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On higher direct images of convergent isocrystals

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Abstract

Let k be a perfect field of characteristic $p>0$ and let W be the ring of Witt vectors of k. In this article, we give a new proof of the Frobenius descent for convergent isocrystals on a variety over k relative to W. This proof allows us to deduce an analogue of the de Rham complexes comparison theorem of Berthelot [*𝒟-modules arithmétiques. II. Descente par Frobenius*, Mém. Soc. Math. Fr. (N.S.) **81** (2000)] without assuming a lifting of the Frobenius morphism. As an application, we prove a version of Berthelot's conjecture on the preservation of convergent isocrystals under the higher direct image by a smooth proper morphism of k-varieties.

1. Introduction

1.1 Let k be a perfect field of characteristic $p>0$. A good p-adic cohomology theory on a variety over k is the rigid cohomology developed by Berthelot [Ber86, Ber96a]. The coefficients for this theory are (over-)convergent F-isocrystals: they play a similar role of the lisse ℓ-adic sheaves in ℓ-adic cohomology. In [Ber86, 4.3] and [Tsu03], Berthelot and Tsuzuki conjectured that under a smooth proper morphism of varieties over k, the higher direct image of an (over-)convergent (F-)isocrystal is still an (over-)convergent (F-)isocrystal analogue to the ℓ-adic case. Various cases and variants of this conjecture have been proved by Tsuzuki [Tsu03], Shiho [Shi07b], Étesse [Éte12], Caro [Car15], etc. We refer to an article of Lazda [Laz16] for a survey of these results and the relation between them. The goal of this article is to prove a version of Berthelot's conjecture for convergent isocrystals in the context of convergent topos developed by Ogus (Theorem 1.9).

1.2 In [Ogu84, Ogu07], Ogus introduced a crystalline-like site: *convergent site*, and defined a *convergent isocrystal* as a crystal on this site. Let us briefly recall his definition.

Let W be the ring of Witt vectors of k, K its fraction field and X a scheme of finite type over k. We denote by $\mathrm{Conv}(X/\mathrm{W})$ the category of couples $(\mathfrak{T},u)$ consisting of an adic flat formal W-scheme of finite type $\mathfrak{T}$ and a k-morphism u from the reduced subscheme T_0 of the special fiber of $\mathfrak{T}$ to X. Morphisms are defined in a natural way. A family of morphisms $\{(\mathfrak{T}_i,u_i)\to(\mathfrak{T},u)\}_{i\in I}$ is a covering if $\{\mathfrak{T}_i\to\mathfrak{T}\}_{i\in I}$ is a Zariski covering.

The functor $(\mathfrak{T},u)\mapsto\Gamma(\mathfrak{T}_{\mathrm{zar}},\mathscr{O}_{\mathfrak{T}}[\frac{1}{p}])$ is a sheaf of rings that we denote by $\mathscr{O}_{X/K}$. An $\mathscr{O}_{X/K}$-module amounts to giving the following data:

(i) for every object $(\mathfrak{T},u)$ of $\mathrm{Conv}(X/\mathrm{W})$, an $\mathscr{O}_{\mathfrak{T}}[\frac{1}{p}]$-module $\mathscr{F}_{\mathfrak{T}}$ of $\mathfrak{T}_{\mathrm{zar}}$;

(ii) for every morphism $f:(\mathfrak{T}_1,u_1)\to(\mathfrak{T}_2,u_2)$ of $\mathrm{Conv}(X/\mathrm{W})$, an $\mathscr{O}_{\mathfrak{T}_1}$-linear morphism $c_f: f^*(\mathscr{F}_{\mathfrak{T}_2})\to\mathscr{F}_{\mathfrak{T}_1}$

satisfying a cocycle condition for the composition of morphisms as in [BO15, 5.1].

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A convergent isocrystal on $\mathrm{Conv}(X/\mathrm{W})$ is a coherent crystal of $\mathscr{O}_{X/K}$-modules $\mathscr{F}$ on $\mathrm{Conv}(X/\mathrm{W})$, i.e. for every object $(\mathfrak{T},u)$ of $\mathrm{Conv}(X/\mathrm{W})$, $\mathscr{F}_{\mathfrak{T}}$ is coherent and, for every morphism f of $\mathrm{Conv}(X/\mathrm{W})$, the transition morphism c_f is an isomorphism. We denote by $\mathrm{Iso}^{\dagger}(X/\mathrm{W})$ the category of convergent isocrystals on $\mathrm{Conv}(X/\mathrm{W})$. If X is smooth over k, there exists a canonical functor ι from $\mathrm{Iso}^{\dagger}(X/\mathrm{W})$ to the category of crystals of $\mathscr{O}_{X/\mathrm{W}}$-modules on the crystalline site $\mathrm{Crys}(X/\mathrm{W})$ up to isogeny [Ogu07, 0.7.2]. Its essential image satisfies certain convergent conditions (cf. [Ber96a, 2.2.14]).

1.3 In [Ogu84, 4.6], Ogus showed that the category $\mathrm{Iso}^{\dagger}(X/\mathrm{W})$ satisfies the descent property under a proper and surjective morphism of k-schemes. Then, if X' denotes the base change of X by the Frobenius morphism of k, the functorial morphism of convergent topoi induced by the relative Frobenius morphism $F_{X/k}: X\to X'$ gives an equivalence of categories:

$$F^{*}_{X/k,\mathrm{conv}}:\mathrm{Iso}^{\dagger}(X'/\mathrm{W})\xrightarrow{\sim}\mathrm{Iso}^{\dagger}(X/\mathrm{W}) \tag{1.3.1}$$

that we call *Frobenius descent.*

A *convergent F-isocrystal* on $\mathrm{Conv}(X/\mathrm{W})$ is a couple $(\mathscr{E},\varphi)$ of a convergent isocrystal $\mathscr{E}$ on $\mathrm{Conv}(X/\mathrm{W})$ and an isomorphism φ between $\mathscr{E}$ and its pullback via the absolute Frobenius morphism of X (cf. § 5.14 for a precise definition).

1.4 To study the higher direct image of convergent (F-)isocrystals, we need the notion of convergent topos over a p-adic base developed by Shiho [Shi02, Shi07a].[1] Let $\mathfrak{S}$ be an adic flat formal W-scheme of finite type, S_0 the reduced subscheme of its special fiber and X an S_0-scheme. We define the convergent site $\mathrm{Conv}(X/\mathfrak{S})$ of X relative to $\mathfrak{S}$ and the category $\mathrm{Iso}^{\dagger}(X/\mathfrak{S})$ of convergent isocrystals on $\mathrm{Conv}(X/\mathfrak{S})$ as in § 1.2 (cf. Definition 3.1 and § 3.16). Shiho generalized Ogus' proper surjective descent for convergent isocrystals in this setting [Shi07b, 7.3].

We denote by $(X/\mathfrak{S})_{\mathrm{conv,fppf}}$ the topos of fppf sheaves on the category $\mathrm{Conv}(X/\mathfrak{S})$ (§ 3.4). As a first step towards Berthelot's conjecture, we show the following result.

Theorem 1.5 (Theorem 5.2). *Suppose that the Frobenius morphism $F_{S_0}: S_0\to S_0$ is flat. Let X be an S_0-scheme locally of finite type, $X' = X\times_{S_0,F_{S_0}} S_0$ and $F_{X/S_0}: X\to X'$ the relative Frobenius morphism. The functorial morphism of topoi $F_{X/S_0,\mathrm{conv}}:(X/\mathfrak{S})_{\mathrm{conv,fppf}}\to (X'/\mathfrak{S})_{\mathrm{conv,fppf}}$ is an equivalence of topoi.*

Our proof is inspired by a site-theoretic construction of the Cartier transform of Ogus–Vologodsky due to Oyama [OV07, Oya17] and its lifting modulo p^n developed by the author [Xu19]. By Gabber–Bosch–Görtz's faithfully flat descent theory for coherent sheaves in rigid geometry [BG98], we obtain a new proof of the Frobenius descent (1.3.1).

Corollary 1.6 (Proposition 5.7). *Keep the hypotheses of Theorem 1.5. The direct image and inverse image functors of $F_{X/S_0,\mathrm{conv}}$ induce equivalences of categories quasi-inverse to each other:*

$$\mathrm{Iso}^{\dagger}(X/\mathfrak{S})\leftrightarrows\mathrm{Iso}^{\dagger}(X'/\mathfrak{S}). \tag{1.6.1}$$

[1] Actually, Shiho developed a theory of a log convergent site and log convergent cohomology over a p-adic base with log structure.

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1.7 Keep the notation of Theorem 1.5 and suppose that there exist smooth liftings $\mathfrak{X}$ of X and $\mathfrak{X}'$ of X' over $\mathfrak{S}$ (in particular, X is smooth over S_0). We denote by $\widehat{\Omega}^1_{\mathfrak{X}/\mathfrak{S}}$ the $\mathscr{O}_{\mathfrak{X}}$-module of differentials of $\mathfrak{X}$ relative to $\mathfrak{S}$. Given a convergent isocrystal $\mathscr{E} \in \mathbf{Ob}(\mathrm{Iso}^{\dagger}(X/\mathfrak{S}))$, there exists an integrable connection $\nabla : \mathscr{E}_{\mathfrak{X}} \to \mathscr{E}_{\mathfrak{X}} \otimes_{\mathscr{O}_{\mathfrak{X}}} \widehat{\Omega}^1_{\mathfrak{X}/\mathfrak{S}}$ on the coherent $\mathscr{O}_{\mathfrak{X}}[\frac{1}{p}]$-module $\mathscr{E}_{\mathfrak{X}}$ (Proposition 3.17). We denote by $\mathscr{E}_{\mathfrak{X}} \otimes_{\mathscr{O}_{\mathfrak{X}}} \widehat{\Omega}^{\bullet}_{\mathfrak{X}/\mathfrak{S}}$ the associated de Rham complex. We deduce from Theorem 1.5 the following result about comparing de Rham complexes for the Frobenius descent.

Corollary 1.8 (Corollary 5.9). *Keep the above notation and let $f : X \to S_0$ be the canonical morphism. There exists a canonical isomorphism between de Rham complexes of $\mathscr{E}$ and of $F_{X/S_0,\mathrm{conv}\,*}(\mathscr{E})$ in $\mathrm{D}(X'_{\mathrm{zar}}, f^{-1}(\mathscr{O}_{\mathfrak{S}}))$:*

$$F_{X/S_0*}(\mathscr{E}_{\mathfrak{X}} \otimes_{\mathscr{O}_{\mathfrak{X}}} \widehat{\Omega}^{\bullet}_{\mathfrak{X}/\mathfrak{S}}) \xrightarrow{\sim} (F_{X/S_0,\mathrm{conv}\,*}(\mathscr{E}))_{\mathfrak{X}'} \otimes_{\mathscr{O}_{\mathfrak{X}'}} \widehat{\Omega}^{\bullet}_{\mathfrak{X}'/\mathfrak{S}}. \tag{1.8.1}$$

In [Ber96b], Berthelot introduced a sheaf $\mathscr{D}^{\dagger}$ of differential operators over $\mathfrak{X}$ and described (over-)convergent isocrystals in terms of arithmetic $\mathscr{D}^{\dagger}$-modules. If there exists a lifting $F : \mathfrak{X} \to \mathfrak{X}'$ of the relative Frobenius morphism F_{X/S_0}, he used F to establish a version of Frobenius descent for arithmetic $\mathscr{D}^{\dagger}$-modules and a comparison result for de Rham complexes (cf. [Ber00, 4.2.4 and 4.3.5]). The above results can be viewed as a counterpart of Berthelot's results for convergent isocrystals without assuming a lifting of Frobenius morphism.

Our main result is the following.

Theorem 1.9 (Corollary 8.3). *Let $g : X \to Y$ be a smooth proper morphism of k-schemes locally of finite type. The higher direct image of a convergent isocrystal (respectively F-isocrystal) on $\mathrm{Conv}(X/\mathrm{W})$ (§ 1.3) is a convergent isocrystal (respectively F-isocrystal) on $\mathrm{Conv}(Y/\mathrm{W})$.*

Our proof relies on a preprint of Shiho [Shi07a] on relative crystalline cohomology of convergent isocrystals.

In [Ogu84], Ogus described $\mathrm{R}^i\, g_{\mathrm{conv}\,*}(\mathscr{O}_{X/K})$ in terms of relative crystalline cohomology. Morrow showed that $\mathrm{R}^i\, g_{\mathrm{conv}\,*}(\mathscr{O}_{X/K})$ coincides with the higher direct images in crystalline cohomology when Y is smooth [Mor19]. Our approach follows a similar line of their work.

In a recent preprint of Di Proietto, Tonini and Zhang [DTZ18], they showed that the higher direct image of an isocrystal (crystal on the crystalline site $(X/\mathrm{W})_{\mathrm{crys}}$ up to isogeny) via g_{crys} is still an isocrystal when Y is smooth. Our result is compatible with theirs via the functor ι (§ 1.2). However, two results are independent and are proved in different methods.

1.10 In the following, we explain the structure of this article and the strategy for proving Theorem 1.9.

Section 2 contains general notation and a review of isocrystals on crystalline sites. In § 3, we recall the definition of the convergent topos over a p-adic base and results on the cohomology of convergent isocrystals following Shiho [Shi07a]. In § 4, we show that under a smooth proper morphism of smooth k-schemes $X \to Y$, the higher direct image of a convergent isocrystal on $\mathrm{Conv}(X/\mathrm{W})$ is a 'p-adic convergent isocrystal' on $\mathrm{Conv}(Y/\mathrm{W})$, i.e. it satisfies the property of a coherent crystal in a certain subcategory of $\mathrm{Conv}(Y/\mathrm{W})$ (Proposition 4.8). Section 5 is devoted to the Frobenius descent (Theorem 1.5 and Corollary 1.6). Using Dwork's trick and Theorem 1.5, we deduce Theorem 1.9 in the case where Y is smooth over k (Theorem 5.10). In § 6, we briefly review Raynaud's approach to rigid geometry following Abbes' book [Abb10]. Section 7 is devoted to a modification of convergent topos which allows us to apply the faithfully flat descent in rigid geometry in the full extent. Based on previous results and Ogus' proper surjective descent, we complete the proof of Theorem 1.9 in the non-smooth case in § 8.

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2. Preliminary

2.1 In this article, p denotes a prime number, k denotes a perfect field of characteristic p, W the ring of Witt vectors of k and K the fraction field of W.

Let $\mathfrak{X}$ be an adic formal W-scheme. For any $n \geqslant 1$, we denote by $\mathfrak{X}_n$ the reduction modulo p^n of $\mathfrak{X}$. If we use a gothic letter $\mathfrak{X}$ to denote an adic formal W-scheme, the corresponding roman letter X will denote its special fiber $\mathfrak{X}_1$.

We denote by $\mathbf{S}$ the category whose objects are adic formal W-schemes of finite type [Abb10, 2.3.13] and morphisms are adic morphisms [Abb10, 2.2.7]. By [EGAI, 6.1.5(v)], morphisms of $\mathbf{S}$ are of finite type. We denote by $\mathbf{S}^{\diamond}$ the full subcategory of $\mathbf{S}$ consisting of *flat* formal W-schemes of finite type.

2.2 Let $\mathscr{A}$ be an abelian category. We denote by $\mathscr{A}_{\mathbb{Q}}$ the category with the same objects as $\mathscr{A}$ such that the set of morphisms is given for any object M, N of $\mathscr{A}$ by

$$\mathrm{Hom}_{\mathscr{A}_{\mathbb{Q}}}(M,N) = \mathrm{Hom}_{\mathscr{A}}(M,N) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

For any object M of $\mathscr{A}$, we denote its image in $\mathscr{A}_{\mathbb{Q}}$ by $M_{\mathbb{Q}}$.

2.3 Let $\mathfrak{X}$ be an object of $\mathbf{S}$. For any $\mathscr{O}_{\mathfrak{X}}$-module $\mathscr{F}$, we set $\mathscr{F}[\frac{1}{p}] = \mathscr{F} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$.

We denote by $\mathbf{Coh}(\mathscr{O}_{\mathfrak{X}})$ (respectively $\mathbf{Coh}(\mathscr{O}_{\mathfrak{X}}[\frac{1}{p}])$) the category of coherent $\mathscr{O}_{\mathfrak{X}}$-modules (respectively $\mathscr{O}_{\mathfrak{X}}[\frac{1}{p}]$-modules). The canonical functor $\mathbf{Coh}(\mathscr{O}_{\mathfrak{X}}) \to \mathbf{Coh}(\mathscr{O}_{\mathfrak{X}}[\frac{1}{p}])$ defined by $\mathscr{F} \mapsto \mathscr{F}[\frac{1}{p}]$ induces an equivalence of categories $\mathbf{Coh}(\mathscr{O}_{\mathfrak{X}})_{\mathbb{Q}} \xrightarrow{\sim} \mathbf{Coh}(\mathscr{O}_{\mathfrak{X}}[\frac{1}{p}])$ [AGT16, III.6.16].

2.4 Let $\mathscr{C}$ and $\mathscr{D}$ be two sites, $\widehat{\mathscr{C}}$ (respectively $\widehat{\mathscr{D}}$) the category of presheaves of sets on $\mathscr{C}$ (respectively $\mathscr{D}$) and $u : \mathscr{C} \to \mathscr{D}$ a functor. We have a functor

$$\widehat{u}^* : \widehat{\mathscr{D}} \to \widehat{\mathscr{C}}, \quad \mathscr{G} \mapsto \widehat{u}^*(\mathscr{G}) = \mathscr{G} \circ u.$$

It admits a right adjoint $\widehat{u}_* : \widehat{\mathscr{C}} \to \widehat{\mathscr{D}}$ [SGA4, I 5.1].

If $u : \mathscr{C} \to \mathscr{D}$ is a cocontinuous (respectively continuous) functor and $\mathscr{F}$ (respectively $\mathscr{G}$) is a sheaf on $\mathscr{C}$ (respectively $\mathscr{D}$), then $\widehat{u}_*(\mathscr{F})$ (respectively $\widehat{u}^*(\mathscr{G})$) is a sheaf on $\mathscr{D}$ (respectively $\mathscr{C}$) [SGA4, III 1.2, 2.1 and 2.2].

Let $\widetilde{\mathscr{C}}$ (respectively $\widetilde{\mathscr{D}}$) be the topos of the sheaves of sets on $\mathscr{C}$ (respectively $\mathscr{D}$) and $u : \mathscr{C} \to \mathscr{D}$ a *cocontinuous* functor. Then u induces a morphism of topoi $g : \widetilde{\mathscr{C}} \to \widetilde{\mathscr{D}}$ defined by $g_* = \widehat{u}_*$ and $g^* = a \circ \widehat{u}^*$, where a is the sheafification functor (cf. [SGA4, III 2.3]).

PROPOSITION 2.5 [Oya17, 4.2.1]. *Let $\mathscr{C}$ be a site, $\mathscr{D}$ a site whose topology is defined by a pretopology, $\widetilde{\mathscr{C}}$ (respectively $\widetilde{\mathscr{D}}$) the topos of sheaves on $\mathscr{C}$ (respectively $\mathscr{D}$) and $u : \mathscr{C} \to \mathscr{D}$ a functor. Assume that:*

(i) *u is fully faithful;*

(ii) *u is continuous and cocontinuous;*

(iii) *for every object V of $\mathscr{D}$, there exists a covering $\{u(U_i) \to V\}_{i \in I}$ of V in $\mathscr{D}$ with objects U_i of $\mathscr{C}$.*

Then the morphism of topoi $g : \widetilde{\mathscr{C}} \to \widetilde{\mathscr{D}}$ defined by $g^* = \widehat{u}^*$ and $g_* = \widehat{u}_*$ (2.4) is an equivalence of topoi.

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2.6 In the following, $\mathfrak{S}$ denotes an object of $\mathbf{S}^{\diamond}$ and X an S-scheme.

We equip $p\mathscr{O}_{\mathfrak{S}}$ with the canonical PD-structure γ. Recall that the *crystalline site* $\mathrm{Crys}(X/\mathfrak{S})$ is defined as follows [BO15, 7.17]: an object is a quadruple (U,T,ι,δ) consisting of an open subscheme U of X, a scheme T over $\mathfrak{S}_n$ for some integer $n\geqslant 1$, a closed immersion $\iota:U\to T$ and a PD-structure δ on $\mathrm{Ker}(\mathscr{O}_T\to\mathscr{O}_U)$ compatible with γ. A morphism from (U',T',ι',δ') to (U,T,ι,δ) of $\mathrm{Crys}(X/\mathfrak{S})$ consists of an open immersion $U'\to U$ and an $\mathfrak{S}$-morphism $T'\to T$ compatible with ι',ι and the PD-structures. A family of morphisms $\{(U_i,T_i)\to(U,T)\}_{i\in I}$ is a covering if each morphism $T_i\to T$ is an open immersion and $|T|=\bigcup_{i\in I}|T_i|$. We denote by $(X/\mathfrak{S})_{\mathrm{crys}}$ the topos of sheaves of sets on $\mathrm{Crys}(X/\mathfrak{S})$.

The presheaf of rings defined by $(U,T)\mapsto\Gamma(T,\mathscr{O}_T)$ is a sheaf that we denote by $\mathscr{O}_{X/\mathfrak{S}}^{\mathrm{crys}}$. For an $\mathscr{O}_{X/\mathfrak{S}}^{\mathrm{crys}}$-module $\mathscr{F}$ and an object (U,T) of $\mathrm{Crys}(X/\mathfrak{S})$, we denote by $\mathscr{F}_T$ the evaluation of $\mathscr{F}$ at (U,T) [BO15, 5.1].

