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GROUPS OF HEIGHT FOUR

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Abstract

If G and H are infinite groups then G is said to be larger than H ($H \preccurlyeq G$) if there are subgroups A of G, B of H, each of finite index, such that B is an epimorphic image of A. Pride (1979) showed that if G has finite 'height' with respect to the quasi-order $\preccurlyeq$ then there are only finitely many (classes of) minimal groups H with $H \preccurlyeq G$, and asked whether this were true without the minimality restriction on H. This paper gives a negative answer to his question by exhibiting a group G of height four with infinitely many (classes of) groups H satisfying $H \preccurlyeq G$.

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1. Introduction

Pride (1976) defined a quasi-order of the class of infinite groups as follows: If G and H are infinite groups, then G is *larger* than H ($H \preccurlyeq G$) if there are subgroups A of G, B of H, each of finite index, such that B is an epimorphic image of A. Then G and H are said to be *equally large* ($G \simeq H$) if each is larger than the other, and $\preccurlyeq$ induces a partial order on the $\simeq$ classes of infinite groups. Groups G lying in minimal classes under this partial order are said to have height one and, more generally, a group has height n if n is maximal such that there exists a sequence $G_1 \preccurlyeq G_2 \preccurlyeq \dots \preccurlyeq G_n = G$ of infinite groups no two of which are equally large.

Pride (1979) showed that if G has finite height then there are only finitely many (up to $\simeq$) height one groups H with $H \preccurlyeq G$. He then asked whether this result was still true without the minimality restriction on H. In this paper we give a negative answer to his question by exhibiting a group G of height four such that there are infinitely many (up to $\simeq$) H of height three with $H \preccurlyeq G$.

2. Construction

The group we construct will in fact be abelian. Hence, from now on, we assume all groups under consideration to be abelian and use additive notation.

Let $\mathbf{Z}$ and $\mathbf{Q}$ denote the additive groups of integers and rationals, respectively. Fix a prime p and let $p^{-\infty}\mathbf{Z}$ denote the subgroup of $\mathbf{Q}$ consisting of those fractions whose denominators are powers of p. Then the quasicyclic quotient group $p^{-\infty}\mathbf{Z}/\mathbf{Z}$ is usually denoted by $\mathbf{Z}(p^\infty)$. There is a natural action of the p-adic integers $\mathbf{Z}_p$ on $\mathbf{Z}(p^\infty)$ which we write as left multiplication. Let α, β be p-adic integers and define a map $\varphi_{\alpha,\beta}$ from $\mathbf{Z}(p^\infty)$ to $\mathbf{Z}(p^\infty) \oplus \mathbf{Z}(p^\infty) \oplus \mathbf{Z}(p^\infty)$ by $\varphi_{\alpha,\beta}(x) = x \oplus \alpha x \oplus \beta x$. Then $\varphi_{\alpha,\beta}(\mathbf{Z}(p^\infty))$ is a subgroup of $\mathbf{Z}(p^\infty) \oplus \mathbf{Z}(p^\infty) \oplus \mathbf{Z}(p^\infty)$ isomorphic to $\mathbf{Z}(p^\infty)$. But the quotient group $(p^{-\infty}\mathbf{Z} \oplus p^{-\infty}\mathbf{Z} \oplus p^{-\infty}\mathbf{Z})/(\mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z})$ is naturally isomorphic to $\mathbf{Z}(p^\infty) \oplus \mathbf{Z}(p^\infty) \oplus \mathbf{Z}(p^\infty)$. Hence, by the isomorphism theorems, $\varphi_{\alpha,\beta}(\mathbf{Z}(p^\infty))$ corresponds to a subgroup of $p^{-\infty}\mathbf{Z} \oplus p^{-\infty}\mathbf{Z} \oplus p^{-\infty}\mathbf{Z}$ containing $\mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z}$. We denote this subgroup by $G(\alpha, \beta)$. Then we have:

THEOREM. (a) *For all α, β in $\mathbf{Z}_p$, $G(\alpha, \beta)$ has height four.*

(b) *If α, β are algebraically independent over $\mathbf{Q}$ then there are infinitely many (up to $\simeq$) H of height three with $H \preccurlyeq G(\alpha, \beta)$.*

