \widetilde{\mathcal{A}}_{\mathbb{C}}(\mathsf{X}) p_A} = p_A \widetilde{\mathcal{O}}_{\mathsf{X}} p_A.$$

Hence, $p_A \widetilde{\mathcal{O}}_{\mathsf{X}} p_A = \mathbb{C} p_A$, and therefore, p_A is also a minimal idempotent in $\widetilde{\mathcal{O}}_{\mathsf{X}}$. □

PROPOSITION 6.4. *Let* X *be a subshift. Then,*

$$Soc(\widetilde{\mathcal{A}}_{\mathbb{C}}(\mathsf{X})) \subseteq Soc(\widetilde{\mathcal{O}}_{\mathsf{X}}).$$

Proof. According to corollary 4.3, $Soc(\widetilde{\mathcal{A}}_{\mathbb{C}}(\mathsf{X}))$ is the two-sided ideal of $\widetilde{\mathcal{A}}_{\mathbb{C}}(\mathsf{X})$ generated by the set $\{p_A : A \in \mathcal{Q}_{\mathsf{X}}\}$. To prove that $Soc(\widetilde{\mathcal{A}}_{\mathbb{C}}(\mathsf{X})) \subseteq Soc(\widetilde{\mathcal{O}}_{\mathsf{X}})$, since $Soc(\widetilde{\mathcal{O}}_{\mathsf{X}})$ is also a two-sided ideal, it is enough to show that $p_A \in Soc(\widetilde{\mathcal{O}}_{\mathsf{X}})$ for each $A \in \mathcal{Q}_{\mathsf{X}}$. By lemma 6.3, p_A is a minimal idempotent element in $\widetilde{\mathcal{O}}_{\mathsf{X}}$ for each $A \in \mathcal{Q}_{\mathsf{X}}$. Therefore, from [12, proposition 30.6], we obtain that $\widetilde{\mathcal{O}}_{\mathsf{X}} p_A$ is a minimal left ideal in $\widetilde{\mathcal{O}}_{\mathsf{X}}$ and so, in particular, $p_A \in Soc(\widetilde{\mathcal{O}}_{\mathsf{X}})$ for each $p_A \in \mathcal{Q}_{\mathsf{X}}$. Consequently, $Soc(\widetilde{\mathcal{A}}_{\mathbb{C}}(\mathsf{X})) \subseteq Soc(\widetilde{\mathcal{O}}_{\mathsf{X}})$. $\square$
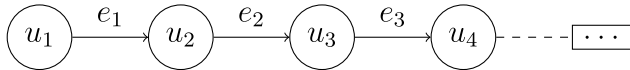
We conclude the article with an example illustrating that the inclusion $Soc(\widetilde{\mathcal{A}}_{\mathbb{C}}(\mathsf{X})) \subseteq Soc(\widetilde{\mathcal{O}}_{\mathsf{X}})$ may be strict.

EXAMPLE 6.5. Let X be the subshift of Example 5.16, that is,

$$\mathsf{X} := \{(1j)^{\infty}, (j1)^{\infty} : j \in \mathbb{N}^* \setminus \{2\}\} \cup \{(k+j)_{j \in \mathbb{N}^*} : k \in \mathbb{N}^*\}.$$

Let $p = 1234...$ and $A = \{p\}$.

The graph E_p associated with the element p is the graph

Let H be a separable Hilbert space, with an orthonormal Schauder basis $\{\delta_n : n \geq 1\}$, and let $K(H)$ denote the C*-algebra of all the compact operators in H. For each $n \geq 1$, define $S_{e_n}, P_{u_n} \in K(H)$ as follows: $S_{e_n}(\delta_{n+1}) = \delta_n$ and $S_{e_n}(\delta_k) = 0$ for each $k \neq n+1$, and $P_{u_n}(\delta_n) = \delta_n$ and $P_{u_n}(\delta_k) = 0$ for each $k \neq n$. Notice that the Hilbert adjoint operator $S_{e_n}^*$ of S_{e_n} is such that $S_{e_n}^*(\delta_n) = \delta_{n+1}$, and $S_{e_n}^*(\delta_k) = 0$ for $k \neq n$.

It is well known that there exits an injective homomorphism $\Phi : L_{\mathbb{C}}(E_p) \to K(H)$ such that $\Phi(e_n) = S_{e_n}$, $\Phi(e_n^*) = S_{e_n}^*$, and $\Phi(u_n) = P_{u_n}$. Moreover, this homomorphism extends to an isomorphism of C*-algebras $\Psi : C^*(E_p) \to K(H)$.