Definition 2.7 ([BO15, 6.1], [Shi07a, 1.8]). (i) We say that an $\mathscr{O}_{X/\mathfrak{S}}^{\mathrm{crys}}$-module $\mathscr{F}$ is *a crystal* if, for every morphism $f:(U',T')\to(U,T)$ of $\mathrm{Crys}(X/\mathfrak{S})$, the transition morphism $f^*(\mathscr{F}_T)\to\mathscr{F}_{T'}$ is an isomorphism.

(ii) We say that a crystal $\mathscr{F}$ is *a crystal of $\mathscr{O}_{X/\mathfrak{S}}^{\mathrm{crys}}$-modules of finite presentation* if $\mathscr{F}_T$ is an $\mathscr{O}_T$-module of finite presentation for every object (U,T) of $\mathrm{Crys}(X/\mathfrak{S})$.

(iii) We denote by $\mathscr{C}(\mathscr{O}_{X/\mathfrak{S}}^{\mathrm{crys}})$ the category of crystals of $\mathscr{O}_{X/\mathfrak{S}}^{\mathrm{crys}}$-modules of finite presentation. Objects of $\mathscr{C}(\mathscr{O}_{X/\mathfrak{S}}^{\mathrm{crys}})_{\mathbb{Q}}$ (§ 2.2) are called *isocrystals*.

2.8 Crystals have an equivalent description in terms of modules equipped with hyper-PD-stratification and of modules with integrable connection. Let us briefly recall these notions.

Let $\mathfrak{X}$ be an adic formal $\mathfrak{S}$-scheme of finite type and $\mathfrak{X}^2=\mathfrak{X}\times_{\mathfrak{S}}\mathfrak{X}$. Let G be an adic formal $\mathfrak{X}^2$-scheme and let $q_1,q_2:\mathrm{G}\to\mathfrak{X}$ be the canonical projections. *A formal $\mathfrak{X}$-groupoid structure over $\mathfrak{S}$ on* G are three adic morphisms $\alpha:\mathrm{G}\times_{\mathfrak{X}}\mathrm{G}\to\mathrm{G}$, $\iota:\mathfrak{X}\to\mathrm{G}$ and $\eta:\mathrm{G}\to\mathrm{G}$, where the fibered product $\mathrm{G}\times_{\mathfrak{X}}\mathrm{G}$ is taken on the left (respectively right) for the $\mathfrak{X}$-structure defined by q_2 (respectively q_1), satisfying the compatibility conditions for groupoids (cf. [Xu19, 4.7]). We set $q_{13}=\alpha$ and $q_{12},q_{23}:\mathrm{G}\times_{\mathfrak{X}}\mathrm{G}\to\mathrm{G}$, the projections in the first and second components, respectively.

Let G be a formal $\mathfrak{X}$-groupoid over $\mathfrak{S}$ and M an $\mathscr{O}_{\mathfrak{X}}$-module. *An $\mathscr{O}_{\mathrm{G}}$-stratification on M* is an $\mathscr{O}_{\mathrm{G}}$-linear isomorphism $\varepsilon:q_2^*(M)\xrightarrow{\sim}q_1^*(M)$ satisfying $\iota^*(\varepsilon)=\mathrm{id}_M$ and the cocycle condition $q_{12}^*(\varepsilon)\circ q_{23}^*(\varepsilon)=q_{13}^*(\varepsilon)$.

2.9 Suppose that X is smooth over S and admits a smooth lifting $\mathfrak{X}$ over $\mathfrak{S}$. We denote by $\mathrm{P}_{\mathfrak{X}/\mathfrak{S}}$ the adic formal $\mathfrak{X}^2$-scheme defined by the PD-envelope of the diagonal immersion $\mathfrak{X}\to\mathfrak{X}^2$ compatible with the canonical PD-structure γ (§ 2.6). By the universal property of the PD-envelope, the formal $\mathfrak{X}^2$-scheme $\mathrm{P}_{\mathfrak{X}/\mathfrak{S}}$ is equipped with a formal $\mathfrak{X}$-groupoid structure (§ 2.8).

Given an object $\mathscr{F}$ of $\mathscr{C}(\mathscr{O}_{X/\mathfrak{S}}^{\mathrm{crys}})$ (Definition 2.7), the coherent $\mathscr{O}_{\mathfrak{X}}$-module $\mathscr{F}_{\mathfrak{X}}=\varprojlim_{n\geqslant 1}\mathscr{F}_{\mathfrak{X}_n}$ is equipped with an $\mathscr{O}_{\mathrm{P}_{\mathfrak{X}/\mathfrak{S}}}$-stratification and then an integrable connection relative to $\mathfrak{S}$. Moreover, the following categories are canonically equivalent (see [BO15, 6.6] and [Sta, 07JH]).

(i) The category $\mathscr{C}(\mathscr{O}_{X/\mathfrak{S}}^{\mathrm{crys}})$.

(ii) The category of coherent $\mathscr{O}_{\mathfrak{X}}$-modules equipped with an $\mathscr{O}_{\mathrm{P}_{\mathfrak{X}/\mathfrak{S}}}$-stratification.

(iii) The category of coherent $\mathscr{O}_{\mathfrak{X}}$-modules equipped with a topologically quasi-nilpotent integrable connection relative to $\mathfrak{S}$ [BO15, 6.1].

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Proposition 2.10 ([Ogu07, 0.7.5], [Shi07a, 1.23]). *Keep the assumption of § 2.9. Let M be a coherent $\mathscr{O}_{\mathfrak{X}}[\frac{1}{p}]$-module and ε an $\mathscr{O}_{\mathrm{P}_{\mathfrak{X}/\mathfrak{S}}}$-stratification on M. There exist a coherent $\mathscr{O}_{\mathfrak{X}}$-module M° and an $\mathscr{O}_{\mathrm{P}_{\mathfrak{X}/\mathfrak{S}}}$-stratification ε° on M° such that $(M^\circ[\frac{1}{p}], \varepsilon^\circ \otimes \mathrm{id})$ is isomorphic to (M, ε).*

Definition 2.11 [SGA6, I 1.3.1]. Let $(\mathscr{T}, A)$ be a ringed topos. We say that an A-module M of $\mathscr{T}$ is *locally projective of finite type* if the following equivalent conditions are satisfied:

(i) M is of finite type and the functor $\mathscr{H}om_A(M, -)$ is exact;
(ii) M is of finite type and every epimorphism of A-modules $N \to M$ admits locally a section;
(iii) M is locally a direct summand of a free A-module of finite type.

When $\mathscr{T}$ has enough points and, for every point x of $\mathscr{T}$, the stalk of A at x is a local ring, the locally projective A-modules of finite type are locally free A-modules of finite type [SGA6, I 2.15.1].

Lemma 2.12. *Let $\mathfrak{X}$ be a smooth formal* W*-scheme, M a coherent $\mathscr{O}_{\mathfrak{X}}$-module and ∇ an integrable connection on M relative to* W*. Then $M[\frac{1}{p}]$ is a locally projective $\mathscr{O}_{\mathfrak{X}}[\frac{1}{p}]$-module of finite type (2.11). In particular, given a coherent $\mathscr{O}_{\mathfrak{X}}[\frac{1}{p}]$-module with an $\mathscr{O}_{\mathrm{P}_{\mathfrak{X}/\mathrm{W}}}$-stratification (M, ε) (respectively an object $\mathscr{E}$ of $\mathscr{C}(\mathscr{O}_{X/\mathrm{W}}^{\mathrm{crys}})$), M (respectively $\mathscr{E}_{\mathfrak{X}}[\frac{1}{p}]$) is a locally projective $\mathscr{O}_{\mathfrak{X}}[\frac{1}{p}]$-module of finite type.*

Proof. The first assertion is a standard result (cf. [Kat70, 8.8] and [Ked10, 1.2]). Then the second assertion follows from § 2.9 and Proposition 2.10. □

2.13 We denote by $u_{X/\mathfrak{S},\mathrm{crys}} : (X/\mathfrak{S})_{\mathrm{crys}} \to X_{\mathrm{zar}}$ the canonical morphism of topoi [BO15, 5.12] and by $g_{X/\mathfrak{S},\mathrm{crys}}$ the composition

$$g_{X/\mathfrak{S},\mathrm{crys}} : (X/\mathfrak{S})_{\mathrm{crys}} \to X_{\mathrm{zar}} \to \mathfrak{S}_{\mathrm{zar}},$$

which is ringed by $\mathscr{O}_{X/\mathfrak{S}}^{\mathrm{crys}}$ and $\mathscr{O}_{\mathfrak{S}}$. We call $\mathrm{R}^\bullet g_{X/\mathfrak{S},\mathrm{crys}\,*}(-)$ *the relative crystalline cohomology*.

For an isocrystal $\mathscr{E} = \mathscr{F}_{\mathbb{Q}}$ with $\mathscr{F} \in \mathbf{Ob}(\mathscr{C}(\mathscr{O}_{X/\mathfrak{S}}^{\mathrm{crys}}))$, we set (§ 2.1)

$$\mathrm{R}^q g_{X/\mathfrak{S},\mathrm{crys}\,*}(\mathscr{E}) = \mathrm{R}^q g_{X/\mathfrak{S},\mathrm{crys}\,*}(\mathscr{F})\left[\frac{1}{p}\right], \quad \mathrm{R}\, g_{X/\mathfrak{S},\mathrm{crys}\,*}(\mathscr{E}) = \mathrm{R}\, g_{X/\mathfrak{S},\mathrm{crys}\,*}(\mathscr{F})\left[\frac{1}{p}\right].$$

It is clear that the above definition is independent of the choice of $\mathscr{F}$.

When X is smooth and proper over S and $\mathfrak{S}$ is separated, Shiho proved that the relative crystalline cohomology $\mathrm{R}^q g_{X/\mathfrak{S},\mathrm{crys}\,*}(\mathscr{E})$ is a coherent $\mathscr{O}_{\mathfrak{S}}[\frac{1}{p}]$-module [Shi07a, 1.15]. Moreover, he showed a base-change result for relative crystalline cohomology (cf. [Shi07a, 1.16 and 1.19]).

3. Convergent topos and convergent isocrystals

In this section, $\mathfrak{S}$ denotes an adic flat formal W-scheme of finite type and X denotes an S-scheme. For any scheme T, we denote by T_0 the reduced subscheme of T.

Definition 3.1 ([Ogu84, 2.1], [Shi07a, 2.4]). We define a category $\mathrm{Conv}(X/\mathfrak{S})$ as follows.

(i) An object of $\mathrm{Conv}(X/\mathfrak{S})$ is a pair $(\mathfrak{T}, u)$ consisting of an adic formal $\mathfrak{S}$-scheme of finite type which is flat over W and an S-morphism $u : T_0 \to X$.

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(ii) Let $(\mathfrak{T}', u')$ and $(\mathfrak{T}, u)$ be two objects of $\mathrm{Conv}(X/\mathfrak{S})$. A morphism from $(\mathfrak{T}', u')$ to $(\mathfrak{T}, u)$ is a $\mathfrak{S}$-morphism $f : \mathfrak{T}' \to \mathfrak{T}$ such that the induced morphism $f_0 : T'_0 \to T_0$ is compatible with u' and u.

We denote an object $(\mathfrak{T}, u)$ of $\mathrm{Conv}(X/\mathfrak{S})$ simply by $\mathfrak{T}$ if there is no risk of confusion.

It is clear that if $X \to Y$ is a nilpotent immersion of S-schemes, the category $\mathrm{Conv}(X/\mathfrak{S})$ is canonically equivalent to $\mathrm{Conv}(Y/\mathfrak{S})$.

3.2 Let $f : (\mathfrak{T}', u') \to (\mathfrak{T}, u)$ and $g : (\mathfrak{T}'', u'') \to (\mathfrak{T}, u)$ be two morphisms of $\mathrm{Conv}(X/\mathfrak{S})$. We denote by $\mathfrak{Z}$ the closed formal subscheme of $\mathfrak{T}' \times_{\mathfrak{T}} \mathfrak{T}''$ defined by the ideal of p-torsion elements of $\mathscr{O}_{\mathfrak{T}' \times_{\mathfrak{T}} \mathfrak{T}''}$. The fibered product of f and g in $\mathrm{Conv}(X/\mathfrak{S})$ is represented by $\mathfrak{Z}$, which is flat over W, equipped with the composition $Z_0 \to T'_0 \times_{T_0} T''_0 \to X$ induced by u' and u''.

If either $\mathfrak{T}'' \to \mathfrak{T}$ or $\mathfrak{T}' \to \mathfrak{T}$ is flat, then $\mathfrak{Z}$ is equal to $\mathfrak{T}' \times_{\mathfrak{T}} \mathfrak{T}''$.

3.3 Let $\mathfrak{T}$ be an object of $\mathbf{S}$. We denote by $\mathbf{Zar}_{/\mathfrak{T}}$ (respectively $\mathfrak{T}_{\mathrm{zar}}$) the Zariski site (respectively topos) of $\mathfrak{T}$.

We say that a family of morphisms $\{f_i : \mathfrak{T}_i \to \mathfrak{T}\}_{i \in I}$ of $\mathbf{S}$ is an *fppf covering* if each morphism f_i is flat and if $|\mathfrak{T}| = \bigcup_{i \in I} f_i(|\mathfrak{T}_i|)$. Since $\mathfrak{T}$ is quasi-compact, each fppf covering of $\mathfrak{T}$ admits a finite fppf subcovering of $\mathfrak{T}$. Note that fppf coverings in $\mathbf{S}$ are stable by base change and by composition.

We denote by $\mathbf{Fft}_{/\mathfrak{T}}$ the full subcategory of $\mathbf{S}_{/\mathfrak{T}}$ consisting of adic flat formal $\mathfrak{T}$-schemes and by $\mathfrak{T}_{\mathrm{fppf}}$ the topos of sheaves of sets on $\mathbf{Fft}_{/\mathfrak{T}}$, equipped with the topology generated by fppf coverings.

Given a morphism $f : \mathfrak{T}' \to \mathfrak{T}$ of $\mathbf{S}$, the canonical functor $\mathbf{Fft}_{/\mathfrak{T}} \to \mathbf{Fft}_{/\mathfrak{T}'}$ (respectively $\mathbf{Zar}_{/\mathfrak{T}} \to \mathbf{Zar}_{/\mathfrak{T}'}$) defined by $\mathfrak{Y} \mapsto \mathfrak{Y} \times_{\mathfrak{T}} \mathfrak{T}'$ is continuous and left exact. For $\tau \in \{\mathrm{zar}, \mathrm{fppf}\}$, it induces the functorial morphism of topoi $f_\tau : \mathfrak{T}'_\tau \to \mathfrak{T}_\tau$.

3.4 We say that a family of morphisms $\{(\mathfrak{T}_i, u_i) \to (\mathfrak{T}, u)\}_{i \in I}$ of $\mathrm{Conv}(X/\mathfrak{S})$ is a *Zariski* (respectively *fppf*) covering if the family of morphisms $\{\mathfrak{T}_i \to \mathfrak{T}\}_{i \in I}$ of $\mathbf{S}$ is a Zariski (respectively fppf) covering. By § 3.3, Zariski (respectively fppf) coverings in $\mathrm{Conv}(X/\mathfrak{S})$ form a pretopology. For $\tau \in \{\mathrm{zar}, \mathrm{fppf}\}$, we denote by $(X/\mathfrak{S})_{\mathrm{conv},\tau}$ the topos of sheaves of sets on $\mathrm{Conv}(X/\mathfrak{S})$, equipped with the τ-topology.

Remark 3.5. The above definition of a convergent site is slightly different from that of [Ogu84, Shi07a], where the authors considered a category whose objects are triples $(\mathfrak{T}, Z, u)$, where $\mathfrak{T}$ is the same as above, Z is a closed subscheme of definition of $\mathfrak{T}$ such that $T_0 \to \mathfrak{T}$ factors through Z and $u : Z \to X$ is an S-morphism. However, it follows from Proposition 2.5 that the convergent topoi (with Zariski topology) defined in two different ways are equivalent and we freely use results of [Shi07a] in our setting.

3.6 Let $(\mathfrak{T}, u)$ be an object of $\mathrm{Conv}(X/\mathfrak{S})$ and $\tau \in \{\mathrm{zar}, \mathrm{fppf}\}$. The canonical functor

$$\mathbf{Fft}_{/\mathfrak{T}} \quad (\text{respectively } \mathbf{Zar}_{/\mathfrak{T}}) \to \mathrm{Conv}(X/\mathfrak{S}), \quad (f : \mathfrak{T}' \to \mathfrak{T}) \mapsto (\mathfrak{T}', u \circ f_0)$$

is cocontinuous and induces a morphism of topoi (§ 2.4)

$$s_{\mathfrak{T}} : \mathfrak{T}_\tau \to (X/\mathfrak{S})_{\mathrm{conv},\tau}, \quad \forall \tau \in \{\mathrm{zar}, \mathrm{fppf}\}. \tag{3.6.1}$$

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For any sheaf $\mathscr{F}$ of $(X/\mathfrak{S})_{\mathrm{conv},\tau}$, we set $\mathscr{F}_{\mathfrak{T}} = s_{\mathfrak{T}}^{*}(\mathscr{F})$, called the *evaluation of* $\mathscr{F}$ *at* $\mathfrak{T}$. For any morphism $f : \mathfrak{T}' \to \mathfrak{T}$ of $\mathrm{Conv}(X/\mathfrak{S})$, we have a canonical morphism (§ 3.3)

$$\beta_f : \mathscr{F}_{\mathfrak{T}} \to f_{\tau *}(\mathscr{F}_{\mathfrak{T}'}) \tag{3.6.2}$$

and we denote its adjoint by

$$\gamma_f : f_{\tau}^{*}(\mathscr{F}_{\mathfrak{T}}) \to \mathscr{F}_{\mathfrak{T}'}. \tag{3.6.3}$$

It is clear that $\gamma_{\mathrm{id}} = \mathrm{id}$. If f is a morphism of $\mathbf{Fft}_{/\mathfrak{T}}$ (respectively $\mathbf{Zar}_{/\mathfrak{T}}$), f_τ is the localization morphism at $\mathfrak{T}'$ and then γ_f is an isomorphism. If $g : \mathfrak{T}'' \to \mathfrak{T}'$ is another morphism of $\mathrm{Conv}(X/\mathfrak{S})$, one verifies that $\gamma_{g\circ f} = \gamma_f \circ f_{\tau}^{*}(\gamma_g)$.

Proposition 3.7. *For $\tau \in \{\mathrm{zar}, \mathrm{fppf}\}$, a sheaf $\mathscr{F}$ of $\mathrm{Conv}(X/\mathfrak{S})_{\mathrm{conv},\tau}$ is equivalent to the following data:*

(i) *for every object $(\mathfrak{T}, u)$ of $\mathrm{Conv}(X/\mathfrak{S})$, a sheaf $\mathscr{F}_{\mathfrak{T}}$ of $\mathfrak{T}_\tau$;*
(ii) *for every morphism $f : (\mathfrak{T}', u') \to (\mathfrak{T}, u)$, a transition morphism γ_f (3.6.3)*

subject to the following conditions.

(a) *If f is the identity morphism of $(\mathfrak{T}, u)$, then γ_f is the identity morphism.*
(b) *If $f : \mathfrak{T}' \to \mathfrak{T}$ is a morphism of $\mathbf{Zar}_{/\mathfrak{T}}$ (respectively $\mathbf{Fft}_{/\mathfrak{T}}$), then γ_f is an isomorphism.*
(c) *If f and g are two composable morphisms, then we have $\gamma_{g\circ f} = \gamma_f \circ f_{\tau}^{*}(\gamma_g)$.*

Proof. Given a datum $\{\mathscr{F}_{\mathfrak{T}}, \gamma_f\}$ as in the proposition, for any morphism $f : \mathfrak{T}' \to \mathfrak{T}$ of $\mathrm{Conv}(X/\mathfrak{S})$, the morphism γ_f induces a morphism $\mathscr{F}_{\mathfrak{T}}(\mathfrak{T}) \to \mathscr{F}_{\mathfrak{T}'}(\mathfrak{T}')$. In view of conditions (a) and (c), the correspondence $\mathfrak{T} \mapsto \mathscr{F}_{\mathfrak{T}}(\mathfrak{T})$ defines a presheaf $\mathscr{F}$ on $\mathrm{Conv}(X/\mathfrak{S})$. In view of condition (b), $\mathscr{F}$ is a sheaf and the above construction is quasi-inverse to § 3.6. Then the proposition follows. □

3.8 Note that the fppf topology on $\mathrm{Conv}(X/\mathfrak{S})$ is finer than the Zariski topology. Equipped with the fppf topology on the source and the Zariski topology on the target, the identical functor $\mathrm{id} : \mathrm{Conv}(X/\mathfrak{S}) \to \mathrm{Conv}(X/\mathfrak{S})$ is cocontinuous and induces a morphism of topoi (§ 2.4)

$$\alpha : (X/\mathfrak{S})_{\mathrm{conv,fppf}} \to (X/\mathfrak{S})_{\mathrm{conv,zar}}. \tag{3.8.1}$$

If $\mathscr{F}$ is a sheaf of $(X/\mathfrak{S})_{\mathrm{conv,fppf}}$, then $\alpha_*(\mathscr{F})$ is equal to $\mathscr{F}$ as presheaves. If $\mathscr{G}$ is a sheaf of $(X/\mathfrak{S})_{\mathrm{conv,zar}}$, then $\alpha^*(\mathscr{G})$ is the sheafification of $\mathscr{G}$ with respect to the fppf topology.