Before proceeding to the proof, observe that $G(\alpha, \beta)$ can alternatively be described as the set of all $x \oplus y \oplus z$ in $p^{-\infty}\mathbf{Z} \oplus p^{-\infty}\mathbf{Z} \oplus p^{-\infty}\mathbf{Z}$ such that $\alpha x - y$ and $\beta x - z$ are p-adic integers. Furthermore, if we denote by $G(\alpha)$ the subgroup of $p^{-\infty}\mathbf{Z} \oplus p^{-\infty}\mathbf{Z}$ obtained by projecting $G(\alpha, \beta)$ onto its first two 'coordinates', then $G(\alpha)$ is the set of all $x \oplus y$ such that $\alpha x - y$ is a p-adic integer. $G(\alpha)$ is, of course, the subgroup of $p^{-\infty}\mathbf{Z} \oplus p^{-\infty}\mathbf{Z}$ corresponding to the image of the map $\varphi_\alpha : \mathbf{Z}(p^\infty) \to \mathbf{Z}(p^\infty) \oplus \mathbf{Z}(p^\infty)$ given by $\varphi_\alpha(x) = x \oplus \alpha x$.

3. Proof

(a) It is easy to see that $G(\alpha, \beta)$ has height at least four from the sequence

$$\mathbf{Z}(p^\infty) \preccurlyeq p^{-\infty}\mathbf{Z} \preccurlyeq G(\alpha) \preccurlyeq G(\alpha, \beta).$$

Here we consider $G(\alpha, \beta)$ as mapped onto $G(\alpha)$ by projection onto its first two coordinates and $G(\alpha)$ as mapped onto $p^{-\infty}\mathbf{Z}$ by projection onto its first coordinate. Furthermore $\mathbf{Z}(p^\infty)$ is an epimorphic image of $p^{-\infty}\mathbf{Z}$. To see that no two of these groups are equally large note that their torsion-free ranks are 0, 1, 2 and 3 respectively, and that $H \preccurlyeq G$ with H, G abelian implies that the torsion-free rank of G is at least as big as that of H.

Now suppose that $G(\alpha, \beta)$ has height bigger than four, so that there is a sequence of infinite groups

$$G_1 \preccurlyeq G_2 \preccurlyeq G_3 \preccurlyeq G_4 \preccurlyeq G_5 = G(\alpha, \beta)$$

with no two equally large. Without loss of generality (by passing to subgroups of finite index) we may assume that each G_i is an epimorphic image of G_{i+1}. Now $G(\alpha, \beta)$ is an extension of a finitely generated group by a copy of $\mathbf{Z}(p^\infty)$. If we denote by $\mathscr{C}$ the class of abelian groups which are either finitely generated or an extension of a finitely generated group by a copy of $\mathbf{Z}(p^\infty)$, then it is routine (see Fuchs (1970, 1973) or Kaplansky (1969) for the basic theory of abelian groups) to check that:

(1) $\mathscr{C}$ is closed under subgroups and epimorphic images;

(2) a torsion group in $\mathscr{C}$ is either finite or the direct sum of a finite group and a copy of $\mathbf{Z}(p^\infty)$;

(3) an arbitrary group in $\mathscr{C}$ is either the direct sum of a finite group and a torsion-free group or the direct sum of a finite group, a copy of $\mathbf{Z}(p^\infty)$ and a finitely generated torsion-free group.

Hence, without loss of generality, we may further assume that each G_i is either torsion-free or the direct sum of a copy of $\mathbf{Z}(p^\infty)$ and a finitely generated torsion-free group. Now consider the kernels K_{i+1} of the epimorphisms from G_{i+1} onto G_i. Since G_5 has torsion-free rank three and, whenever K_{i+1} is not torsion, the torsion-free rank of G_i is strictly less than that of G_{i+1}, we must have K_{i+1} torsion for at least one i. Let i_0 be the biggest such i. Then G_{i_0+1} must be of the form $A \oplus B$ with A a copy of $\mathbf{Z}(p^\infty)$ and B finitely generated torsion-free. Hence $K_{i_0+1} = A$ (otherwise $G_{i_0+1}/K_{i_0+1} = G_{i_0}$ would be isomorphic to G_{i_0+1}), and G_{i_0} is isomorphic to B and hence is finitely generated torsion-free. This implies that G_i is finitely generated torsion-free for all $i \leqslant i_0$. Hence K_i is torsion-free for all $i \leqslant i_0$, so the torsion-free rank of G_1 must be zero, that is G_1 is torsion. But G_1 is also torsion-free, so we have obtained a contradiction (remember G_1 is infinite). This concludes the proof of (a).