Now, let $\varphi : L_{\mathbb{C}}(E_p) \to \widetilde{\mathcal{A}}_{\mathbb{C}}(\mathsf{X})$ be as in proposition 5.6 (that is, φ is defined on the generators as follows: $\varphi(u_i) = p_{\{z_i\}}$, $\varphi(e_i) = p_{\{z_i\}} s_i$ and $\varphi(e_i^*) = s_i^* p_{\{z_i\}}$, where $z_i = i(i+1)(i+2)... \in \mathsf{X}$). By corollary 5.7, we have that $Im(\varphi) = Soc(\widetilde{\mathcal{A}}_R(\mathsf{X}))$. Let $B = \Phi(L_{\mathbb{C}}(E_p))$, so that $\varphi \circ \Phi^{-1} : B \to Soc(\widetilde{\mathcal{A}}_R(\mathsf{X}))$ is an isomorphism. Through direct calculations, we obtain that the matrix associated with each element of B, relative to the basis $\delta_n : n \geq 1$, belongs to $M_{\infty}(\mathbb{C})$. Furthermore, for every element $T \in M_{\infty}(\mathbb{C})$, there exists an element in B whose associated matrix is precisely T. Consequently, there exists an isomorphism between B and $M_{\infty}(\mathbb{C})$, implying that $Soc(\widetilde{\mathcal{A}}_R(\mathsf{X}))$ is isomorphic to $M_{\infty}(\mathbb{C})$.

Recall that φ is injective, and hence, it extends to an (injective) homomorphism $\psi : C^*(E_p) \to \widetilde{\mathcal{O}}_{\mathsf{X}}$. So, we get an injective homomorphism $f : K(H) \to \widetilde{\mathcal{O}}_{\mathsf{X}}$ defined by $f = \psi \circ \Psi^{-1}$. Notice that $f(B) = Soc(\widetilde{\mathcal{A}}_R(\mathsf{X}))$.

Next, we show that $Soc(\widetilde{\mathcal{A}}_R(\mathsf{X}))$ is strictly contained in $Soc(\widetilde{\mathcal{O}}_\mathsf{X})$. To prove this, we show that there exists an element $L \in K(H)$ such that $f(L) \in Soc(\widetilde{\mathcal{O}}_\mathsf{X})$ and L has the property that its associated matrix, relative to the basis $\{\delta_n : n \geq 1\}$, does not belong to $M_\infty(\mathbb{C})$.

Define, for each $x, y \in H$, the operator $L_{x,y} \in K(H)$ by $L_{x,y}(z) = x\langle z, y\rangle$. Through a few straightforward computations, we obtain that $L_{\delta_i, \delta_j} = \Psi(e_i e_{i+1}...e_{j-1})$ for each $i < j$, that $L_{\delta_i, \delta_i} = \Psi(u_i)$, and that $L_{\delta_i, \delta_j} = \Psi(e_{i-1}^* e_{i-2}^*...e_j^*)$ if $i > j$.

Notice that $f(L_{\delta_1, \delta_j}) = p_{z_1} s_1...s_{j-1}$ for each $j > 1$, and that $f(L_{\delta_1, \delta_1}) = p_{\{z_1\}}$. So, $f(L_{\delta_1, \delta_j}) \in p_{\{z_1\}}\widetilde{\mathcal{O}}_\mathsf{X}$ for each $j \geq 1$. Now, let $h = \sum_{j=1}^{\infty} \frac{1}{j}\delta_j$, and for each $n \geq 1$, define $h_n = \sum_{j=1}^{n} \frac{1}{j}\delta_j$. Observe that $f(L_{\delta_1, h_n}) = \sum_{j=1}^{n} \frac{1}{j} f(L_{\delta_1, \delta_j}) \in p_{\{z_1\}}\widetilde{\mathcal{O}}_\mathsf{X}$. Since $h_n \to h$ (in H), we obtain that $L_{\delta_1, h_n} \to L_{\delta_1, h}$ in $K(H)$, so that $f(L_{\delta_1, h})$ belongs to the closure of $p_{\{z_1\}}\widetilde{\mathcal{O}}_\mathsf{X}$ (in $\widetilde{\mathcal{O}}_\mathsf{X}$). As the set $p_{\{z_1\}}\widetilde{\mathcal{O}}_\mathsf{X}$ is closed in $\widetilde{\mathcal{O}}_\mathsf{X}$, we conclude that $f(L_{\delta_1, h}) \in p_{\{z_1\}}\widetilde{\mathcal{O}}_\mathsf{X}$.

Now, by lemma 6.3, we get that $p_{\{z_1\}}$ is a minimal idempotent. Hence, from [12, proposition 30.6], we get that $p_{\{z_1\}} \in Soc(\widetilde{\mathcal{O}}_\mathsf{X})$. Since $Soc(\widetilde{\mathcal{O}}_\mathsf{X})$ is a two-sided ideal, we obtain that $p_{\{z_1\}}\widetilde{\mathcal{O}}_\mathsf{X} \subseteq Soc(\widetilde{\mathcal{O}}_\mathsf{X})$, and thus $f(L_{\delta_1, h}) \in Soc(\widetilde{\mathcal{O}}_\mathsf{X})$.

To finish the proof, notice that the matrix of $L_{\delta_1, h}$, relative to the basis $\{\delta_n : n \geq 1\}$, has infinitely many non-zero elements, and so, this matrix does not belong to $M_\infty(\mathbb{C})$. So, the element $L = L_{\delta_1, h}$ has the property that its associated matrix is not an element of $M_\infty(\mathbb{C})$, but $f(L) \in Soc(\widetilde{\mathcal{O}}_\mathsf{X})$. Therefore, $Soc(\widetilde{\mathcal{A}}_R(\mathsf{X}))$ (which is isomorphic to $M_\infty(\mathbb{C})$) is strictly contained in $Soc(\widetilde{\mathcal{O}}_\mathsf{X})$.

Funding

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