3.9 Let $g : \mathfrak{S}' \to \mathfrak{S}$ be a morphism of $\mathbf{S}^{\diamond}$ (§ 2.1), X' an S'-scheme and $f : X' \to X$ a morphism compatible with g, i.e. the diagram

$$\begin{array}{ccccc} X' & \longrightarrow & S' & \longrightarrow & \mathfrak{S}' \\ {\scriptstyle f}\downarrow & & \downarrow & & \downarrow{\scriptstyle g} \\ X & \longrightarrow & S & \longrightarrow & \mathfrak{S} \end{array} \tag{3.9.1}$$

is commutative. For any object $(\mathfrak{T}, u)$ of $\mathrm{Conv}(X'/\mathfrak{S}')$, $(\mathfrak{T}, f \circ u)$ defines an object of $\mathrm{Conv}(X/\mathfrak{S})$. We obtain a functor that we denote by

$$\varphi : \mathrm{Conv}(X'/\mathfrak{S}') \to \mathrm{Conv}(X/\mathfrak{S}), \quad (\mathfrak{T}, u) \mapsto (\mathfrak{T}, f \circ u). \tag{3.9.2}$$

It is clear that φ commutes with the fibered product (§ 3.2).

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Lemma 3.10. (i) *Let* $(\mathfrak{T}, u)$ *be an object of* $\mathrm{Conv}(X'/\mathfrak{S}')$ *and* $g:(\mathfrak{Z}, w) \to \varphi(\mathfrak{T}, u)$ *a morphism of* $\mathrm{Conv}(X/\mathfrak{S})$. *Then there exist an object* $(\mathfrak{Z}, v)$ *of* $\mathrm{Conv}(X'/\mathfrak{S}')$ *and a morphism* $h:(\mathfrak{Z}, v) \to (\mathfrak{T}, u)$ *of* $\mathrm{Conv}(X'/\mathfrak{S}')$ *such that* $g = \varphi(h)$.

(ii) *Equipped with the Zariski topology (respectively fppf topology) (§ 3.4) on both sides, the functor* φ *is continuous and cocontinuous.*

Proof. (i) By considering compositions $\mathfrak{Z} \to \mathfrak{T} \to \mathfrak{S}'$ and $Z_0 \to T_0 \to X'$, we obtain an object $(\mathfrak{Z}, v)$ of $\mathrm{Conv}(X'/\mathfrak{S})$ and a morphism $h:(\mathfrak{Z}, v) \to (\mathfrak{T}, u)$ of $\mathrm{Conv}(X'/\mathfrak{S}')$ such that $g = \varphi(h)$.

(ii) A family of morphisms $\{(\mathfrak{T}_i, u_i) \to (\mathfrak{T}, u)\}_{i\in I}$ of $\mathrm{Conv}(X'/\mathfrak{S}')$ is a Zariski (respectively fppf) covering if and only if its image by φ is a Zariski (respectively fppf) covering. Since φ commutes with the fibered product, the continuity of φ follows from [SGA4, III 1.6].

Let $\{(\mathfrak{T}_i, u_i) \to \varphi(\mathfrak{T}, u)\}_{i\in I}$ be a Zariski (respectively fppf) covering. By (i), there exists a Zariski (respectively fppf) covering $\{(\mathfrak{T}_i, v_i) \to (\mathfrak{T}, u)\}_{i\in I}$ mapping by φ to the given element. Then φ is cocontinuous by [SGA4, III 2.1]. □

3.11 By § 2.4 and Lemma 3.10, the functor φ (3.9.2) induces a morphism of topoi

$$f_{\mathrm{conv},\tau}: (X'/\mathfrak{S}')_{\mathrm{conv},\tau} \to (X/\mathfrak{S})_{\mathrm{conv},\tau} \tag{3.11.1}$$

such that the pullback functor is induced by the composition with φ. For a sheaf $\mathscr{F}$ of $(X/\mathfrak{S})_{\mathrm{conv},\tau}$ and an object $\mathfrak{T}$ of $\mathrm{Conv}(X'/\mathfrak{S}')$, we have (Proposition 3.7)

$$(f^*_{\mathrm{conv},\tau}(\mathscr{F}))_{\mathfrak{T}} = \mathscr{F}_{\varphi(\mathfrak{T})}. \tag{3.11.2}$$

For any morphism h of $\mathrm{Conv}(X'/\mathfrak{S}')$, the transition morphism of $f^*_{\mathrm{conv},\tau}(\mathscr{F})$ associated to g (Proposition 3.7) is equal to the transition morphism of $\mathscr{F}$ associated to $\varphi(h)$.

By considering inverse image functors, one verifies that the following diagram commutes (3.8.1):

$$\begin{array}{ccc} (X'/\mathfrak{S}')_{\mathrm{conv,fppf}} & \xrightarrow{f_{\mathrm{conv,fppf}}} & (X/\mathfrak{S})_{\mathrm{conv,fppf}} \\ \downarrow{\scriptstyle \alpha'} & & \downarrow{\scriptstyle \alpha} \\ (X'/\mathfrak{S}')_{\mathrm{conv,zar}} & \xrightarrow{f_{\mathrm{conv,zar}}} & (X/\mathfrak{S})_{\mathrm{conv,zar}} \end{array} \tag{3.11.3}$$

3.12 Let $\mathfrak{T}$ be an object of $\mathbf{S}$ and $\mathscr{F}$ a coherent $\mathscr{O}_{\mathfrak{T}}[\frac{1}{p}]$-module. Since $\mathfrak{T}$ is quasi-compact, by Gabber–Bosch–Görtz's fppf descent for coherent $\mathscr{O}_{\mathfrak{T}}[\frac{1}{p}]$-modules [Abb10, 5.11.11], the presheaf on $\mathbf{Fft}_{/\mathfrak{T}}$

$$(f:\mathfrak{T}' \to \mathfrak{T}) \mapsto \Gamma(\mathfrak{T}', f^*_{\mathrm{zar}}(\mathscr{F}))$$

is a sheaf for the fppf topology. In particular, $\mathscr{O}_{\mathfrak{T}}[\frac{1}{p}]$ defines a sheaf of rings of $\mathfrak{T}_{\mathrm{fppf}}$ that we still denote by $\mathscr{O}_{\mathfrak{T}}[\frac{1}{p}]$. We call abusively *a coherent* $\mathscr{O}_{\mathfrak{T}}[\frac{1}{p}]$*-module of* $\mathfrak{T}_{\mathrm{fppf}}$ a sheaf of $\mathfrak{T}_{\mathrm{fppf}}$ associated to a coherent $\mathscr{O}_{\mathfrak{T}}[\frac{1}{p}]$-module.

3.13 We define a presheaf of rings $\mathscr{O}_{X/\mathfrak{S}}[\frac{1}{p}]$ on $\mathrm{Conv}(X/\mathfrak{S})$ by

$$(\mathfrak{T}, u) \mapsto \Gamma\bigg(\mathfrak{T}, \mathscr{O}_{\mathfrak{T}}\bigg[\frac{1}{p}\bigg]\bigg). \tag{3.13.1}$$

By fppf descent (§ 3.12), $\mathscr{O}_{X/\mathfrak{S}}[\frac{1}{p}]$ is a sheaf for the fppf topology. Since the fppf topology is finer than the Zariski topology, it is also a sheaf for the Zariski topology.

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For any object $(\mathfrak{T}, u)$ of $\mathrm{Conv}(X/\mathfrak{S})$, we have $(\mathscr{O}_{X/\mathfrak{S}}[\frac{1}{p}])_{\mathfrak{T}} = \mathscr{O}_{\mathfrak{T}}[\frac{1}{p}]$. If $\mathscr{F}$ is an $\mathscr{O}_{X/\mathfrak{S}}[\frac{1}{p}]$-module of $(X/\mathfrak{S})_{\mathrm{conv},\tau}$, $\mathscr{F}_{\mathfrak{T}}$ is an $\mathscr{O}_{\mathfrak{T}}[\frac{1}{p}]$-module of $\mathfrak{T}_\tau$. For any morphism $f : \mathfrak{T}' \to \mathfrak{T}$ of $\mathrm{Conv}(X/\mathfrak{S})$, the transition morphism γ_f (Proposition 3.7) extends to an $\mathscr{O}_{\mathfrak{T}'}[\frac{1}{p}]$-linear morphism (§ 3.12)

$$c_f : f_\tau^*(\mathscr{F}_{\mathfrak{T}}) \to \mathscr{F}_{\mathfrak{T}'}. \tag{3.13.2}$$

In view of Proposition 3.7, we deduce the following description for $\mathscr{O}_{X/\mathfrak{S}}[\frac{1}{p}]$-modules.

Proposition 3.14. *For $\tau \in \{\mathrm{zar}, \mathrm{fppf}\}$, an $\mathscr{O}_{X/\mathfrak{S}}[\frac{1}{p}]$-module $\mathscr{F}$ of $(X/\mathfrak{S})_{\mathrm{conv},\tau}$ is equivalent to the following data:*

(i) *for every object $\mathfrak{T}$ of $\mathrm{Conv}(X/\mathfrak{S})$, an $\mathscr{O}_{\mathfrak{T}}[\frac{1}{p}]$-module $\mathscr{F}_{\mathfrak{T}}$ of $\mathfrak{T}_\tau$;*

(ii) *for every morphism $f : \mathfrak{T}' \to \mathfrak{T}$ of $\mathrm{Conv}(X/\mathfrak{S})$, an $\mathscr{O}_{\mathfrak{T}'}$-linear morphism c_f (3.13.2),*

which is subject to the following conditions.

(a) *If f is the identity morphism, then c_f is the identity.*

(b) *If $f : \mathfrak{T}' \to \mathfrak{T}$ is a morphism of $\mathbf{Zar}_{/\mathfrak{T}}$ (respectively $\mathbf{Fft}_{/\mathfrak{T}}$), then c_f is an isomorphism.*

(c) *If f and g are two composable morphisms, then we have $c_{g\circ f} = c_f \circ f_\tau^*(c_g)$.*

Definition 3.15. Let $\mathscr{F}$ be an $\mathscr{O}_{X/\mathfrak{S}}[\frac{1}{p}]$-module of $(X/\mathfrak{S})_{\mathrm{conv},\tau}$ for $\tau \in \{\mathrm{zar}, \mathrm{fppf}\}$.

(i) We say that $\mathscr{F}$ is *coherent* if for every object $\mathfrak{T}$ of $\mathrm{Conv}(X/\mathfrak{S})$, $\mathscr{F}_{\mathfrak{T}}$ is coherent (§ 3.12).

(ii) We say that $\mathscr{F}$ is a *crystal* if for every morphism f of $\mathrm{Conv}(X/\mathfrak{S})$, c_f is an isomorphism.

With the notation of § 3.11, the morphism $f_{\mathrm{conv},\tau}$ is ringed by $\mathscr{O}_{X'/\mathfrak{S}}[\frac{1}{p}]$ and $\mathscr{O}_{X/\mathfrak{S}}[\frac{1}{p}]$. The inverse image functor of modules $f^*_{\mathrm{conv},\tau}$ sends coherent $\mathscr{O}_{X/\mathfrak{S}}[\frac{1}{p}]$-modules (respectively crystals) to coherent $\mathscr{O}_{X'/\mathfrak{S}'}[\frac{1}{p}]$-modules (respectively crystals).

3.16 Let $\mathscr{E}$ be a coherent crystal of $\mathscr{O}_{X/\mathfrak{S}}[\frac{1}{p}]$-modules of $(X/\mathfrak{S})_{\mathrm{conv,zar}}$. By fppf descent (§ 3.12), $\mathscr{E}$ is also a sheaf for the fppf topology. In particular, the direct image and inverse image functors of α (§ 3.8) induce an equivalence between the category of coherent crystals of $\mathscr{O}_{X/\mathfrak{S}}[\frac{1}{p}]$-modules of $(X/\mathfrak{S})_{\mathrm{conv,zar}}$ and the category of coherent crystals of $\mathscr{O}_{X/\mathfrak{S}}[\frac{1}{p}]$-modules of $(X/\mathfrak{S})_{\mathrm{conv,fppf}}$.

Following [Ogu84, Shi07a], for $\tau \in \{\mathrm{zar}, \mathrm{fppf}\}$, a coherent crystal of $\mathscr{O}_{X/\mathfrak{S}}[\frac{1}{p}]$-modules of $(X/\mathfrak{S})_{\mathrm{conv},\tau}$ is called *a convergent isocrystal of* $(X/\mathfrak{S})_{\mathrm{conv},\tau}$. We denote the full subcategory of $\mathscr{O}_{X/\mathfrak{S}}[\frac{1}{p}]$-modules consisting of these objects by $\mathrm{Iso}^\dagger(X/\mathfrak{S})$.

We say that a convergent isocrystal $\mathscr{E}$ is *locally projective* if for every object $\mathfrak{T}$ of $\mathrm{Conv}(X/\mathfrak{S})$, $\mathscr{E}_{\mathfrak{T}}$ is locally projective of finite type (Definition 2.11).

Proposition 3.17 ([Ogu07, 0.7.2], [Shi07a, 2.35]). *Suppose that X is smooth over S. There exists a canonical functor (Definition 2.7)*

$$\iota : \mathrm{Iso}^\dagger(X/\mathfrak{S}) \to \mathscr{C}(\mathscr{O}^{\mathrm{crys}}_{X/\mathfrak{S}})_{\mathbb{Q}}. \tag{3.17.1}$$

The construction of ι, that we will briefly recall in § 3.20, is based on the following construction in formal geometry.

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3.18 Let $\mathfrak{Y}$ be an object of $\mathbf{S}^{\diamond}$ and $\mathscr{A}$ an open ideal of finite type of $\mathfrak{Y}$ containing p [Abb10, 2.1.19]. We denote by $\mathfrak{Y}'$ the admissible blow-up of $\mathscr{A}$ in $\mathfrak{Y}$ [Abb10, 3.1.2]. The ideal $\mathscr{A}\mathscr{O}_{\mathfrak{Y}'}$ is invertible [Abb10, 3.1.4(i)] and $\mathfrak{Y}'$ is flat over W [Abb10, 3.1.4(ii)]. We denote by $\mathfrak{Y}_{(\mathscr{A}/p)}$ the maximal open formal subscheme of $\mathfrak{Y}'$ on which

$$(\mathscr{A}\mathscr{O}_{\mathfrak{X}'})|\mathfrak{X}_{(\mathscr{A}/p)} = (p\mathscr{O}_{\mathfrak{X}'})|\mathfrak{X}_{(\mathscr{A}/p)} \tag{3.18.1}$$

and we call it *the dilatation of* $\mathscr{A}$ *with respect to* p [Abb10, 3.2.3.4 and 3.2.7]. Note that $\mathfrak{Y}_{(\mathscr{A}/p)}$ is the complement of $\mathrm{Supp}(\mathscr{A}\mathscr{O}_{\mathfrak{Y}'}/p\mathscr{O}_{\mathfrak{Y}'})$ in $\mathfrak{Y}'$ [EGAI, 0.5.2.2].

Let Z be a closed subscheme of Y and $\mathscr{I}$ the ideal sheaf associated to the canonical morphism $Z \to \mathfrak{Y}$. For any $n \geqslant 1$, we denote by $\mathfrak{T}_{Z,n}(\mathfrak{Y})$ the dilatation of $\mathscr{I}^n + p\mathscr{O}_{\mathfrak{Y}}$ with respect to p [Ogu84, 2.5].[2] By (3.18.1), there exists a morphism from the reduced subscheme of $(\mathfrak{T}_{Z,n}(\mathfrak{Y}))_1$ to Z which fits into the following diagram:

$$\begin{array}{ccc} (\mathfrak{T}_{Z,n}(\mathfrak{Y}))_{1,\mathrm{red}} & \longrightarrow & \mathfrak{T}_{Z,n}(\mathfrak{Y}) \\ \downarrow & & \downarrow \\ Z & \longrightarrow & \mathfrak{Y} \end{array} \tag{3.18.2}$$

In particular, $\mathfrak{T}_{Z,n}(\mathfrak{Y})$ defines an object of $\mathrm{Conv}(Z/\mathrm{W})$.

The universal property of dilatation [Abb10, 3.2.6] can be reinterpreted in the following way.

Proposition 3.19 [Xu19, 3.5]. *Keep the notation and assumptions of § 3.18. Let $\mathfrak{T}$ be an adic flat formal* W*-scheme, $T^{(1)}$ the closed subscheme of T defined by the ideal sheaf $\{x \in \mathscr{O}_T | x^p = 0\}$ and $f : \mathfrak{T} \to \mathfrak{Y}$ an adic morphism. Suppose that there exists a morphism $T \to Z$ (respectively $T^{(1)} \to Z$) which fits into the following diagram:*

$$\begin{array}{ccc} T & \longrightarrow & \mathfrak{T} \\ \downarrow & & \downarrow f \\ Z & \longrightarrow & \mathfrak{Y} \end{array} \qquad \left(\textit{respectively} \quad \begin{array}{ccc} T^{(1)} & \longrightarrow & \mathfrak{T} \\ \downarrow & & \downarrow f \\ Z & \longrightarrow & \mathfrak{Y} \end{array}\right)$$

Then there exists a unique adic morphism $g : \mathfrak{T} \to \mathfrak{T}_{Z,1}(\mathfrak{Y})$ (respectively $g : \mathfrak{T} \to \mathfrak{T}_{Z,p}(\mathfrak{Y})$) lifting f.

3.20 We briefly review the construction of ι (Proposition 3.17) in the case where X is separated and admits a smooth lifting $\mathfrak{X}$ over $\mathfrak{S}$.

We set $\mathrm{Q}_{\mathfrak{X}/\mathfrak{S}} = \mathfrak{T}_{X,p}(\mathfrak{X}^2)$, the dilatation of the diagonal immersion $X \to \mathfrak{X}^2 = \mathfrak{X} \times_{\mathfrak{S}} \mathfrak{X}$ (§ 3.18). Using its universal property (Proposition 3.19), one verifies that $\mathrm{Q}_{\mathfrak{X}/\mathfrak{S}}$ is equipped with a formal $\mathfrak{X}$-groupoid structure (§ 2.8) (cf. [Xu19, 4.11]).

Let $\mathscr{E}$ be an object of $\mathrm{Iso}^{\dagger}(X/\mathfrak{S})$. The canonical morphisms $p_1, p_2 : \mathrm{Q}_{\mathfrak{X}/\mathfrak{S}} \to \mathfrak{X}$ give rise to morphisms of $\mathrm{Conv}(X/\mathfrak{S})$. Since $\mathscr{E}$ is a crystal, the composition of transition morphisms $\varepsilon = c_{p_1}^{-1} \circ c_{p_2}$ defines an $\mathscr{O}_{\mathrm{Q}_{\mathfrak{X}/\mathfrak{S}}}$-stratification on $\mathscr{E}_{\mathfrak{X}}$ (§ 2.8).

Using 3.19, the canonical morphism $\mathrm{P}_{\mathfrak{X}/\mathfrak{S}} \to \mathfrak{X}^2$ induces a morphism of formal $\mathfrak{X}$-groupoids $\mathrm{P}_{\mathfrak{X}/\mathfrak{S}} \to \mathrm{Q}_{\mathfrak{X}/\mathfrak{S}}$ (cf. [Xu19, 5.12]). By taking the inverse image, we obtain an $\mathscr{O}_{\mathrm{P}_{\mathfrak{X}/\mathfrak{S}}}$-stratification on $\mathscr{E}_{\mathfrak{X}}$ and then a crystal of $\mathscr{O}_{X/\mathfrak{S}}^{\mathrm{crys}}$-modules of finite presentation up to isogeny by § 2.9 and Proposition 2.10. The construction is clearly functorial.

[2] The union of rigid space $\bigcup_{n \geqslant 1} \mathfrak{T}_{Z,n}(\mathfrak{Y})^{\mathrm{rig}}$ is the same as the tube of Z in $\mathfrak{Y}$ introduced by Berthelot (cf. [Ber96a, 1.1.2 and 1.1.10]).

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3.21 There exists a canonical morphism of topoi $u_{X/\mathfrak{S}} : (X/\mathfrak{S})_{\mathrm{conv,zar}} \to X_{\mathrm{zar}}$ (cf. [Ogu07, § 4]). Suppose that $\mathfrak{S}$ is separated and that X admits a smooth lifting $f : \mathfrak{X} \to \mathfrak{S}$. Let $\mathscr{E}$ be a convergent isocrystal of $(X/\mathfrak{S})_{\mathrm{conv,zar}}$. By § 2.9 and Proposition 2.10, there exists an integrable connection on $\mathscr{E}_{\mathfrak{X}}$ relative to $\mathfrak{S}$ and we denote by $\mathscr{E}_{\mathfrak{X}} \otimes_{\mathscr{O}_{\mathfrak{X}}} \widehat{\Omega}^{\bullet}_{\mathfrak{X}/\mathfrak{S}}$ the associated de Rham complex. Then there exists a canonical isomorphism in the derived category $\mathrm{D}(\mathfrak{X}_{\mathrm{zar}}, f^{-1}(\mathscr{O}_{\mathfrak{S}}[\frac{1}{p}]))$ [Shi07a, 2.33]

$$\mathrm{R}\, u_{X/\mathfrak{S}*}(\mathscr{E}) \xrightarrow{\sim} \mathscr{E}_{\mathfrak{X}} \otimes_{\mathscr{O}_{\mathfrak{X}}} \widehat{\Omega}^{\bullet}_{\mathfrak{X}/\mathfrak{S}}. \tag{3.21.1}$$

Based on the above isomorphism, the finiteness and the base-change property of relative crystalline cohomology (§ 2.13), Shiho showed the following results.