(b) Let n be an integer. Then the automorphism $x \oplus y \oplus z \mapsto x \oplus (y + nz) \oplus z$ of $p^{-\infty}\mathbf{Z} \oplus p^{-\infty}\mathbf{Z} \oplus p^{-\infty}\mathbf{Z}$ is easily seen to restrict to an isomorphism from $G(\alpha, \beta)$ to $G(\alpha + n\beta, \beta)$. Hence, since $G(\alpha) \preccurlyeq G(\alpha, \beta)$, we have $G(\alpha + n\beta) \preccurlyeq G(\alpha, \beta)$ for all n. It remains to be shown that, when α and β are algebraically independent over $\mathbf{Q}$, $G(\alpha + m\beta)$ and $G(\alpha + n\beta)$ cannot be equally large unless $m = n$.

Now if G and H are torsion-free abelian groups of finite rank which are equally large then it must be the case that they have isomorphic subgroups of finite index. Then G and H are said to be *quasi-isomorphic*. A very special case of a result of Beaumont and Pierce (1961), Corollary 4.14, implies that, if $G(\alpha_1)$ and $G(\alpha_2)$ are quasi-isomorphic, then α_2 is a rational (in fact linear fractional) function of α_1. This can be seen directly as follows: If $G(\alpha_1)$ and $G(\alpha_2)$ have isomorphic subgroups

A_1 and A_2, where A_1 has index k in $G(\alpha_1)$, then the isomorphism restricts to an isomorphism from $kG(\alpha_1) \subseteq A_1$ to a subgroup of finite index in $G(\alpha_2)$. But $kG(\alpha_1)$ is isomorphic to $G(\alpha_1)$, so in fact there is an isomorphism φ from $G(\alpha_1)$ to a subgroup of $G(\alpha_2)$. This isomorphism φ extends to an automorphism of $\mathbf{Q} \oplus \mathbf{Q}$ (which is the divisible hull of $G(\alpha_1)$ and of $G(\alpha_2)$), so we have $\varphi(x \oplus y) = (ax+by) \oplus (cx+dy)$ for some a, b, c, d in $\mathbf{Q}$. Since φ carries $G(\alpha_1)$ into $G(\alpha_2)$ we conclude that, for all x, y in $p^{-\infty}\mathbf{Z}$ with $\alpha_1 x - y$ in $\mathbf{Z}_p$, it must be the case that

$$\alpha_2(ax+by)-(cx+dy) = (a\alpha_2-c)x-(-b\alpha_2+d)y$$

lies in $\mathbf{Z}_p$. This is easily seen to imply that $\alpha_1 = (a\alpha_2-c)/(-b\alpha_2+d)$.

If α and β are algebraically independent over $\mathbf{Q}$ then, when $m \neq n$, no such linear fractional relation can hold between $\alpha_1 = \alpha+m\beta$ and $\alpha_2 = \alpha+n\beta$. Hence $G(\alpha+m\beta)$ and $G(\alpha+n\beta)$ cannot be equally large, so the proof of (b) is finished.

4. Conclusion

Of the various questions suggested by the above example, perhaps the most natural is the following: Does there exist a group G of height three such that there are infinitely many (up to $\simeq$) H with $H \preccurlyeq G$? It can be shown that no abelian group G with these properties exists.

References

R. A. Beaumont and R. S. Pierce (1961), 'Torsion-free groups of rank two', *Mem. Amer. Math Soc.* **38**.

L. Fuchs (1970, 1973), *Infinite abelian groups*, Vols. I and II (Academic Press, New York).

I. Kaplansky (1969), *Infinite abelian groups* (University of Michigan Press, Ann Arbor, Michigan).

S. J. Pride (1976), 'The concept of "largeness" in group theory', *Proc. Conf. on Word and Decision Problems in Algebra and Group Theory, Oxford*, 1976 (to appear).

S. J. Pride (1979), 'On groups of finite height', *J. Austral. Math. Soc.* (*Series A*) **28**, 87–99.

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