Theorem 3.22 [Shi07a, 2.36]. *Assume that $\mathfrak{S}$ is separated and that X is smooth and proper over S. Let $\mathscr{E}$ be a convergent isocrystal of $(X/\mathfrak{S})_{\mathrm{conv,zar}}$ and $g_{\mathrm{conv,zar}} : (X/\mathfrak{S})_{\mathrm{conv,zar}} \to (S/\mathfrak{S})_{\mathrm{conv,zar}}$ the functorial morphism. Then there exists a canonical isomorphism in the derived category of $\mathscr{O}_{\mathfrak{S}}[\frac{1}{p}]$-modules (§ 2.13 and Proposition 3.17)*

$$(\mathrm{R}\, g_{\mathrm{conv,zar}\,*}(\mathscr{E}))_{\mathfrak{S}} \xrightarrow{\sim} \mathrm{R}\, g_{X/\mathfrak{S},\mathrm{crys}\,*}(\iota(\mathscr{E})). \tag{3.22.1}$$

In particular, $(\mathrm{R}^i\, g_{\mathrm{conv,zar}\,*}(\mathscr{E}))_{\mathfrak{S}}$ is coherent for any $i \geqslant 0$.

Theorem 3.23 [Shi07a, 2.37]. *Keep the assumption of Theorem 3.22 and suppose moreover that $\mathscr{E}$ is locally projective (§ 3.16). Then $(\mathrm{R}\, g_{\mathrm{conv,zar}\,*}(\mathscr{E}))_{\mathfrak{S}}$ is a perfect complex of $\mathscr{O}_{\mathfrak{S}}[\frac{1}{p}]$-modules.*

Theorem 3.24 [Shi07a, 2.38]. *Let $\varphi : \mathfrak{S}' \to \mathfrak{S}$ be an adic morphism of adic separated flat formal W-schemes of finite type, $X' = X \times_{\mathfrak{S}} \mathfrak{S}'$ and $\varphi_{\mathrm{conv,zar}} : (X'/\mathfrak{S}')_{\mathrm{conv,zar}} \to (X/\mathfrak{S})_{\mathrm{conv,zar}}$ the functorial morphism of convergent topoi (3.11.1). Then, for a locally projective convergent isocrystal $\mathscr{E}$ of $(X/\mathfrak{S})_{\mathrm{conv,zar}}$, we have a canonical isomorphism in the derived category of $\mathscr{O}_{\mathfrak{S}'}[\frac{1}{p}]$-modules:*

$$\mathrm{L}\, \varphi^*_{\mathrm{zar}}((\mathrm{R}\, g_{\mathrm{conv,zar}\,*}(\mathscr{E}))_{\mathfrak{S}}) \xrightarrow{\sim} (\mathrm{R}\, g'_{\mathrm{conv,zar}\,*}(\varphi^*_{\mathrm{conv,zar}}(\mathscr{E})))_{\mathfrak{S}'}. \tag{3.24.1}$$

4. Higher direct images of a convergent isocrystal are *p*-adically convergent

4.1 In this section, we keep the notation of § 3 and let $g : X \to Y$ denote a morphism of S-schemes.

Let $\mathfrak{T}$ be an object of $\mathrm{Conv}(Y/\mathfrak{S})$ and $\tau \in \{\mathrm{zar}, \mathrm{fppf}\}$. By fppf descent for morphisms of formal W-schemes [Abb10, 5.12.1], the presheaf associated to $\mathfrak{T}$ is a sheaf for the fppf (respectively Zariski) topology that we denote by $\widetilde{\mathfrak{T}}$. We set $X_{T_0} = X \times_Y T_0$ and we denote by

$$g_{X/\mathfrak{T},\tau} : (X_{T_0}/\mathfrak{T})_{\mathrm{conv},\tau} \to (T_0/\mathfrak{T})_{\mathrm{conv},\tau},$$
$$\omega_{\mathfrak{T}} : (X_{T_0}/\mathfrak{T})_{\mathrm{conv},\tau} \to (X/\mathfrak{S})_{\mathrm{conv},\tau}$$

the functorial morphisms of topoi (3.11.1).

Lemma 4.2 [Ber74, V 3.2.2]. *There exists a canonical equivalence of topoi:*

$$(X/\mathfrak{S})_{\mathrm{conv},\tau/g^*_{\mathrm{conv},\tau}(\widetilde{\mathfrak{T}})} \xrightarrow{\sim} (X_{T_0}/\mathfrak{T})_{\mathrm{conv},\tau}, \tag{4.2.1}$$

which identifies the localization morphism and $\omega_{\mathfrak{T}}$.

The lemma can be verified in the same way as [Ber74, V 3.2.2].

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Lemma 4.3 [Ber74, V 3.2.3]. *For any $\mathscr{O}_{X/\mathfrak{S}}[\frac{1}{p}]$-module E of $(X/\mathfrak{S})_{\mathrm{conv},\tau}$, there exists a canonical isomorphism in $\mathrm{D}^+(\mathfrak{T}_\tau, \mathscr{O}_{\mathfrak{T}}[\frac{1}{p}])$:*

$$(\mathrm{R}\, g_{\mathrm{conv},\tau *}(E))_{\mathfrak{T}} \xrightarrow{\sim} (\mathrm{R}\, g_{X/\mathfrak{T},\tau *}(\omega_{\mathfrak{T}}^*(E)))_{\mathfrak{T}}. \tag{4.3.1}$$

Proof. Let E be an abelian sheaf of $(X/\mathfrak{S})_{\mathrm{conv},\tau}$ and $f:\mathfrak{T}'\to\mathfrak{T}$ a morphism of $\mathrm{Conv}(Y/\mathfrak{S})$ (§ 4.1). The morphism f induces a functorial morphism of topoi $\varphi:(X_{T_0'}/\mathfrak{T}')_{\mathrm{conv},\tau}\to(X_{T_0}/\mathfrak{T})_{\mathrm{conv},\tau}$, which fits into the following commutative diagram:

$$\begin{array}{ccc} (X_{T_0'}/\mathfrak{T}')_{\mathrm{conv},\tau} & \xrightarrow{\varphi} & (X_{T_0}/\mathfrak{T})_{\mathrm{conv},\tau} \\ {\scriptstyle g_{X/\mathfrak{T}',\tau}}\downarrow & & \downarrow{\scriptstyle g_{X/\mathfrak{T},\tau}} \\ (T_0'/\mathfrak{T}')_{\mathrm{conv},\tau} & \xrightarrow{f_{\mathrm{conv},\tau}} & (T_0/\mathfrak{T})_{\mathrm{conv},\tau} \end{array} \tag{4.3.2}$$

We have $\omega_{\mathfrak{T}'}=\omega_{\mathfrak{T}}\circ\varphi$. By Lemma 4.2, φ (respectively $f_{\mathrm{conv},\tau}$) coincides with the localization morphism on the sheaf $g_{X/\mathfrak{T},\tau}^*(\widetilde{\mathfrak{T}}')$ (respectively $\widetilde{\mathfrak{T}}'$). Then $g_{X/\mathfrak{T},\tau*}(\omega_{\mathfrak{T}}^*(E))$ is the sheaf associated to the presheaf on $\mathrm{Conv}(T_0/\mathfrak{T})$:

$$(f:\mathfrak{T}'\to\mathfrak{T})\mapsto\Gamma((X_{T_0'}/\mathfrak{T}')_{\mathrm{conv},\tau},\omega_{\mathfrak{T}'}^*(E)).$$

The sheaf $(g_{\mathrm{conv},\tau*}(E))_{\mathfrak{T}}$ is associated to the presheaf

$$(f:\mathfrak{T}'\to\mathfrak{T})\mapsto\Gamma((X/\mathfrak{S})_{\mathrm{conv},\tau/g_{\mathrm{conv},\tau}^*(\widetilde{\mathfrak{T}}')}, E|_{g_{\mathrm{conv},\tau}^*(\widetilde{\mathfrak{T}}')}).$$

By Lemma 4.2, we deduce a canonical isomorphism of $\mathfrak{T}_\tau$:

$$(g_{\mathrm{conv},\tau*}(E))_{\mathfrak{T}}\xrightarrow{\sim}(g_{X/\mathfrak{T},\tau*}(\omega_{\mathfrak{T}}^*(E)))_{\mathfrak{T}}. \tag{4.3.3}$$

Since $\omega_{\mathfrak{T}}$ coincides with a localization morphism, if $I^\bullet$ is an injective resolution of E, $\omega_{\mathfrak{T}}^*(I^\bullet)$ is an injective resolution of $\omega_{\mathfrak{T}}^*(E)$. Then the isomorphism (4.3.1) follows from (4.3.3). □

Remark 4.4. Keep the notation of Lemma 4.3. Let $f:\mathfrak{T}'\to\mathfrak{T}$ be a morphism of $\mathrm{Conv}(Y/\mathfrak{S})$. It induces morphisms of topoi (4.3.2). We consider canonical morphisms

$$f_\tau^*(\mathrm{R}^i\, g_{X/\mathfrak{T},\tau*}(\omega_{\mathfrak{T}}^*(E))_{\mathfrak{T}})\to(f_{\mathrm{conv},\tau}^*(\mathrm{R}^i\, g_{X/\mathfrak{T},\tau*}(\omega_{\mathfrak{T}}^*(E))))_{\mathfrak{T}'}\xrightarrow{\sim}\mathrm{R}^i\, g_{X/\mathfrak{T}',\tau*}(\omega_{\mathfrak{T}'}^*(E))_{\mathfrak{T}'}, \tag{4.4.1}$$

where the first morphism is the transition morphism of $\mathrm{R}^i\, g_{X/\mathfrak{T},\tau*}(\omega_{\mathfrak{T}}^*(E))$ associated to f and the second one is an isomorphism because $\omega_{\mathfrak{T}'}=\omega_{\mathfrak{T}}\circ\varphi,\varphi,f_{\mathrm{conv},\tau}$ are localization morphisms [SGA4, V 5.1].

In view of the proof of Lemma 4.3, via (4.3.1), the above composition is compatible with the transition morphism of $\mathrm{R}^i\, g_{\mathrm{conv},\tau*}(E)$ associated to f:

$$f_\tau^*((\mathrm{R}^i\, g_{\mathrm{conv},\tau*}(E))_{\mathfrak{T}})\to(\mathrm{R}^i\, g_{\mathrm{conv},\tau*}(E))_{\mathfrak{T}'}. \tag{4.4.2}$$

Corollary 4.5 [BBM82, 1.1.19]. *Let $\mathscr{E}$ be a convergent isocrystal of $(X/\mathfrak{S})_{\mathrm{conv,fppf}}$. We have (§ 3.8)*

$$\mathrm{R}^i\,\alpha_*(\mathscr{E})=0,\quad\forall i\geqslant 1. \tag{4.5.1}$$

The assertion can be verified in the same way as [BBM82, 1.1.19].

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COROLLARY 4.6. *Let $\mathfrak{S}' \to \mathfrak{S}$ be a morphism of $\mathbf{S}^{\diamond}$ (§ 2.1), Y' an S'-scheme and $h : Y' \to Y$ a morphism compatible with $S' \to S$. We set $X' = X \times_Y Y'$ and we denote by $g' : X' \to Y'$ and $h' : X' \to X$ the canonical morphisms:*

$$\begin{array}{ccc} X' & \xrightarrow{h'} & X \\ {\scriptstyle g'}\downarrow & \square & \downarrow{\scriptstyle g} \\ Y' & \xrightarrow{h} & Y \end{array}$$

Then, for any $\mathscr{O}_{X/\mathfrak{S}}[\frac{1}{p}]$-module E of $(X/\mathfrak{S})_{\mathrm{conv},\tau}$, the base-change morphism

$$h^{*}_{\mathrm{conv},\tau}(\mathrm{R}\, g_{\mathrm{conv},\tau *}(E)) \xrightarrow{\sim} \mathrm{R}\, g'_{\mathrm{conv},\tau *}(h'^{*}_{\mathrm{conv},\tau}(E)) \tag{4.6.1}$$

is an isomorphism.

Proof. Let $\mathfrak{T}$ be an object of $\mathrm{Conv}(Y'/\mathfrak{S}')$. We denote abusively the image of $\mathfrak{T}$ in $\mathrm{Conv}(Y/\mathfrak{S})$ by $\mathfrak{T}$. We set $X_{T_0} = T_0 \times_Y X (= T_0 \times_{Y'} X')$. By applying Lemma 4.3 to g and g', one verifies that the evaluations of the two sides of (4.6.1) at $\mathfrak{T}$ are both isomorphic to $(\mathrm{R}\, g_{X/\mathfrak{T},\tau *}(\omega^{*}_{\mathfrak{T}}(E)))_{\mathfrak{T}}$ and that (4.6.1) induces an isomorphism between them. Then the assertion follows. □

4.7 In the remainder of this section, we consider the case where $\mathfrak{S} = \mathrm{Spf}(\mathrm{W})$ and X, Y are schemes over $S = \mathrm{Spec}(k)$. We denote by $\mathrm{pConv}(X/\mathrm{W})$ the full subcategory of $\mathrm{Conv}(X/\mathrm{W})$ consisting of objects $(\mathfrak{T}, u)$ such that u can be lifted to a k-morphism $\widetilde{u} : T \to X$. Given an object $\mathfrak{T}$ of $\mathrm{pConv}(X/\mathrm{W})$ and a morphism $f : \mathfrak{T}' \to \mathfrak{T}$ of $\mathrm{Conv}(X/\mathrm{W})$, then $\mathfrak{T}'$ is still an object of $\mathrm{pConv}(X/\mathrm{W})$. Objects of $\mathrm{pConv}(X/\mathrm{W})$ are closely related to 'p-adic enlargements' in [Ogu84].

We end this section by showing the following result.

PROPOSITION 4.8. *Suppose that Y is smooth over k and that $g : X \to Y$ is smooth and proper. Let $\mathscr{E}$ be a convergent isocrystal of $(X/\mathrm{W})_{\mathrm{conv},\tau}$ for $\tau \in \{\mathrm{zar}, \mathrm{fppf}\}$ and i an integer $\geqslant 0$. We have:*

(i) *for every object $\mathfrak{T}$ of $\mathrm{pConv}(Y/\mathrm{W})$ (§ 4.7), $\mathrm{R}^i\, g_{\mathrm{conv},\tau *}(\mathscr{E})_{\mathfrak{T}}$ is coherent (§ 3.12);*

(ii) *for every morphism f of $\mathrm{pConv}(Y/\mathrm{W})$, the associated transition morphism c_f of $\mathrm{R}^i\, g_{\mathrm{conv},\tau *}(\mathscr{E})$ is an isomorphism.*

LEMMA 4.9. *Suppose that X is smooth over k. A convergent isocrystal $\mathscr{E}$ of $(X/\mathrm{W})_{\mathrm{conv},\mathrm{zar}}$ is locally projective (§ 3.16).*

Proof. The question being local, we may assume that X admits a smooth lifting $\mathfrak{X}$ over W. By Lemma 2.12 and Proposition 3.17, $\mathscr{E}_{\mathfrak{X}}$ is locally projective. Since every object $\mathfrak{T}$ of $\mathrm{Conv}(X/\mathrm{W})$ locally admits a morphism to $\mathfrak{X}$, we deduce that $\mathscr{E}$ is locally projective. □

LEMMA 4.10. *Keep the assumption of Proposition 4.8 and assume moreover that Y admits a formal smooth lifting $\mathfrak{Y}$ over W. Let $\mathscr{E}$ be a convergent isocrystal of $(X/\mathrm{W})_{\mathrm{conv},\mathrm{zar}}$. Then there exists an $\mathscr{O}_{\mathrm{P}_{\mathfrak{Y}/\mathfrak{S}}}$-stratification on $(\mathrm{R}^i\, g_{\mathrm{conv},\mathrm{zar}\,*}(\mathscr{E}))_{\mathfrak{Y}}$. In particular, $(\mathrm{R}^i\, g_{\mathrm{conv},\mathrm{zar}\,*}(\mathscr{E}))_{\mathfrak{Y}}$ is locally projective of finite type (Definition 2.11).*

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Proof. We take again the notation of § 3.20 for $\mathfrak{Y} \to \mathrm{Spf}(\mathrm{W})$ and we set $\mathfrak{Z} = \mathrm{Q}_{\mathfrak{Y}/\mathrm{W}}$, which we consider as an object of $\mathrm{Conv}(Y/\mathrm{W})$, and $\mathscr{F} = \mathrm{R}^i\, g_{\mathrm{conv,zar}\,*}(\mathscr{E})$. By Lemma 4.3, we have canonical isomorphisms

$$\mathscr{F}_{\mathfrak{Y}} \xrightarrow{\sim} (\mathrm{R}^i\, g_{X/\mathfrak{Y},\mathrm{zar}\,*}(\omega_{\mathfrak{Y}}^*(\mathscr{E})))_{\mathfrak{Y}}, \quad \mathscr{F}_{\mathfrak{Z}} \xrightarrow{\sim} (\mathrm{R}^i\, g_{X/\mathfrak{Z},\mathrm{zar}\,*}(\omega_{\mathfrak{Z}}^*(\mathscr{E})))_{\mathfrak{Z}}. \tag{4.10.1}$$

By Theorem 3.22, $\mathscr{F}_{\mathfrak{Y}}$ is coherent. The projections $p_1, p_2 : \mathfrak{Z} \to \mathfrak{Y}$ define two morphisms of $\mathrm{Conv}(Y/\mathrm{W})$ and induce two morphisms of topoi

$$(X_{Z_0}/\mathfrak{Z})_{\mathrm{conv,zar}} \xrightarrow{\sim} (X \times_{Y,p_1} Z/\mathfrak{Z})_{\mathrm{conv,zar}} \to (X/\mathfrak{Y})_{\mathrm{conv,zar}},$$
$$(X_{Z_0}/\mathfrak{Z})_{\mathrm{conv,zar}} \xrightarrow{\sim} (X \times_{Y,p_2} Z/\mathfrak{Z})_{\mathrm{conv,zar}} \to (X/\mathfrak{Y})_{\mathrm{conv,zar}}.$$

Since X is smooth over k, $\mathscr{E}$ is locally projective by Lemma 4.9. The projections p_1, p_2 are rig-flat [Abb10, 5.4.12]. By Theorem 3.24 and Remark 4.4, p_1, p_2 induce isomorphisms

$$p_2^*(\mathscr{F}_{\mathfrak{Y}}) \xrightarrow[c_{p_2}]{\sim} \mathscr{F}_{\mathfrak{Z}} \xleftarrow[c_{p_1}]{\sim} p_1^*(\mathscr{F}_{\mathfrak{Y}}). \tag{4.10.2}$$

By a standard argument, the isomorphism $c_{p_1}^{-1} \circ c_{p_2}$ defines an $\mathscr{O}_{\mathrm{Q}_{\mathfrak{Y}/\mathrm{W}}}$-stratification on $\mathscr{F}_{\mathfrak{Y}}$. Taking pullback by $\mathrm{P}_{\mathfrak{Y}/\mathrm{W}} \to \mathrm{Q}_{\mathfrak{Y}/\mathrm{W}}$ (Proposition 3.17), we obtain an $\mathscr{O}_{\mathrm{P}_{\mathfrak{Y}/\mathrm{W}}}$-stratification on $\mathscr{F}_{\mathfrak{Y}}$. The second assertion follows from Lemma 2.12. □

4.11 In the following, we prove Proposition 4.8. By Lemma 4.9, $\mathscr{E}$ is locally projective. We set $\mathscr{F}_{\tau}^i = \mathrm{R}^i\, g_{\mathrm{conv},\tau *}(\mathscr{E})$ and $\mathscr{G}_{\tau}^i = (\mathrm{R}^i\, g_{X/\mathfrak{T},\tau *}(\omega_{\mathfrak{T}}^*(\mathscr{E})))$ (§ 4.1). By Lemma 4.3, we have a canonical isomorphism

$$\mathscr{F}_{\tau,\mathfrak{T}}^i \xrightarrow{\sim} \mathscr{G}_{\tau,\mathfrak{T}}^i. \tag{4.11.1}$$

Proof of Proposition 4.8 for Zariski topology. (i) Since $(\mathfrak{T}, u)$ is an object of $\mathrm{pConv}(Y/\mathrm{W})$, we take a lifting $\widetilde{u} : T \to Y$ of u and we set $X_T = X \times_Y T$. Then we have a canonical equivalence $(X_{T_0}/\mathfrak{T})_{\mathrm{conv},\tau} \xrightarrow{\sim} (X_T/\mathfrak{T})_{\mathrm{conv},\tau}$ (Definition 3.1) and the assertion follows from Theorem 3.22.

(ii) The question being local, by Corollary 4.6, we may therefore assume that Y is affine and admits a smooth lifting $\mathfrak{Y}$ over W. Then $\mathscr{F}_{\mathrm{zar},\mathfrak{Y}}^i$ and $\mathscr{G}_{\mathrm{zar},\mathfrak{Y}}^i$ are locally projective of finite type by Lemma 4.10.

We first prove the assertion (ii) for a morphism $h : \mathfrak{T} \to \mathfrak{Y}$ of $\mathrm{pConv}(Y/\mathrm{W})$ with target $\mathfrak{Y}$. By Theorem 3.24, we have a spectral sequence

$$\mathrm{E}_2^{i-j,j} = \mathrm{L}_{i-j}\, h_{\mathrm{zar}}^*(\mathscr{G}_{\mathrm{zar},\mathfrak{Y}}^j) \Rightarrow \mathscr{G}_{\mathrm{zar},\mathfrak{T}}^i. \tag{4.11.2}$$

Since each $\mathscr{G}_{\mathrm{zar},\mathfrak{Y}}^j$ is locally projective of finite type, we deduce that $\mathrm{E}_2^{i-j,j} = 0$ for $i \neq j$. Then the transition morphism of $\mathscr{F}_{\mathrm{zar},\mathfrak{Y}}^i$ associated to f is an isomorphism by Remark 4.4 and (4.11.1).

Since the question is local, for a general morphism $f : (\mathfrak{T}', u') \to (\mathfrak{T}, u)$ of $\mathrm{pConv}(Y/\mathrm{W})$, we may assume that u can be lifted to a morphism $h : \mathfrak{T} \to \mathfrak{Y}$ of $\mathrm{pConv}(Y/\mathrm{W})$. By the previous result, c_h and $c_{h\circ f}$ are isomorphisms. Then we deduce that c_f is an isomorphism by Proposition 3.14(c).

4.12 *Proof of Proposition 4.8 for fppf topology.* We consider the presheaf $\mathscr{P}$ on $\mathrm{Conv}(Y/\mathrm{W})$ defined by

$$(\mathfrak{T}, u) \mapsto \mathrm{H}^i((X_{T_0}/\mathfrak{T})_{\mathrm{conv,fppf}}, \omega_{\mathfrak{T}}^*(\mathscr{E})).$$

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By Corollary 4.5, the right-hand side is isomorphic to $\mathrm{H}^i((X_{T_0}/\mathfrak{T})_{\mathrm{conv,zar}}, \omega_{\mathfrak{T}}^*(\alpha_*(\mathscr{E})))$. We set $\mathscr{F}^i_{\mathrm{zar}} = \mathrm{R}^i\, g_{\mathrm{conv,zar}\,*}(\alpha_*(\mathscr{E}))$. By Lemma 4.3, the fppf (respectively Zariski) sheaf associated to $\mathscr{P}$ is $\mathscr{F}^i_{\mathrm{fppf}}$ (respectively $\mathscr{F}^i_{\mathrm{zar}}$). Then we deduce a canonical isomorphism (3.8.1)

$$\alpha^*(\mathscr{F}^i_{\mathrm{zar}}) \xrightarrow{\sim} \mathscr{F}^i_{\mathrm{fppf}}. \tag{4.12.1}$$

Let $\mathfrak{T}$ be an object of $\mathrm{pConv}(Y/\mathrm{W})$. By Proposition 4.8 for Zariski topology and fppf descent, we deduce that $\mathscr{F}^i_{\mathrm{fppf},\mathfrak{T}}$ is the fppf sheaf associated to the coherent $\mathscr{O}_{\mathfrak{T}}[\frac{1}{p}]$-module $\mathscr{F}^i_{\mathrm{zar},\mathfrak{T}}$ (§ 3.12) and hence is coherent. Assertion (i) follows.

Since $\mathscr{F}^i_{\mathrm{fppf},\mathfrak{T}}$ is the fppf sheaf associated to $\mathscr{F}^i_{\mathrm{zar},\mathfrak{T}}$, assertion (ii) follows from Proposition 4.8(ii) for Zariski topology and (4.12.1). □

5. Frobenius descents

5.1 In this section, $\mathfrak{S}$ denotes an adic flat formal W-scheme of finite type. We suppose that the Frobenius morphism $F_{S_0}: S_0 \to S_0$ of the reduced subscheme of S is *flat* (and hence faithfully flat). Let X be an S_0-scheme locally of finite type. We denote by $(-)'$ the base-change functor F_{S_0} and by $F_{X/S_0}: X \to X'$ the relative Frobenius morphism of X relative to S_0. Then we have a commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{F_{X/S_0}} & X' & \longrightarrow & X \\ & \searrow & \downarrow & \Box & \downarrow \\ & & S_0 & \xrightarrow{F_{S_0}} & S_0 \end{array} \tag{5.1.1}$$

The following theorem is one of the main results in this section.

Theorem 5.2. *Suppose that the Frobenius morphism $F_{S_0}: S_0 \to S_0$ is flat. For every S_0-scheme locally of finite type X, the functorial morphism of convergent topoi (3.11.1) of F_{X/S_0} induces an equivalence of topoi*

$$F_{X/S_0,\mathrm{conv,fppf}}: (X/\mathfrak{S})_{\mathrm{conv,fppf}} \xrightarrow{\sim} (X'/\mathfrak{S})_{\mathrm{conv,fppf}}. \tag{5.2.1}$$

Proof. The morphism $F_{X/S_0,\mathrm{conv,fppf}}$ is induced by the functor (3.9.2):

$$\rho: \mathrm{Conv}(X/\mathfrak{S}) \to \mathrm{Conv}(X'/\mathfrak{S}), \quad (\mathfrak{T}, u) \mapsto (\mathfrak{T}, F_{X/S_0} \circ u). \tag{5.2.2}$$

Note that $F_{X/S_0} \circ u = u' \circ F_{T_0/S_0}$.

By 3.10 and Lemmas 5.4 and 5.5 in the following, the functor ρ satisfies the conditions of Proposition 2.5. Then the theorem follows from Proposition 2.5. □

Lemma 5.3. *Let Y be a reduced S_0-scheme, Z an S_0-scheme and $g_1, g_2: Y \to Z$ two S_0-morphisms. We put $h_i = g_i' \circ F_{Y/S_0}: Y \to Y' \to Z'$ for $i = 1, 2$. If $h_1 = h_2$, then $g_1 = g_2$.*

Proof. Since F_{Y/S_0} is a homeomorphism and $h_1 = h_2$, then $|g_1| = |g_2|$ on the underlying topological spaces. Since the question is local, we can reduce to the case where Y, Z, S_0 are affine.

Since Y is reduced and separated over S_0, F_{Y/S_0} is schematically dominant [EGAI, 5.4.2] and we deduce that $g_1' = g_2'$ [EGAI, 5.4.1]. The Frobenius morphism F_{S_0} is faithfully flat. Then the functor $Y \mapsto Y'$ from the category of affine S_0-schemes to itself is faithful. The lemma follows. □

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Lemma 5.4. *The functor ρ is fully faithful.*

Proof. The functor ρ is clearly faithful. We prove its fullness. Let $(\mathfrak{T}_1, u_1)$ and $(\mathfrak{T}_2, u_2)$ be two objects of $\mathrm{Conv}(X/\mathfrak{S})$ and $g : \rho(\mathfrak{T}_1, u_1) \to \rho(\mathfrak{T}_2, u_2)$ a morphism of $\mathrm{Conv}(X'/\mathfrak{S})$. We set $g_0 : T_{1,0} \to T_{2,0}$, the induced morphism. To show that the morphism $\mathfrak{T}_1 \to \mathfrak{T}_2$ defines a morphism of $\mathrm{Conv}(X/\mathfrak{S})$ which is sent to g by ρ, it suffices to show that $u_1 = u_2 \circ g_0$. Since g is a morphism of $\mathrm{Conv}(X'/\mathfrak{S})$, we have a commutative diagram

$$\begin{array}{ccc} T_{1,0} & \xrightarrow{g_0} & T_{2,0} \\ \downarrow{\scriptstyle F_{T_{1,0}/S_0}} & & \downarrow{\scriptstyle F_{T_{2,0}/S_0}} \\ (T_{1,0})' & \xrightarrow{g_0'} & (T_{2,0})' \\ & {\scriptstyle u_1'}\searrow \quad \swarrow{\scriptstyle u_2'} & \\ & X' & \end{array}$$

Then the assertion follows from Lemma 5.3 applied to u_1 and $u_2 \circ g_0$. □

Lemma 5.5. (i) *Let $(\mathfrak{T}, u)$ be an object of $\mathrm{Conv}(X'/\mathfrak{S})$ such that $\mathfrak{T}$ is affine and that $u : T_0 \to X'$ factors through an affine open subscheme U' of X'. Then there exist an object $(\mathfrak{Z}, v)$ of $\mathrm{Conv}(X/\mathfrak{S})$ and an fppf covering $\{f : \rho(\mathfrak{Z}, v) \to (\mathfrak{T}, u)\}$ in $\mathrm{Conv}(X'/\mathfrak{S})$.*

(ii) *Keep the assumption and notation of (i). Let $g : (\mathfrak{T}_1, u_1) \to (\mathfrak{T}, u)$ be a morphism of $\mathrm{Conv}(X'/\mathfrak{S})$. Then there exist a morphism $h : (\mathfrak{Z}_1, v_1) \to (\mathfrak{Z}, v)$ of $\mathrm{Conv}(X/\mathfrak{S})$ and an fppf covering $\{\varphi : \rho(\mathfrak{Z}_1, v_1) \to (\mathfrak{T}_1, u_1)\}$ such that the following diagram is Cartesian:*

$$\begin{array}{ccc} \rho(\mathfrak{Z}_1, v_1) & \xrightarrow{\varphi} & (\mathfrak{T}_1, u_1) \\ \downarrow{\scriptstyle \rho(h)} & \square & \downarrow{\scriptstyle g} \\ \rho(\mathfrak{Z}, v) & \xrightarrow{f} & (\mathfrak{T}, u) \end{array} \tag{5.5.1}$$

(iii) *Every object of $\mathrm{Conv}(X'/\mathfrak{S})$ admits a Zariski covering whose objects satisfy the conditions of (i).*

Proof. (i) We set $U = F_{X/S_0}^{-1}(U')$, which is an affine S_0-scheme of finite type, and we take a closed S_0-immersion $\iota_0 : U \to Y_0 = \mathrm{Spec}(\mathscr{O}_{S_0}[T_1, \dots, T_d])$. We put $\mathfrak{Y} = \mathrm{Spf}(\mathscr{O}_{\mathfrak{S}}\{T_1, \dots, T_d\})$ and denote by $F : \mathfrak{Y} \to \mathfrak{Y}$ the $\mathfrak{S}$-morphism defined by sending each T_i to T_i^p.

Note that $Y_0' = Y_0$ and the restriction of F on Y_0 is the same as the relative Frobenius morphism F_{Y_0/S_0}. We have a commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{\iota_0} & Y_0 \\ \downarrow{\scriptstyle F_{U/S_0}} & & \downarrow{\scriptstyle F|_{Y_0}} \\ U' & \xrightarrow{\iota_0'} & Y_0 \end{array} \tag{5.5.2}$$

and a canonical morphism $U \to U' \times_{Y_0, F} Y_0$. We denote the composition of $\iota_0' : U' \to Y_0$ and $Y_0 \to \mathfrak{Y}$ by ι'. Since $\mathfrak{Y}$ is smooth over $\mathfrak{S}$, there exists an $\mathfrak{S}$-morphism $\tau : \mathfrak{T} \to \mathfrak{Y}$ lifting

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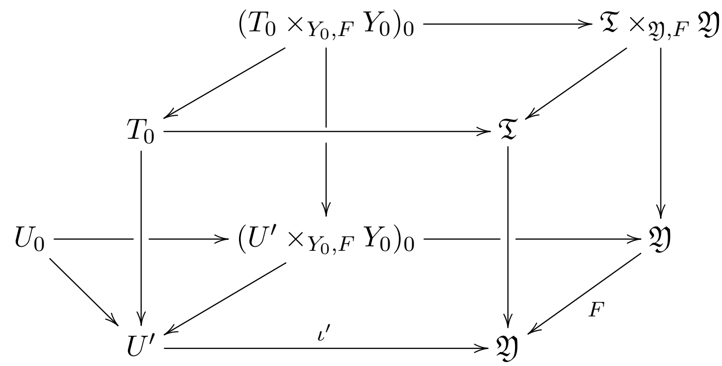
$\iota' \circ u : T_0 \to \mathfrak{Y}$. We consider the following commutative diagram:



(5.5.3)

where $U_0 \to U'$ is induced by F_{U/S_0}.

If the ideal sheaf associated to $\iota_0 : U \hookrightarrow Y_0$ is locally generated by polynomials $\{f_1, \ldots, f_n\}$ of $\mathscr{O}_{S_0}[T_1, \ldots, T_d]$, the ideal sheaf associated to $\iota'_0 \times_{Y_0,F} Y_0 : U' \times_{Y_0,F} Y_0 \hookrightarrow Y_0$ (5.5.2) is locally generated by $\{f_1^p, \ldots, f_n^p\}$. Then the canonical morphism $U \to U' \times_{Y_0,F} Y_0$ induces an isomorphism

$$U_0 \xrightarrow{\sim} (U' \times_{Y_0,F} Y_0)_0.$$

By (5.5.3) and [EGAI, 4.5.11], we obtain an object $(\mathfrak{T} \times_{\mathfrak{Y},F} \mathfrak{Y}, v)$ of $\mathrm{Conv}(X/\mathfrak{S})$ and a morphism $f : \rho(\mathfrak{T} \times_{\mathfrak{Y},F} \mathfrak{Y}, v) \to (\mathfrak{T}, u)$ of $\mathrm{Conv}(X'/\mathfrak{S})$. Since the reduction modulo p of F is faithfully flat of finite type [Ill96, 3.2], so is F (cf. [Xu19, 7.2]). Then f is an fppf covering (§ 3.4).

(ii) We denote by $(\mathfrak{Z}_1, w)$ the fibered product $\rho(\mathfrak{Z}, v) \times_{(\mathfrak{T},u)} (\mathfrak{T}_1, u_1)$ in $\mathrm{Conv}(X'/\mathfrak{S})$. By applying Lemma 3.10 to the projection $(\mathfrak{Z}_1, w) \to \rho(\mathfrak{Z}, v)$, we obtain the Cartesian diagram (5.5.1). Since φ is the base change of f, φ is an fppf covering.

(iii) Let $(\mathfrak{T}, u)$ be an object of $\mathrm{Conv}(X'/\mathfrak{S})$ and U' an affine open subscheme of X'. We denote by $\mathfrak{T}_{U'}$ the open formal subscheme of $\mathfrak{T}$ associated to the open subset $u^{-1}(|U'|)$ of $|T_0| = |T|$. The assertion follows by taking an affine covering of $\mathfrak{T}_{U'}$ for every U'. □

Lemma 5.6. *Let $\mathfrak{T}$ be an object of $\mathrm{Conv}(X'/\mathfrak{S})$, $\mathfrak{Z}$ an object of $\mathrm{Conv}(X/\mathfrak{S})$ and $\{\rho(\mathfrak{Z}) \to \mathfrak{T}\}$ a morphism of $\mathrm{Conv}(X'/\mathfrak{S})$. Then there exist an object $\mathfrak{Z} \times_{\mathfrak{T}} \mathfrak{Z}$ of $\mathrm{Conv}(X/\mathfrak{S})$ and two morphisms $p_1, p_2 : \mathfrak{Z} \times_{\mathfrak{T}} \mathfrak{Z} \to \mathfrak{Z}$ of $\mathrm{Conv}(X/\mathfrak{S})$ such that $\rho(\mathfrak{Z} \times_{\mathfrak{T}} \mathfrak{Z}) = \rho(\mathfrak{Z}) \times_{\mathfrak{T}} \rho(\mathfrak{Z})$ and that $\rho(p_1)$ (respectively $\rho(p_2)$) is the projection $\rho(\mathfrak{Z}) \times_{\mathfrak{T}} \rho(\mathfrak{Z}) \to \rho(\mathfrak{Z})$ on the first (respectively second) component.*

Proof. By applying Lemma 3.10(i) to the projection $\rho(\mathfrak{Z}) \times_{\mathfrak{T}} \rho(\mathfrak{Z}) \to \rho(\mathfrak{Z})$ on the first component, we obtain an object $\mathfrak{Z} \times_{\mathfrak{T}} \mathfrak{Z}$ of $\mathrm{Conv}(X/\mathfrak{S})$ and a morphism $p_1 : \mathfrak{Z} \times_{\mathfrak{T}} \mathfrak{Z} \to \mathfrak{Z}$ as in the proposition. The existence of p_2 follows from the fullness of ρ (Lemma 5.4). □

We deduce from Theorem 5.2 a new proof of Frobenius descent for convergent isocrystals (Proposition 5.7) and a comparison of de Rham complexes for the Frobenius descent (Corollary 5.9).

Proposition 5.7. *Keep the assumption of Theorem 5.2. The inverse image and the direct image functors of $F_{X/S_0,\mathrm{conv},\mathrm{zar}}$ induce equivalences of categories quasi-inverse to each other (Definition 3.15):*

$$\mathrm{Iso}^{\dagger}(X/\mathfrak{S}) \rightleftarrows \mathrm{Iso}^{\dagger}(X'/\mathfrak{S}). \tag{5.7.1}$$

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Proof. By § 3.16, convergent isocrystals are sheaves for fppf topology and we work with fppf topology in this proof. We write simply (5.2.1) for F_{X/S_0} and we will show that the direct image and inverse image functors of F_{X/S_0} send coherent crystals of $\mathscr{O}_{X/\mathfrak{S}}[\frac{1}{p}]$-modules to coherent crystals of $\mathscr{O}_{X'/\mathfrak{S}}[\frac{1}{p}]$-modules. The assertion for the inverse image follows from (3.11.2) and we will prove it for the direct image.

Let $\mathscr{F}$ be a coherent crystal of $\mathscr{O}_{X/\mathfrak{S}}[\frac{1}{p}]$-modules and $(\mathfrak{T},u)$ an object of $\mathrm{Conv}(X'/\mathfrak{S})$. We first show that $(F_{X/S_0*}(\mathscr{F}))_{\mathfrak{T}}$ is coherent. By Lemma 5.5(iii), we may assume that $(\mathfrak{T},u)$ satisfies the conditions of Lemma 5.5(i). Then, by Lemmas 5.5(ii) and 5.6, there exist objects $\mathfrak{Z}$ and $\mathfrak{Z}\times_{\mathfrak{T}}\mathfrak{Z}$ of $\mathrm{Conv}(X/\mathfrak{S})$, an fppf covering $\{f:\rho(\mathfrak{Z})\to\mathfrak{T}\}$ and two morphisms $p_1,p_2:\mathfrak{Z}\times_{\mathfrak{T}}\mathfrak{Z}\to\mathfrak{Z}$ such that $\rho(\mathfrak{Z}\times_{\mathfrak{T}}\mathfrak{Z})=\rho(\mathfrak{Z})\times_{\mathfrak{T}}\rho(\mathfrak{Z})$ and that $\rho(p_1)$ and $\rho(p_2)$ are the canonical projections of $\rho(\mathfrak{Z})\times_{\mathfrak{T}}\rho(\mathfrak{Z})$. In particular, the morphism of formal schemes $\mathfrak{Z}\times_{\mathfrak{T}}\mathfrak{Z}\to\mathfrak{Z}$ attached to p_1 (respectively p_2) is the projection on the first (respectively second) component.

Since the adjunction morphism $F^*_{X/S_0}F_{X/S_0*}\to\mathrm{id}$ is an isomorphism (Theorem 5.2), we have (3.11.2)

$$(F_{X/S_0*}(\mathscr{F}))_{\rho(\mathfrak{Z})}=\mathscr{F}_{\mathfrak{Z}},\quad (F_{X/S_0*}(\mathscr{F}))_{\rho(\mathfrak{Z}\times_{\mathfrak{T}}\mathfrak{Z})}=\mathscr{F}_{\mathfrak{Z}\times_{\mathfrak{T}}\mathfrak{Z}}. \tag{5.7.2}$$

Since $\mathscr{F}$ is a crystal, we have $\mathscr{O}_{\mathfrak{Z}\times_{\mathfrak{T}}\mathfrak{Z}}$-linear isomorphisms

$$p_2^*(\mathscr{F}_{\mathfrak{Z}})\xrightarrow[\sim]{c_{p_2}}\mathscr{F}_{\mathfrak{Z}\times_{\mathfrak{T}}\mathfrak{Z}}\xleftarrow[\sim]{c_{p_1}}p_1^*(\mathscr{F}_{\mathfrak{Z}}). \tag{5.7.3}$$

Then we obtain a descent datum $(\mathscr{F}_{\mathfrak{Z}},c_{p_1}^{-1}\circ c_{p_2})$ for the fppf covering $\{f:\mathfrak{Z}\to\mathfrak{T}\}$. By fppf descent [Abb10, 5.11.11], there exist a coherent $\mathscr{O}_{\mathfrak{T}}[\frac{1}{p}]$-module $\mathscr{M}$ and a canonical $\mathscr{O}_{\mathfrak{Z}}$-linear isomorphism $f^*(\mathscr{M})\xrightarrow{\sim}\mathscr{F}_{\mathfrak{Z}}$.

On the other hand, since $F_{X/S_0*}(\mathscr{F})$ is a sheaf in fppf topology, there exists an exact sequence

$$0\to(F_{X/S_0*}(\mathscr{F}))(\mathfrak{T})\to(F_{X/S_0*}(\mathscr{F}))(\rho(\mathfrak{Z}))\to(F_{X/S_0*}(\mathscr{F}))(\rho(\mathfrak{Z}\times_{\mathfrak{T}}\mathfrak{Z})). \tag{5.7.4}$$

By (5.7.2), we deduce an $\mathscr{O}_{\mathfrak{T}}$-linear isomorphism $\mathscr{M}\xrightarrow{\sim}(F_{X/S_0*}(\mathscr{F}))_{\mathfrak{T}}$. In particular, $(F_{X/S_0*}(\mathscr{F}))_{\mathfrak{T}}$ is coherent. Hence, $F_{X/S_0*}(\mathscr{F})$ is coherent.

Following the same argument as in the second part of the proof of [Xu19, 9.13], we show that for every morphism g of $\mathrm{Conv}(X/\mathfrak{S})$, the transition morphism c_g associated to $F_{X/S_0*}(\mathscr{F})$ is an isomorphism, i.e. $F_{X/S_0*}(\mathscr{F})$ is a crystal. □

Proposition 5.8. *We consider the following diagram:*

$$\begin{array}{ccc} (X/\mathfrak{S})_{\mathrm{conv,zar}} & \xrightarrow{F_{X/S_0,\mathrm{conv,zar}}} & (X'/\mathfrak{S})_{\mathrm{conv,zar}} \\ {\scriptstyle u_{X/\mathfrak{S}}}\downarrow & & \downarrow{\scriptstyle u_{X'/\mathfrak{S}}} \\ X_{\mathrm{zar}} & \xrightarrow{F_{X/S_0}} & X'_{\mathrm{zar}} \end{array} \tag{5.8.1}$$

where vertical arrows are defined in § 3.21. Let $\mathscr{E}$ be a convergent isocrystal of $(X/\mathrm{W})_{\mathrm{conv,zar}}$ and denote the structure morphism $X\to S_0$ by f. Then there exists a canonical isomorphism in the derived category $\mathrm{D}(X_{\mathrm{zar}},f^{-1}(\mathscr{O}_{\mathfrak{S}}))$:

$$F_{X/S_0*}(\mathrm{R}\,u_{X/\mathfrak{S}*}(\mathscr{E}))\xrightarrow{\sim}\mathrm{R}\,u_{X'/\mathfrak{S}*}(F_{X/S_0\,\mathrm{conv,zar}\,*}(\mathscr{E})). \tag{5.8.2}$$

Proof. We consider $\mathscr{E}$ as a coherent crystal of $\mathscr{O}_{X/\mathfrak{S}}[\frac{1}{p}]$-modules of $(X/\mathfrak{S})_{\mathrm{conv,fppf}}$. Then $\alpha_*(\mathscr{E})$ and $\mathscr{E}$ are equal as presheaves and $\mathrm{R}^i\,\alpha_*(\mathscr{E})=0$ for $i\geqslant 1$ (Corollary 4.5). Then the assertion follows from Theorem 5.2 and the fact that $F_{X/S_0}:X_{\mathrm{zar}}\to X'_{\mathrm{zar}}$ is an equivalence of topoi. □

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Corollary 5.9. *Keep the assumption of Proposition 5.8 and suppose that there exist smooth liftings $\mathfrak{X}$ of X and $\mathfrak{X}'$ of X' over $\mathfrak{S}$. Let $f : X \to S_0$ be the canonical morphism. Then there exists a canonical isomorphism between the de Rham complexes of $\mathscr{E}$ and of $F_{X/S_0,\mathrm{conv},\mathrm{zar}\,*}(\mathscr{E})$ in $\mathrm{D}(X'_{\mathrm{zar}}, f^{-1}(\mathscr{O}_{\mathfrak{S}}))$:*

$$F_{X/S_0 *}(\mathscr{E}_{\mathfrak{X}} \otimes_{\mathscr{O}_{\mathfrak{X}}} \widehat{\Omega}^{\bullet}_{\mathfrak{X}/\mathfrak{S}}) \xrightarrow{\sim} (F_{X/S_0,\mathrm{conv},\mathrm{zar}\,*}\mathscr{E})_{\mathfrak{X}'} \otimes_{\mathscr{O}_{\mathfrak{X}'}} \widehat{\Omega}^{\bullet}_{\mathfrak{X}'/\mathfrak{S}}. \tag{5.9.1}$$

Proof. It follows from (3.21.1) and Proposition 5.8. □

Theorem 5.10. *Let $g : X \to Y$ be a smooth proper morphism of smooth k-schemes and $\mathscr{E}$ a convergent isocrystal of $\mathrm{Conv}(X/\mathrm{W})_{\mathrm{conv},\tau}$. Then $\mathrm{R}^i\, g_{\mathrm{conv},\tau *}(\mathscr{E})$ is a convergent isocrystal of $\mathrm{Conv}(Y/\mathrm{W})_{\mathrm{conv},\tau}$ for every $i \geqslant 0$.*

Inspired by Ogus' arguments in [Ogu84], we use Proposition 4.8 and Dwork's trick to prove Theorem 5.10. The proof of Proposition 4.8 relies on the local projectiveness of $\mathscr{E}$, which is obtained from the smoothness of X and of Y (Lemma 4.9). In Theorem 8.2, we will improve the above theorem to the non-smooth case.

We first introduce certain subcategories of $\mathrm{Conv}(X/\mathrm{W})$.

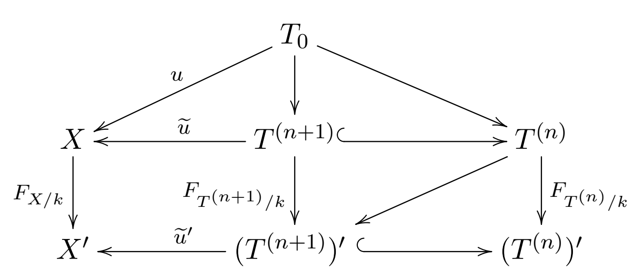
Definition 5.11. (i) Let n be an integer $\geqslant 0$ and T a k-scheme. We denote by $T^{(n)}$ the closed subscheme of T defined by the ideal sheaf $\{x \in \mathscr{O}_T | x^{p^n} = 0\}$.

(ii) We denote by $\mathrm{Conv}^{(n)}(X/\mathrm{W})$ the full subcategory of $\mathrm{Conv}(X/\mathrm{W})$ consisting of objects $(\mathfrak{T}, u)$ such that $u : T_0 \to X$ can be lifted to a k-morphism $\widetilde{u} : T^{(n)} \to X$.

Given an object $(\mathfrak{T}, u)$ of $\mathrm{Conv}^{(n)}(X/\mathrm{W})$ and a morphism $(\mathfrak{T}', u') \to (\mathfrak{T}, u)$ of $\mathrm{Conv}(X/\mathrm{W})$, then $(\mathfrak{T}', u')$ is also an object of $\mathrm{Conv}^{(n)}(X/\mathrm{W})$. In particular, $T^{(0)} = T$ and $\mathrm{Conv}^{(0)}(X/\mathrm{W})$ coincides with $\mathrm{pConv}(X/\mathrm{W})$ (§ 4.7).

Lemma 5.12. *The functor ρ (5.2.2) sends $\mathrm{Conv}^{(n+1)}(X/\mathrm{W})$ to $\mathrm{Conv}^{(n)}(X'/\mathrm{W})$.*

Proof. Let $(\mathfrak{T}, u)$ be an object of $\mathrm{Conv}^{(n+1)}(X/\mathrm{W})$ and $\widetilde{u} : T^{(n+1)} \to X$ a lifting of u. The absolute Frobenius morphism $T^{(n)} \to T^{(n)}$ factors through the closed subscheme $T^{(n+1)}$ and then the relative Frobenius morphism $F_{T^{(n)}/k}$ factors through $(T^{(n+1)})'$. We have a commutative diagram



(5.12.1)

Then the morphism $F_{X/k} \circ u$ can be lifted to a k-morphism $T^{(n)} \to X'$ and the lemma follows. □

5.13 *Proof of Theorem 5.10.* By Corollary 4.5, it suffices to prove the assertion for fppf topology. There exists an object $\mathscr{G}$ of $\mathrm{Iso}^{\dagger}(X'/\mathrm{W})$ with $F^*_{X/k,\mathrm{conv},\mathrm{fppf}}(\mathscr{G}) \simeq \mathscr{E}$ (Proposition 5.7). If we set $\mathscr{F} = \mathrm{R}^i\, g_{\mathrm{conv},\mathrm{fppf}\,*}(\mathscr{E})$ and $\mathscr{H} = \mathrm{R}^i\, g'_{\mathrm{conv},\mathrm{fppf}\,*}(\mathscr{G})$, then we have $F^*_{Y/k,\mathrm{conv},\mathrm{fppf}}(\mathscr{H}) \simeq \mathscr{F}$ by Theorem 5.2.

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Each object (respectively morphism) of $\mathrm{Conv}(Y/\mathrm{W})$ belongs to a subcategory $\mathrm{Conv}^{(n)}(Y/\mathrm{W})$ (Definition 5.11) for some integer n. We prove the following assertions by induction:

(i) for every object $\mathfrak{T}$ of $\mathrm{Conv}^{(n)}(Y/\mathrm{W})$, $\mathscr{F}_{\mathfrak{T}}$ is coherent;

(ii) for every morphism f of $\mathrm{Conv}^{(n)}(Y/\mathrm{W})$, the transition morphism c_f associated to $\mathscr{F}$ is an isomorphism.

The assertions for $n=0$ are proved in Proposition 4.8. Suppose that the assertions hold for $n \geqslant 0$; we prove them for $n+1$. Let $(\mathfrak{T},u)$ be an object of $\mathrm{Conv}^{(n+1)}(Y/\mathrm{W})$. By (3.11.2), we deduce that

$$\mathscr{H}_{\rho(\mathfrak{T})} \xrightarrow{\sim} \mathscr{F}_{\mathfrak{T}}.$$

By the induction hypotheses, for any object $\mathfrak{Z}$ of $\mathrm{Conv}^{(n)}(X'/\mathrm{W})$, $\mathscr{H}_{\mathfrak{Z}}$ is coherent. Then assertion (i) follows from Lemma 5.12 and the induction hypotheses.

Assertion (ii) can be verified in the same way by § 3.11 and Lemma 5.12. □

5.14 The Frobenius homomorphism $\sigma:\mathrm{W}\to\mathrm{W}$ induces a morphism of topoi $(X'/\mathrm{W})_{\mathrm{conv},\tau}\to(X/\mathrm{W})_{\mathrm{conv},\tau}$ for $\tau\in\{\mathrm{zar},\mathrm{fppf}\}$. For any sheaf $\mathscr{E}$ of $(X/\mathrm{W})_{\mathrm{conv},\tau}$, we denote by $\mathscr{E}'$ the inverse image of $\mathscr{E}$ to $(X'/\mathrm{W})_{\mathrm{conv},\tau}$.

A *convergent F-isocrystal of* $\mathrm{Conv}(X/\mathrm{W})_{\mathrm{conv},\tau}$ is a pair $(\mathscr{E},\varphi)$ consisting of a convergent isocrystal $\mathscr{E}$ of $(X/\mathrm{W})_{\mathrm{conv},\tau}$ and an isomorphism, called a *Frobenius structure* of $\mathscr{E}$,

$$\varphi: F^*_{X/k,\mathrm{conv},\tau}(\mathscr{E}') \xrightarrow{\sim} \mathscr{E}. \tag{5.14.1}$$

Corollary 5.15. *Keep the assumption of Theorem 5.10 and let φ be a Frobenius structure on $\mathscr{E}$. Then, for any $i\geqslant 0$, the pair $(\mathrm{R}^i g_{\mathrm{conv},\tau *}(\mathscr{E}),\mathrm{R}^i g_{\mathrm{conv},\tau *}(\varphi))$ is a convergent F-isocrystal of* $\mathrm{Conv}(Y/\mathrm{W})_{\mathrm{conv},\tau}$.

Proof. By Theorem 5.10, it suffices to prove the assertion for fppf topology. Consider the isomorphism

$$\mathrm{R}^i g_{\mathrm{conv},\mathrm{fppf}\,*}(\varphi):\mathrm{R}^i g_{\mathrm{conv},\mathrm{fppf}\,*}(F^*_{X/k,\mathrm{conv},\mathrm{fppf}}(\mathscr{E}')) \xrightarrow{\sim} \mathrm{R}^i g_{\mathrm{conv},\mathrm{fppf}\,*}(\mathscr{E}). \tag{5.15.1}$$

By Corollary 4.6 and Theorem 5.2, the left-hand side is isomorphic to

$$F^*_{Y/k,\mathrm{conv},\mathrm{fppf}}((\mathrm{R}^i g_{\mathrm{conv},\mathrm{fppf}\,*}(\mathscr{E}))').$$

Then the assertion follows. □

6. Review of rigid geometry

6.1 Recall that $\mathbf{S}$ denotes the category of adic formal W-schemes of finite type whose morphisms are W-morphisms of finite type (§ 2.1). The set $\mathbf{B}$ of admissible blow-ups in $\mathbf{S}$ forms a right multiplicative system in $\mathbf{S}$ [Abb10, 4.1.4]. We denote by $\mathbf{R}$ the localized category of $\mathbf{S}$ relative to $\mathbf{B}$. Objects of $\mathbf{R}$ are called *coherent rigid spaces (over $K=\mathrm{W}[\frac{1}{p}]$)*.

Let $\mathcal{X}$ be an object of $\mathbf{R}$. We denote by $\langle\mathcal{X}\rangle$ the set of rigid points of $\mathcal{X}$ [Abb10, 4.3.1], by $\mathbf{Ad}_{/\mathcal{X}}$ the full subcategory of $\mathbf{R}_{/\mathcal{X}}$ consisting of open immersions to $\mathcal{X}$, by $\mathcal{X}_{\mathrm{ad}}$ the topos of sheaves of sets on $\mathbf{Ad}_{/\mathcal{X}}$ for the admissible topology [Abb10, 4.3.8] and by $\mathscr{O}_{\mathcal{X}}$ the structure sheaf of $\mathcal{X}_{\mathrm{ad}}$ [Abb10, 4.7.4].

For any object $\mathfrak{X}$ (respectively morphism f) of $\mathbf{S}$, we denote its image in $\mathbf{R}$ by $\mathfrak{X}^{\mathrm{rig}}$ (respectively f^{rig}) and we set $\mathbf{B}_{\mathfrak{X}}$, the full subcategory of $\mathbf{S}_{/\mathfrak{X}}$ consisting of admissible blow-ups.

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6.2 Recall that $\mathbf{S}^\diamond$ denotes the full subcategory of $\mathbf{S}$ consisting of *flat* formal W-schemes of finite type (§ 2.1). For any object $\mathfrak{X}$ of $\mathbf{S}^\diamond$ and any admissible blow-up $\varphi:\mathfrak{X}'\to\mathfrak{X}$, $\mathfrak{X}'$ is still an object of $\mathbf{S}^\diamond$ [Abb10, 3.1.4]. Then the set $\mathbf{B}^\diamond$ of admissible blow-ups in $\mathbf{S}^\diamond$ forms a right multiplicative system in $\mathbf{S}^\diamond$. By [Abb10, 4.1.15(iii)], the canonical functor $\mathbf{S}^\diamond\to\mathbf{R}$ is essentially surjective and hence induces an equivalence of categories between the localized category of $\mathbf{S}^\diamond$ relative to $\mathbf{B}^\diamond$ and $\mathbf{R}$.

6.3 Let $\mathfrak{X}$ be an object of $\mathbf{S}$. We denote the specialization morphism of topoi [Abb10, 4.5.2] by

$$\rho_{\mathfrak{X}}:\mathfrak{X}^{\mathrm{rig}}_{\mathrm{ad}}\to\mathfrak{X}_{\mathrm{zar}}. \tag{6.3.1}$$

For any object $(\mathfrak{X}',\varphi)$ of $\mathbf{B}_{\mathfrak{X}}$, we denote by μ_φ the composition

$$\mu_\varphi:\mathfrak{X}^{\mathrm{rig}}_{\mathrm{ad}}\xrightarrow{\sim}\mathfrak{X}'^{\mathrm{rig}}_{\mathrm{ad}}\xrightarrow{\rho_{\mathfrak{X}'}}\mathfrak{X}'_{\mathrm{zar}}. \tag{6.3.2}$$

Let $\mathscr{F}$ be an $\mathscr{O}_{\mathfrak{X}}$-module. We denote by $\mathscr{F}^{\mathrm{rig}}$ the rigid fiber associated to $\mathscr{F}$ [Abb10, 4.7.4], which is a sheaf of $\mathfrak{X}^{\mathrm{rig}}_{\mathrm{ad}}$. We have a functorial isomorphism [Abb10, 4.7.4.2]

$$\mathscr{F}^{\mathrm{rig}}\xrightarrow{\sim}\varinjlim_{(\mathfrak{X}',\varphi)\in\mathbf{B}^{\circ}_{\mathfrak{X}}}\mu_\varphi^*\biggl((\varphi^*_{\mathrm{zar}}(\mathscr{F}))\biggl[\frac{1}{p}\biggr]\biggr). \tag{6.3.3}$$

In particular, $\mathscr{O}_{\mathfrak{X}^{\mathrm{rig}}}$ is defined by $(\mathscr{O}_{\mathfrak{X}})^{\mathrm{rig}}$. The morphism $\rho_{\mathfrak{X}}$ (respectively μ_φ) is ringed by $\mathscr{O}_{\mathfrak{X}^{\mathrm{rig}}}$ (respectively $\mathscr{O}^{\mathrm{rig}}_{\mathfrak{X}'}$) and $\mathscr{O}_{\mathfrak{X}}$ [Abb10, 4.7.5].

If $\mathscr{F}$ is moreover coherent, we have a canonical isomorphism $\rho^*_{\mathfrak{X}}(\mathscr{F}[\frac{1}{p}])\xrightarrow{\sim}\mathscr{F}^{\mathrm{rig}}$ [Abb10, 4.7.2.8].

6.4 Let $\mathbf{Coh}(\mathscr{O}_{\mathfrak{X}^{\mathrm{rig}}})$ be the category of coherent $\mathscr{O}_{\mathfrak{X}^{\mathrm{rig}}}$-modules over $\mathfrak{X}^{\mathrm{rig}}_{\mathrm{ad}}$ [Abb10, 4.8.16]. The inverse image functor of modules $\rho^*_{\mathfrak{X}}$ induces an equivalence of categories [Abb10, 4.7.8.2, 4.7.29.2 and 4.8.18]

$$\rho^*_{\mathfrak{X}}:\mathbf{Coh}\biggl(\mathscr{O}_{\mathfrak{X}}\biggl[\frac{1}{p}\biggr]\biggr)\xrightarrow{\sim}\mathbf{Coh}(\mathscr{O}_{\mathfrak{X}^{\mathrm{rig}}}). \tag{6.4.1}$$

The functor $\rho_{\mathfrak{X}*}$ sends coherent $\mathscr{O}_{\mathfrak{X}^{\mathrm{rig}}}$-modules to coherent $\mathscr{O}_{\mathfrak{X}}[\frac{1}{p}]$-modules [Abb10, 4.7.8.1] and defines a quasi-inverse to (6.4.1).

6.5 Let $f:\mathfrak{X}\to\mathfrak{Y}$ be a morphism of $\mathbf{S}$. It induces a morphism of ringed topoi $f^{\mathrm{rig}}_{\mathrm{ad}}:(\mathfrak{X}^{\mathrm{rig}}_{\mathrm{ad}},\mathscr{O}_{\mathfrak{X}^{\mathrm{rig}}})\to(\mathfrak{Y}^{\mathrm{rig}}_{\mathrm{ad}},\mathscr{O}_{\mathfrak{Y}^{\mathrm{rig}}})$ [Abb10, 4.7.2.1]. The diagram

$$\begin{array}{ccc}(\mathfrak{X}^{\mathrm{rig}}_{\mathrm{ad}},\mathscr{O}_{\mathfrak{X}^{\mathrm{rig}}}) & \xrightarrow{f^{\mathrm{rig}}_{\mathrm{ad}}} & (\mathfrak{Y}^{\mathrm{rig}}_{\mathrm{ad}},\mathscr{O}_{\mathfrak{Y}^{\mathrm{rig}}})\\ \downarrow{\scriptstyle\rho_{\mathfrak{X}}} & & \downarrow{\scriptstyle\rho_{\mathfrak{Y}}}\\ (\mathfrak{X}_{\mathrm{zar}},\mathscr{O}_{\mathfrak{X}}[\frac{1}{p}]) & \xrightarrow{f_{\mathrm{zar}}} & (\mathfrak{Y}_{\mathrm{zar}},\mathscr{O}_{\mathfrak{Y}}[\frac{1}{p}])\end{array} \tag{6.5.1}$$

is commutative up to canonical isomorphisms [Abb10, 4.7.24.2].

Let $\mathscr{F}$ be a coherent $\mathscr{O}_{\mathfrak{Y}^{\mathrm{rig}}}$-module. By § 6.4, there exist canonical isomorphisms

$$\rho^*_{\mathfrak{X}}(f^*_{\mathrm{zar}}(\rho_{\mathfrak{Y}_*}(\mathscr{F})))\xrightarrow{\sim}f^{\mathrm{rig}*}_{\mathrm{ad}}(\rho^*_{\mathfrak{Y}}(\rho_{\mathfrak{Y}*}(\mathscr{F})))\xrightarrow{\sim}f^{\mathrm{rig}*}_{\mathrm{ad}}(\mathscr{F}).$$

Then we deduce that the following base-change morphism is an isomorphism:

$$f^*_{\mathrm{zar}}\rho_{\mathfrak{Y}*}(\mathscr{F})\xrightarrow{\sim}\rho_{\mathfrak{X}*}f^{\mathrm{rig}*}_{\mathrm{ad}}(\mathscr{F}). \tag{6.5.2}$$

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6.6 We say that a family $\{\mathcal{X}_i \to \mathcal{X}\}_{i\in I}$ of flat morphisms of $\mathbf{R}$ [Abb10, 5.10.1] is an *fppf covering* if it admits a finite subfamily $\{\mathcal{X}_j \to \mathcal{X}\}_{j\in J}$ such that $\bigcup_{j\in J} f_j(\langle \mathcal{X}_j\rangle) = \langle \mathcal{X}\rangle$, i.e. $\coprod_{j\in J} \mathcal{X}_j \to \mathcal{X}$ is faithfully flat [Abb10, 5.10.11]. In view of [Abb10, 5.10.12], fppf coverings are stable by composition and by base change in $\mathbf{R}$.

Let $\mathcal{X}$ be a coherent rigid space. We denote by $\mathbf{Rf}_{/\mathcal{X}}$ the full subcategory of $\mathbf{R}_{/\mathcal{X}}$ consisting of flat morphisms to $\mathcal{X}$. We call *fppf topology* the topology on $\mathbf{Rf}_{/\mathcal{X}}$ generated by the pretopology for which coverings are fppf coverings. We denote by $\mathcal{X}_{\mathrm{fppf}}$ the topos of sheaves of sets on this site.

By fppf descent of morphisms [Abb10, 5.12.4], the fppf topology on $\mathbf{Rf}_{/\mathcal{X}}$ is subcanonical, i.e. the presheaf associated to each object of $\mathbf{Rf}_{/\mathcal{X}}$ is a sheaf for the fppf topology.

6.7 Let $\mathscr{F}$ be a coherent $\mathscr{O}_{\mathcal{X}}$-module. The presheaf on $\mathbf{Rf}_{/\mathcal{X}}$

$$(f : \mathcal{X}' \to \mathcal{X}) \mapsto \Gamma(\mathcal{X}', f_{\mathrm{ad}}^*(\mathscr{F}))$$

is a sheaf for the fppf topology by fppf descent for coherent modules on rigid spaces [Abb10, 5.11.11]. In particular, $\mathscr{O}_{\mathcal{X}}$ defines a sheaf of rings of $\mathcal{X}_{\mathrm{fppf}}$ that we still denote by $\mathscr{O}_{\mathcal{X}}$. We call abusively *a coherent $\mathscr{O}_{\mathcal{X}}$-module of $\mathcal{X}_{\mathrm{fppf}}$* a sheaf of $\mathcal{X}_{\mathrm{fppf}}$ associated to a coherent $\mathscr{O}_{\mathcal{X}}$-module of $\mathcal{X}_{\mathrm{ad}}$.

Given a morphism $f : \mathcal{X}' \to \mathcal{X}$ of $\mathbf{R}$, the canonical functor $\mathbf{Rf}_{/\mathcal{X}} \to \mathbf{Rf}_{/\mathcal{X}'}$ defined by $\mathcal{Y} \mapsto \mathcal{Y} \times_{\mathcal{X}} \mathcal{X}'$ is continuous and left exact. It induces the functorial morphism of topoi $f_{\mathrm{fppf}} : \mathcal{X}'_{\mathrm{fppf}} \to \mathcal{X}_{\mathrm{fppf}}$, which is ringed by $\mathscr{O}_{\mathcal{X}}$ and $\mathscr{O}_{\mathcal{X}'}$.

7. Rigid convergent topos and convergent isocrystals

7.1 In this section, $\mathfrak{S}$ denotes an adic flat formal W-scheme of finite type and X an S-scheme.

We will introduce a full subcategory of $(X/\mathfrak{S})_{\mathrm{conv,zar}}$ consisting of sheaves $\mathscr{F} = \{\mathscr{F}_{\mathfrak{T}}, \beta_f\}$ (§ 3.6) such that the morphism β_f is an isomorphism if the underlying morphism of formal schemes of f is an admissible blow-up. It turns out that this category forms a topos $(X/\mathfrak{S})_{\mathrm{rconv,ad}}$ (§ 7.3 and Corollary 7.13) and admits a canonical morphism to $(X/\mathfrak{S})_{\mathrm{conv,zar}}$ (§ 7.11). Convergent isocrystals lie in $(X/\mathfrak{S})_{\mathrm{rconv,ad}}$ and their cohomologies remain unchanged in this topos (Proposition 7.19 and Corollary 7.25).

We begin by introducing $(X/\mathfrak{S})_{\mathrm{rconv,ad}}$ and its fppf variant.

Lemma 7.2. *We denote by $\mathbf{B}_{X/\mathfrak{S}}$ the set of morphisms in* Conv$(X/\mathfrak{S})$ *(Definition 3.1) whose underlying morphism on formal schemes is an admissible blow-up. Then it forms a right multiplicative system in* Conv$(X/\mathfrak{S})$.

Proof. For any object $(\mathfrak{T}, u)$ of Conv$(X/\mathfrak{S})$, we have a canonical functor $s_{\mathfrak{T}} : \mathbf{B}_{\mathfrak{T}} \to \mathrm{Conv}(X/\mathfrak{S})$ sending $(\mathfrak{T}', \varphi)$ to $(\mathfrak{T}', u \circ \varphi_0)$. Then the assertion follows from the facts that admissible blow-ups form a right multiplicative system in $\mathbf{S}^{\diamond}_{/\mathfrak{S}}$ (§ 6.2) and that the canonical functor $\mathrm{Conv}(X/\mathfrak{S}) \to \mathbf{S}^{\diamond}_{\mathfrak{S}}$ is faithful. □

7.3 We denote by RConv$(X/\mathfrak{S})$ the localized category of Conv$(X/\mathfrak{S})$ relative to $\mathbf{B}_{X/\mathfrak{S}}$. More precisely, objects of RConv$(X/\mathfrak{S})$ are the same as those of Conv$(X/\mathfrak{S})$. For two objects $(\mathfrak{Z}, v)$, $(\mathfrak{T}, u)$ of RConv$(X/\mathfrak{S})$, we have

$$\mathrm{Hom}_{\mathrm{RConv}(X/\mathfrak{S})}((\mathfrak{Z}, v), (\mathfrak{T}, u)) = \varinjlim_{(\mathfrak{Z}', \varphi) \in \mathbf{B}^{\circ}_{\mathfrak{Z}}} \mathrm{Hom}_{\mathrm{Conv}(X/\mathfrak{S})}((\mathfrak{Z}', v \circ \varphi_0), (\mathfrak{T}, u)). \qquad (7.3.1)$$

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We denote by $\mathrm{Q}_{X/\mathfrak{S}}$ the canonical functor

$$\mathrm{Q}_{X/\mathfrak{S}} : \mathrm{Conv}(X/\mathfrak{S}) \to \mathrm{RConv}(X/\mathfrak{S}). \tag{7.3.2}$$

For an object $\mathfrak{T}$ (respectively a morphism f) of $\mathrm{Conv}(X/\mathfrak{S})$, we write $\mathfrak{T}^{\mathrm{rig}} = \mathrm{Q}_{X/\mathfrak{S}}(\mathfrak{T})$ (respectively $f^{\mathrm{rig}} = \mathrm{Q}_{X/\mathfrak{S}}(f)$) if there is no risk of confusion.

7.4 We denote by $\widehat{\mathrm{Conv}}(X/\mathfrak{S})$ (respectively $\widehat{\mathrm{RConv}}(X/\mathfrak{S})$) the category of presheaves on $\mathrm{Conv}(X/\mathfrak{S})$ (respectively $\mathrm{RConv}(X/\mathfrak{S})$) and by $\mathrm{Q}^*_{X/\mathfrak{S}} : \widehat{\mathrm{RConv}}(X/\mathfrak{S}) \to \widehat{\mathrm{Conv}}(X/\mathfrak{S})$ the functor defined by $\mathscr{F} \mapsto \mathscr{F} \circ \mathrm{Q}_{X/\mathfrak{S}}$. The functor $\mathrm{Q}_{X/\mathfrak{S}*}$ admits a left adjoint $\mathrm{Q}_{X/\mathfrak{S}!}$ [SGA4, I 5.1] defined as follows.

For any object $\mathfrak{T}^{\mathrm{rig}}$ of $\mathrm{RConv}(X/\mathfrak{S})$, we denote by $I^{\mathfrak{T}^{\mathrm{rig}}}_{\mathrm{Q}}$ the category whose objects are pairs $(\mathfrak{Z}, g)$ consisting of an object $\mathfrak{Z}$ of $\mathrm{Conv}(X/\mathfrak{S})$ and a morphism $g : \mathfrak{T}^{\mathrm{rig}} \to \mathfrak{Z}^{\mathrm{rig}}$ of $\mathrm{RConv}(X/\mathfrak{S})$. A morphism $(\mathfrak{Z}', g') \to (\mathfrak{Z}, g)$ is given by a morphism $\mu : \mathfrak{Z}' \to \mathfrak{Z}$ of $\mathrm{Conv}(X/\mathfrak{S})$ such that $g = \mu^{\mathrm{rig}} \circ g'$. Then we have [SGA4, I 5.1.1]

$$\mathrm{Q}_{X/\mathfrak{S}!}(\mathscr{F})(\mathfrak{T}^{\mathrm{rig}}) = \varinjlim_{(\mathfrak{Z},g) \in (I^{\mathfrak{T}^{\mathrm{rig}}}_{\mathrm{Q}})^{\circ}} \mathscr{F}(\mathfrak{Z}). \tag{7.4.1}$$

Moreover, we have a commutative diagram [SGA4, I 1.5.4]

$$\begin{array}{ccc} \mathrm{Conv}(X/\mathfrak{S}) & \xrightarrow{\mathrm{Q}_{X/\mathfrak{S}}} & \mathrm{RConv}(X/\mathfrak{S}) \\ \downarrow & & \downarrow \\ \widehat{\mathrm{Conv}}(X/\mathfrak{S}) & \xrightarrow{\mathrm{Q}_{X/\mathfrak{S}!}} & \widehat{\mathrm{RConv}}(X/\mathfrak{S}) \end{array} \tag{7.4.2}$$

where the vertical functors are the canonical functors.

Proposition 7.5. (i) *The category* $(I^{\mathfrak{T}^{\mathrm{rig}}}_{\mathrm{Q}})^{\circ}$ *is filtered [SGA4, I 2.7].*

(ii) *The functor* $\mathrm{Q}_{X/\mathfrak{S}!}$ *is left exact (and hence is exact).*

(iii) *Fiber products are representable in* $\mathrm{RConv}(X/\mathfrak{S})$ *and* $\mathrm{Q}_{X/\mathfrak{S}}$ *commutes with fiber products.*

Proof. We verify the following conditions of [SGA4, I 2.7] for $(I^{\mathfrak{T}^{\mathrm{rig}}}_{\mathrm{Q}})^{\circ}$.

(PS1) Given two morphisms $u : (\mathfrak{Z}_1, g_1) \to (\mathfrak{Z}_0, g_0)$ and $v : (\mathfrak{Z}_2, g_2) \to (\mathfrak{Z}_0, g_0)$ of $I^{\mathfrak{T}^{\mathrm{rig}}}_{\mathrm{Q}}$, by Lemma 7.2 and (7.3.1), there exist an admissible blow-up $\mathfrak{T}'$ of $\mathfrak{T}$ and morphisms $\mathfrak{g}_i : \mathfrak{T}' \to \mathfrak{Z}_i$ of $\mathrm{Conv}(X/\mathfrak{S})$ such that $\mathfrak{g}_i^{\mathrm{rig}} = g_i$ and that $u \circ \mathfrak{g}_1 = \mathfrak{g}_0 = v \circ \mathfrak{g}_2$. Then we obtain a morphism $\mathfrak{h} : \mathfrak{T}' \to \mathfrak{Z}_1 \times_{\mathfrak{Z}_0} \mathfrak{Z}_2$ of $\mathrm{Conv}(X/\mathfrak{S})$ (§ 3.2) and an object $(\mathfrak{Z}_1 \times_{\mathfrak{Z}_0} \mathfrak{Z}_2, \mathfrak{h}^{\mathrm{rig}})$ of $I^{\mathfrak{T}^{\mathrm{rig}}}_{\mathrm{Q}}$ dominant $(\mathfrak{Z}_i, g_i)$ for $i = 1, 2$. The diagram

$$\begin{array}{ccc} (\mathfrak{Z}_1 \times_{\mathfrak{Z}_0} \mathfrak{Z}_2, \mathfrak{h}^{\mathrm{rig}}) & \longrightarrow & (\mathfrak{Z}_2, g_2) \\ \downarrow & & \downarrow \\ (\mathfrak{Z}_1, g_1) & \longrightarrow & (\mathfrak{Z}_0, g_0) \end{array}$$

commutes. Then condition (PS1) follows.

(PS2) Let $u, v : (\mathfrak{Y}, g) \to (\mathfrak{Z}, h)$ be two morphisms of $I^{\mathfrak{T}^{\mathrm{rig}}}_{\mathrm{Q}}$. There exist an admissible blow-up $(\mathfrak{T}', \varphi)$ of $\mathfrak{T}$ and a morphism $\mathfrak{g} : \mathfrak{T}' \to \mathfrak{Y}$ of $\mathrm{Conv}(X/\mathfrak{S})$ such that $u \circ \mathfrak{g} = v \circ \mathfrak{g}$ in $\mathrm{Conv}(X/\mathfrak{S})$,

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denoted by $\mathfrak{h}$. We have $\mathfrak{g}^{\mathrm{rig}} = g, \mathfrak{h}^{\mathrm{rig}} = h$. Then $(\mathfrak{T}', \varphi^{\mathrm{rig}})$ defines an object of $I_{\mathrm{Q}}^{\mathfrak{T}^{\mathrm{rig}}}$ and $\mathfrak{h}$ (respectively $\mathfrak{g}$) defines a morphism from $(\mathfrak{T}', \varphi^{\mathrm{rig}})$ to $(\mathfrak{Y}, g)$ (respectively $(\mathfrak{Z}, h)$). Condition (PS2) follows from $u \circ \mathfrak{g} = v \circ \mathfrak{g} = \mathfrak{h}$.

It is clear that $I_{\mathrm{Q}}^{\mathfrak{T}^{\mathrm{rig}}}$ is non-empty. Given two objects $(\mathfrak{Z}_1, g_1)$ and $(\mathfrak{Z}_2, g_2)$, there exist an admissible blow-up $\mathfrak{T}'$ of $\mathfrak{T}$ and morphisms $\mathfrak{g}_i : \mathfrak{T}' \to \mathfrak{Z}_i$ of $\mathrm{Conv}(X/\mathfrak{S})$ for $i = 1, 2$ such that $\mathfrak{g}_i^{\mathrm{rig}} = g_i$. Hence, $I_{\mathrm{Q}}^{\mathfrak{T}^{\mathrm{rig}}}$ is connected. Then assertion (i) follows.

Assertion (ii) follows from (i). Assertion (iii) follows from (ii), (7.4.2) and the fact that the fiber product is representable in $\mathrm{Conv}(X/\mathfrak{S})$. □

7.6 The canonical functor $\mathrm{Conv}(X/\mathfrak{S}) \to \mathbf{S}^{\diamond}_{/\mathfrak{S}}$ defined by $(\mathfrak{T}, u) \mapsto \mathfrak{T}$ induces a functor

$$\mathrm{RConv}(X/\mathfrak{S}) \to \mathbf{R}_{/\mathfrak{S}^{\mathrm{rig}}}. \tag{7.6.1}$$

In view of the definition of fiber product in $\mathbf{R}_{/\mathfrak{S}^{\mathrm{rig}}}$ [Abb10, 4.1.13], the above functor commutes with fiber products.

7.7 We say that a family of morphisms $\{(\mathfrak{T}_i, u_i)^{\mathrm{rig}} \to (\mathfrak{T}, u)^{\mathrm{rig}}\}_{i \in I}$ of $\mathrm{RConv}(X/\mathfrak{S})$ is an admissible (respectively fppf) covering if its image $\{\mathfrak{T}_i^{\mathrm{rig}} \to \mathfrak{T}^{\mathrm{rig}}\}_{i \in I}$ in $\mathbf{R}$ is an admissible (respectively fppf) covering (§ 6.6 and [Abb10, 4.3.8]). By §§ 6.6 and 7.6, admissible (respectively fppf) coverings form a pretopology. For $\tau \in \{\mathrm{ad}, \mathrm{fppf}\}$, we call *rigid convergent topos of X over $\mathfrak{S}$ (with τ-topology)* and denote by $(X/\mathfrak{S})_{\mathrm{rconv},\tau}$ the topos of sheaves of sets on $\mathrm{RConv}(X/\mathfrak{S})$, equipped with the topology associated to the pretopology defined by admissible (respectively fppf) coverings.

7.8 Let $(\mathfrak{T}, u)$ be an object of $\mathrm{Conv}(X/\mathfrak{S})$. The canonical functor (§ 6.2)

$$r_{\mathfrak{T}} : \mathbf{S}^{\diamond}_{/\mathfrak{T}} \to \mathrm{Conv}(X/\mathfrak{S}), \quad (f : \mathfrak{T}' \to \mathfrak{T}) \mapsto (\mathfrak{T}', u \circ f_0)$$

sends admissible blow-ups to $\mathbf{B}_{X/\mathfrak{S}}$ and hence induces a functor

$$r_{\mathfrak{T}^{\mathrm{rig}}} : \mathbf{R}_{/\mathfrak{T}^{\mathrm{rig}}} \to \mathrm{RConv}(X/\mathfrak{S}). \tag{7.8.1}$$

The restriction of (7.8.1) to $\mathbf{Ad}_{/\mathfrak{T}^{\mathrm{rig}}}$ (respectively $\mathbf{Rf}_{/\mathfrak{T}^{\mathrm{rig}}}$) is cocontinuous for the admissible (respectively fppf) topology and induces a morphism of topoi

$$s_{\mathfrak{T}^{\mathrm{rig}}} : \mathfrak{T}^{\mathrm{rig}}_{\tau} \to (X/\mathfrak{S})_{\mathrm{conv},\tau}, \quad \tau \in \{\mathrm{ad}, \mathrm{fppf}\}. \tag{7.8.2}$$

For any sheaf $\mathscr{F}$ of $(X/\mathfrak{S})_{\mathrm{conv},\tau}$, we set $\mathscr{F}_{\mathfrak{T}^{\mathrm{rig}}} = s^*_{\mathfrak{T}^{\mathrm{rig}}}(\mathscr{F})$. For any morphism $f : \mathfrak{T}'^{\mathrm{rig}} \to \mathfrak{T}^{\mathrm{rig}}$ of $\mathrm{RConv}(X/\mathfrak{S})$, we have a canonical morphism

$$\beta_f : \mathscr{F}_{\mathfrak{T}^{\mathrm{rig}}} \to f_{\tau *}(\mathscr{F}_{\mathfrak{T}'^{\mathrm{rig}}}) \tag{7.8.3}$$

and we denote its adjoint by

$$\gamma_f : f^*_{\tau}(\mathscr{F}_{\mathfrak{T}^{\mathrm{rig}}}) \to \mathscr{F}_{\mathfrak{T}'^{\mathrm{rig}}}. \tag{7.8.4}$$

If the morphism of underlying rigid spaces of f belongs to $\mathbf{Ad}_{/\mathfrak{T}^{\mathrm{rig}}}$ (respectively $\mathbf{Rf}_{/\mathfrak{T}^{\mathrm{rig}}}$), the functorial morphism f_{τ} is the localization morphism at $\mathfrak{T}'$ and then γ_f is an isomorphism. If $g : \mathfrak{T}''^{\mathrm{rig}} \to \mathfrak{T}'^{\mathrm{rig}}$ is another morphism of $\mathrm{RConv}(X/\mathfrak{S})$, one verifies that $\gamma_{g \circ f} = \gamma_f \circ f^*_{\tau}(\gamma_g)$.

By repeating the proof of Proposition 3.7, we have the following description for a sheaf of $(X/\mathfrak{S})_{\mathrm{rconv},\tau}$.

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Proposition 7.9. *For $\tau \in \{\mathrm{ad}, \mathrm{fppf}\}$, a sheaf $\mathscr{F}$ of $(X/\mathfrak{S})_{\mathrm{rconv},\tau}$ is equivalent to the following data:*

(i) *for every object $\mathfrak{T}^{\mathrm{rig}}$ of $\mathrm{RConv}(X/\mathfrak{S})$, a sheaf $\mathscr{F}_{\mathfrak{T}^{\mathrm{rig}}}$ of $\mathfrak{T}^{\mathrm{rig}}_{\tau}$;*

(ii) *for every morphism $f : \mathfrak{T}'^{\mathrm{rig}} \to \mathfrak{T}^{\mathrm{rig}}$ of $\mathrm{RConv}(X/\mathfrak{S})$, a morphism γ_f (7.8.4)*

subject to the following conditions.

(a) *If f is the identity morphism of $(\mathfrak{T}, u)$, then γ_f is the identity morphism.*

(b) *If the underlying morphism $f : \mathfrak{T}'^{\mathrm{rig}} \to \mathfrak{T}^{\mathrm{rig}}$ of coherent rigid spaces is a morphism of $\mathbf{Ad}_{/\mathfrak{T}^{\mathrm{rig}}}$ (respectively $\mathbf{Rf}_{/\mathfrak{T}^{\mathrm{rig}}}$), then γ_f is an isomorphism.*

(c) *If f and g are two composable morphisms, then we have $\gamma_{g\circ f} = \gamma_f \circ f^*_{\tau}(\gamma_g)$.*

7.10 Note that the fppf topology on $\mathrm{RConv}(X/\mathfrak{S})$ is finer than the admissible topology. Equipped with the fppf topology on the source and the admissible topology on the target, the identical functor $\mathrm{id} : \mathrm{RConv}(X/\mathfrak{S}) \to \mathrm{RConv}(X/\mathfrak{S})$ is cocontinuous and induces a morphism of topoi (§ 2.4)

$$\alpha_r : (X/\mathfrak{S})_{\mathrm{rconv,fppf}} \to (X/\mathfrak{S})_{\mathrm{rconv,ad}}. \tag{7.10.1}$$

If $\mathscr{F}$ is a sheaf of $(X/\mathfrak{S})_{\mathrm{rconv,fppf}}$, then $\alpha_{r*}(\mathscr{F})$ is equal to $\mathscr{F}$ as presheaves. If $\mathscr{G}$ is a sheaf of $(X/\mathfrak{S})_{\mathrm{rconv,ad}}$, then $\alpha_r^*(\mathscr{G})$ is the sheafification of $\mathscr{G}$ with respect to the fppf topology.

7.11 Equipped with the Zariski topology on the source and the admissible topology on the target, the canonical functor $\mathrm{Q}_{X/\mathfrak{S}}$ (7.3.2) is clearly continuous. Since the functor $\mathrm{Q}_{X/\mathfrak{S}!}$ and the sheafification functor are exact (§ 7.6), then we have a morphism of topoi

$$\rho_{X/\mathfrak{S}} : (X/\mathfrak{S})_{\mathrm{rconv,ad}} \to (X/\mathfrak{S})_{\mathrm{conv,zar}} \tag{7.11.1}$$

defined by $\rho_{X/\mathfrak{S}*} = \mathrm{Q}^*_{X/\mathfrak{S}}$ and $\rho^*_{X/\mathfrak{S}} = a\circ \mathrm{Q}_{X/\mathfrak{S}!}$ (§ 7.6), where a denotes the sheafification functor.

For any object $\mathfrak{T}$ of $\mathrm{Conv}(X/\mathfrak{S})$ and any sheaf $\mathscr{F}$ of $(X/\mathfrak{S})_{\mathrm{rconv,ad}}$, we have (6.3.1)

$$(\rho_{X/\mathfrak{S}*}(\mathscr{F}))_{\mathfrak{T}} = \rho_{\mathfrak{T}*}(\mathscr{F}_{\mathfrak{T}^{\mathrm{rig}}}). \tag{7.11.2}$$

Let $f : \mathfrak{Z} \to \mathfrak{T}$ be a morphism of $\mathrm{Conv}(X/\mathfrak{S})$ and $\beta_{f^{\mathrm{rig}}}, \gamma_{f^{\mathrm{rig}}}$ (respectively β_f, γ_f) transition morphisms of $\mathscr{F}$ associated to f^{rig} (respectively $\rho_{X/\mathfrak{S}*}(\mathscr{F})$ associated to f) (§§ 3.6 and 7.8). Via (7.11.2), we have

$$\beta_f = \rho_{\mathfrak{T}*}(\beta_{f^{\mathrm{rig}}}). \tag{7.11.3}$$

Then γ_f coincides with the composition of the base-change morphism and $\rho_{\mathfrak{Z}*}(\gamma_{f^{\mathrm{rig}}})$:

$$f^*_{\mathrm{zar}}(\rho_{\mathfrak{T}*}(\mathscr{F}_{\mathfrak{T}^{\mathrm{rig}}})) \to \rho_{\mathfrak{Z}*}(f^{\mathrm{rig}*}_{\mathrm{ad}}(\mathscr{F}_{\mathfrak{T}^{\mathrm{rig}}})) \xrightarrow{\rho_{\mathfrak{Z}*}(\gamma_{f^{\mathrm{rig}}})} \rho_{\mathfrak{Z}*}(\mathscr{F}_{\mathfrak{Z}^{\mathrm{rig}}}). \tag{7.11.4}$$

Proposition 7.12. *Let $\mathscr{F}$ be a sheaf of $(X/\mathfrak{S})_{\mathrm{conv,zar}}$ and $(\mathfrak{T}, u)$ an object of $\mathrm{Conv}(X/\mathfrak{S})$. There exists a canonical isomorphism (6.3.2)*

$$(\rho^*_{X/\mathfrak{S}}(\mathscr{F}))_{\mathfrak{T}^{\mathrm{rig}}} \xrightarrow{\sim} \varinjlim_{(\mathfrak{T}',\varphi)\in \mathbf{B}^{\circ}_{\mathfrak{T}}} \mu^*_{\varphi}(\mathscr{F}_{\mathfrak{T}'}). \tag{7.12.1}$$

Proof. Let $\mathcal{U}$ be an object of $\mathbf{Ad}_{/\mathfrak{T}^{\mathrm{rig}}}$ that we consider as an object of $\mathrm{RConv}(X/\mathfrak{S})$ via $r_{\mathfrak{T}^{\mathrm{rig}}}$ (7.8.1). By (7.4.1), $(\rho^*_{X/\mathfrak{S}}(\mathscr{F}))_{\mathfrak{T}^{\mathrm{rig}}}$ is the sheaf associated to the presheaf on $\mathbf{Ad}_{/\mathfrak{T}^{\mathrm{rig}}}$:

$$\mathcal{U} \mapsto \varinjlim_{(\mathfrak{Z},g)\in (I^{\mathcal{U}}_{\mathrm{Q}})^{\circ}} \mathscr{F}(\mathfrak{Z}). \tag{7.12.2}$$

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We denote by $J_{\mathrm{Q}}^{\mathcal{U}}$ the category of quadruples $(\mathfrak{T}', \varphi, \mathfrak{U}, g)$ consisting of an admissible blow-up $(\mathfrak{T}', \varphi)$ of $\mathfrak{T}$, an open formal subscheme $\mathfrak{U}$ of $\mathfrak{T}'$ and an open immersion $g : \mathcal{U} \to \mathfrak{U}^{\mathrm{rig}}$ over $\mathfrak{T}^{\mathrm{rig}}$. A morphism $(\mathfrak{T}'_1, \varphi_1, \mathfrak{U}_1, g_1)$ to $(\mathfrak{T}'_2, \varphi_2, \mathfrak{U}_2, g_2)$ is a morphism $\mathfrak{T}'_1 \to \mathfrak{T}'_2$ of $\mathbf{B}_{\mathfrak{T}}$ sending $\mathfrak{U}_1$ to $\mathfrak{U}_2$ compatible with g_1, g_2. The category $J_{\mathrm{Q}}^{\mathcal{U}}$ is clearly fibered over $\mathbf{B}_{\mathfrak{T}}$:

$$J_{\mathrm{Q}}^{\mathcal{U}} \to \mathbf{B}_{\mathfrak{T}}, \quad (\mathfrak{T}', \varphi, \mathfrak{U}, g) \mapsto (\mathfrak{T}', \varphi).$$

For any admissible blow-up $(\mathfrak{T}', \varphi)$ of $\mathfrak{T}$, we denote its fiber by $J_{\mathrm{Q},\mathfrak{T}'}^{\mathcal{U}}$. The sheaf $\mu_{\varphi}^{*}(\mathscr{F}_{\mathfrak{T}'})$ is associated to the presheaf on $\mathbf{Ad}_{/\mathfrak{T}^{\mathrm{rig}}}$:

$$\mathcal{U} \mapsto \varinjlim_{(\mathfrak{U},g) \in (J_{\mathrm{Q},\mathfrak{T}'}^{\mathcal{U}})^{\circ}} \mathscr{F}(r_{\mathfrak{T}}(\mathfrak{U})). \tag{7.12.3}$$

Then the right-hand side of (7.12.1) is the sheaf on $\mathbf{Ad}_{/\mathfrak{T}^{\mathrm{rig}}}$ associated to the presheaf

$$\mathcal{U} \mapsto \varinjlim_{(\mathfrak{T}',\varphi,\mathfrak{U},g) \in (J_{\mathrm{Q}}^{\mathcal{U}})^{\circ}} \mathscr{F}(r_{\mathfrak{T}}(\mathfrak{U})). \tag{7.12.4}$$

We have a canonical functor (§ 7.8)

$$r : J_{\mathrm{Q}}^{\mathcal{U}} \to I_{\mathrm{Q}}^{\mathcal{U}} \quad (\mathfrak{T}', \varphi, \mathfrak{U}, g) \mapsto (r_{\mathfrak{T}}(\mathfrak{U}), r_{\mathfrak{T}^{\mathrm{rig}}}(g)).$$

We denote by J the full subcategory of $J_{\mathrm{Q}}^{\mathcal{U}}$ consisting of objects such that g is an isomorphism. Then each morphism of J is Cartesian. Each category $(J_{\mathrm{Q},\mathfrak{T}'}^{\mathcal{U}})^{\circ}$ is filtered by [SGA4, I 5.2]. We deduce that $(J_{\mathrm{Q}}^{\mathcal{U}})^{\circ}$ is filtered. It is clear that J° is cofinal in $(J_{\mathrm{Q}}^{\mathcal{U}})^{\circ}$ and hence is filtered [SGA4, I 8.1.3a].

To prove the assertion, it suffices to show that the induced functor $r : J^{\circ} \to (I_{\mathrm{Q}}^{\mathcal{U}})^{\circ}$ is cofinal in the sense of [SGA4, I 8.1.1]. By [Abb10, 4.2.2], for any object $(\mathfrak{Z}, g : \mathcal{U} \to \mathfrak{Z}^{\mathrm{rig}})$ of $I_{\mathrm{Q}}^{\mathcal{U}}$, there exists a morphism $h : \mathfrak{U} \to \mathfrak{Z}$ of $\mathrm{Conv}(X/\mathfrak{S})$ with an open formal subscheme $\mathfrak{U}$ of some admissible blow-up $\mathfrak{T}'$ of $\mathfrak{T}$ such that $g = h^{\mathrm{rig}}$, i.e. condition (F1) of [SGA4, I 8.1.3] is satisfied. Given an object $(\mathfrak{Z}, g)$ of $I_{\mathrm{Q}}^{\mathcal{U}}$, an object $(\mathfrak{T}', \varphi, \mathfrak{U}, h)$ of J and two morphisms $f_1, f_2 : (r_{\mathfrak{T}}(\mathfrak{U}), r_{\mathfrak{T}^{\mathrm{rig}}}(h)) \to (\mathfrak{Z}, g)$, then $f_1^{\mathrm{rig}} = f_2^{\mathrm{rig}}$ in $\mathrm{RConv}(X/\mathfrak{S})$ since h is an isomorphism. By [Abb10, 3.5.9], we deduce that $f_1 = f_2$, i.e. condition (F2) of [SGA4, I 8.1.3] is satisfied. Then the assertion follows from [SGA4, I 8.1.3b]. □

COROLLARY 7.13.

(i) *The canonical morphism* $\rho_{X/\mathfrak{S}}^{*}\rho_{X/\mathfrak{S}*} \to \mathrm{id}$ *is an isomorphism.*

(ii) *The functor* $\rho_{X/\mathfrak{S}*}$ *is fully faithful and its essential image consists of sheaves* $\mathscr{F} = \{\mathscr{F}_{\mathfrak{T}}, \beta_f\}$ *(§ 3.6) such that* β_f *is an isomorphism for every morphism* f *of* $\mathbf{B}_{X/\mathfrak{S}}$.

Proof. (i) Let $\mathscr{F}$ be a sheaf of $(X/\mathfrak{S})_{\mathrm{rconv,ad}}$. Via (7.11.2) and (7.12.1), we consider the evaluation of $\rho_{X/\mathfrak{S}}^{*}\rho_{X/\mathfrak{S}*}(\mathscr{F}) \to \mathscr{F}$ at an object $\mathfrak{T}^{\mathrm{rig}}$ of $\mathrm{RConv}(X/\mathfrak{S})$:

$$\varinjlim_{(\mathfrak{T}',\varphi) \in \mathbf{B}_{\mathfrak{T}}^{\circ}} \mu_{\varphi}^{*}(\mu_{\varphi*}(\mathscr{F}_{\mathfrak{T}^{\mathrm{rig}}})) \to \mathscr{F}_{\mathfrak{T}^{\mathrm{rig}}}. \tag{7.13.1}$$

In view of the proof of Proposition 7.12, the morphism $\mu_{\varphi}^{*}(\mu_{\varphi*}(\mathscr{F}_{\mathfrak{T}^{\mathrm{rig}}})) \to \mathscr{F}_{\mathfrak{T}^{\mathrm{rig}}}$ deduced from (7.13.1) is nothing but the adjunction morphism. Then the assertion follows from [Abb10, 4.5.27 and 4.5.28].

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(ii) By (i), the functor $\rho_{X/\mathfrak{S}*}$ is fully faithful. By (7.11.3), the essential image of $\rho_{X/\mathfrak{S}*}$ satisfies the desired property. Let $\mathscr{G}$ be a sheaf of $(X/\mathfrak{S})_{\mathrm{conv,zar}}$ satisfying the desired property. By (7.11.2), Proposition 7.12 and [Abb10, 4.5.22 and 4.5.27], we deduce that the evaluation of the adjunction morphism

$$\mathscr{G} \to \rho_{X/\mathfrak{S}*}\rho^*_{X/\mathfrak{S}}(\mathscr{G}) \tag{7.13.2}$$

at each object of $\mathrm{RConv}(X/\mathfrak{S})$ is an isomorphism. Then the assertion follows. □

7.14 Let $g : \mathfrak{S}' \to \mathfrak{S}$ be a morphism of $\mathbf{S}^\diamond$, X' an S'-scheme and $f : X' \to X$ a morphism compatible with g as in § 3.9. The canonical functor $\varphi : \mathrm{Conv}(X'/\mathfrak{S}') \to \mathrm{Conv}(X/\mathfrak{S})$ defined by $(\mathfrak{T}, u) \mapsto (\mathfrak{T}, f \circ u)$ (3.9.2) sends admissible blow-ups to admissible blow-ups. Then φ induces a functor that we denote by

$$\psi : \mathrm{RConv}(X'/\mathfrak{S}') \to \mathrm{RConv}(X/\mathfrak{S}). \tag{7.14.1}$$

Since φ commutes with fiber products, the same holds for ψ by § 7.6. In view of Lemma 3.10(i) and (7.3.1), one verifies that the functor ψ is continuous and cocontinuous for admissible (respectively fppf) topology in the same way as in Lemma 3.10. By § 2.4, the functor ψ (7.14.1) induces morphisms of topoi

$$f_{\mathrm{rconv},\tau} : (X'/\mathfrak{S}')_{\mathrm{rconv},\tau} \to (X/\mathfrak{S})_{\mathrm{rconv},\tau}, \quad \tau \in \{\mathrm{ad}, \mathrm{fppf}\} \tag{7.14.2}$$

such that the pullback functor is induced by the composition with ψ. For a sheaf $\mathscr{F}$ of $(X/\mathfrak{S})_{\mathrm{rconv},\tau}$ and an object $\mathfrak{T}$ of $\mathrm{RConv}(X'/\mathfrak{S}')$, we have

$$(f^*_{\mathrm{rconv},\tau}(\mathscr{F}))_{\mathfrak{T}^{\mathrm{rig}}} = \mathscr{F}_{\psi(\mathfrak{T}^{\mathrm{rig}})}. \tag{7.14.3}$$

For any morphism g of $\mathrm{RConv}(X'/\mathfrak{S}')$, the transition morphism of $f^*_{\mathrm{rconv},\tau}(\mathscr{F})$ associated to g is equal to the transition morphism of $\mathscr{F}$ associated to $\psi(g)$.

In view of the description of inverse image functors, we deduce the following result.

Corollary 7.15. *Keep the assumption and notation of § 3.9 and of § 7.14. The diagram*

$$\begin{array}{ccc}
(X'/\mathfrak{S}')_{\mathrm{rconv,ad}} & \xrightarrow{f_{\mathrm{rconv,ad}}} & (X/\mathfrak{S})_{\mathrm{rconv,ad}} \\
\Big\downarrow{\scriptstyle \rho_{X'/\mathfrak{S}'}} & & \Big\downarrow{\scriptstyle \rho_{X/\mathfrak{S}}} \\
(X'/\mathfrak{S}')_{\mathrm{conv,zar}} & \xrightarrow{f_{\mathrm{conv,zar}}} & (X/\mathfrak{S})_{\mathrm{conv,zar}}
\end{array}$$

is commutative up to canonical isomorphisms.

7.16 We set $\mathscr{O}^{\mathrm{rig}}_{X/\mathfrak{S}} = \rho^*_{X/\mathfrak{S}}(\mathscr{O}_{X/\mathfrak{S}}[\frac{1}{p}])$. By (6.3.3) and Proposition 7.12, for any object $\mathfrak{T}^{\mathrm{rig}}$ of $\mathrm{RConv}(X/\mathfrak{S})$, we have a canonical isomorphism

$$(\mathscr{O}^{\mathrm{rig}}_{X/\mathfrak{S}})_{\mathfrak{T}^{\mathrm{rig}}} \xrightarrow{\sim} \mathscr{O}_{\mathfrak{T}^{\mathrm{rig}}}. \tag{7.16.1}$$

Then, by fppf descent [Abb10, 5.11.11], the presheaf $\mathscr{O}^{\mathrm{rig}}_{X/\mathfrak{S}}$ is also a sheaf for the fppf topology. For $\tau \in \{\mathrm{ad}, \mathrm{fppf}\}$, if $\mathscr{F}$ is an $\mathscr{O}^{\mathrm{rig}}_{X/\mathfrak{S}}$-module of $(X/\mathfrak{S})_{\mathrm{rconv},\tau}$, $\mathscr{F}_{\mathfrak{T}^{\mathrm{rig}}}$ is an $\mathscr{O}_{\mathfrak{T}^{\mathrm{rig}}}$-module. For any morphism $f : \mathfrak{T}'^{\mathrm{rig}} \to \mathfrak{T}^{\mathrm{rig}}$ of $\mathrm{RConv}(X/\mathfrak{S})$, the transition morphism γ_f (Proposition 7.9) extends to an $(\mathscr{O}_{\mathfrak{T}'^{\mathrm{rig}}})$-linear morphism (§ 6.7)

$$c_f : f^*_\tau(\mathscr{F}_{\mathfrak{T}^{\mathrm{rig}}}) \to \mathscr{F}_{\mathfrak{T}'^{\mathrm{rig}}}. \tag{7.16.2}$$

In view of Proposition 7.9, we deduce the following description for $\mathscr{O}^{\mathrm{rig}}_{X/\mathfrak{S}}$-modules.

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Proposition 7.17. *For $\tau \in \{\text{zar}, \text{fppf}\}$, an $\mathscr{O}^{\text{rig}}_{X/\mathfrak{S}}$-module of $(X/\mathfrak{S})_{\text{rconv},\tau}$ is equivalent to the following data:*

(i) *for every object $\mathfrak{T}^{\text{rig}}$ of $\text{RConv}(X/\mathfrak{S})$, an $\mathscr{O}_{\mathfrak{T}^{\text{rig}}}$-module $\mathscr{F}_{\mathfrak{T}}$ of $\mathfrak{T}_\tau$;*

(ii) *for every morphism $f : \mathfrak{T}'^{\text{rig}} \to \mathfrak{T}^{\text{rig}}$ of $\text{RConv}(X/\mathfrak{S})$, an $\mathscr{O}_{\mathfrak{T}'^{\text{rig}}}$-linear morphism c_f (7.16.2),*

which is subject to the following conditions.

(a) *If f is the identity morphism, then c_f is the identity.*

(b) *If the underlying morphism $f : \mathfrak{T}'^{\text{rig}} \to \mathfrak{T}^{\text{rig}}$ of coherent rigid spaces is a morphism of $\mathbf{Ad}_{/\mathfrak{T}^{\text{rig}}}$ (respectively $\mathbf{Rf}_{/\mathfrak{T}^{\text{rig}}}$), then c_f is an isomorphism.*

(c) *If f and g are two composable morphisms, then we have $c_{g\circ f} = c_f \circ f^*_\tau(c_g)$.*

Definition 7.18. Let $\mathscr{F}$ be an $\mathscr{O}^{\text{rig}}_{X/\mathfrak{S}}$-module of $(X/\mathfrak{S})_{\text{rconv},\tau}$.

(i) We say that $\mathscr{F}$ is *coherent* if for every object $\mathfrak{T}^{\text{rig}}$ of $\text{RConv}(X/\mathfrak{S})$, $\mathscr{F}_{\mathfrak{T}^{\text{rig}}}$ is coherent (§ 6.7).

(ii) We say that $\mathscr{F}$ is a *crystal* if for every morphism f of $\text{RConv}(X/\mathfrak{S})$, c_f is an isomorphism.

By fppf descent [Abb10, 5.11.11], the direct image and inverse image functors of α_r (7.10.1) induce equivalences of categories quasi-inverse to each other between the category of coherent crystals of $\mathscr{O}^{\text{rig}}_{X/\mathfrak{S}}$-modules of $(X/\mathfrak{S})_{\text{rconv,ad}}$ and that of $(X/\mathfrak{S})_{\text{rconv,fppf}}$.

Proposition 7.19. *The direct image and inverse image functors of $\rho_{X/\mathfrak{S}}$ induce equivalences of categories quasi-inverse to each other between the category of coherent crystals of $\mathscr{O}_{X/\mathfrak{S}}[\frac{1}{p}]$-modules of $(X/\mathfrak{S})_{\text{conv,zar}}$ and that of $(X/\mathfrak{S})_{\text{rconv,ad}}$.*

Proof. Let $\mathscr{F}$ be a coherent crystal of $\mathscr{O}^{\text{rig}}_{X/\mathfrak{S}}$-modules of $(X/\mathfrak{S})_{\text{rconv,ad}}$. By § 6.4 and (7.11.2), $\rho_{X/\mathfrak{S}*}(\mathscr{F})$ is coherent. In view of (6.5.2) and (7.11.4), we deduce that it is also a crystal. By Corollary 7.13(i), $\rho^*_{X/\mathfrak{S}}\rho_{X/\mathfrak{S}*}(\mathscr{F}) \to \mathscr{F}$ is an isomorphism.

Let $\mathscr{G}$ be a coherent crystal of $\mathscr{O}_{X/\mathfrak{S}}[\frac{1}{p}]$-modules of $(X/\mathfrak{S})_{\text{conv,zar}}$ and $\mathscr{H} = \rho^*_{X/\mathfrak{S}}(\mathscr{G})$. By Tate's acyclicity [Abb10, 3.5.5], $\mathscr{G}$ is contained in the essential image of $\rho_{X/\mathfrak{S}*}$ (Corollary 7.13(ii)). Then we have a canonical isomorphism $\mathscr{G} \xrightarrow{\sim} \rho_{X/\mathfrak{S}*}(\mathscr{H})$ (7.13.2). We deduce from Proposition 7.12 and Tate's acyclicity [Abb10, 3.5.5] (respectively (6.5.2) and (7.11.4)) that $\mathscr{H}$ is coherent (respectively is a crystal). Then the assertion follows. □

7.20 Let $g : X \to Y$ be a morphism of S-schemes, $\mathfrak{T}$ an object of $\text{Conv}(Y/\mathfrak{S})$ and $\mathfrak{T}^{\text{rig}}$ its image in $\text{RConv}(Y/\mathfrak{S})$. By fppf descent for morphisms of coherent rigid spaces [Abb10, 5.12.1], the presheaf associated to $\mathfrak{T}^{\text{rig}}$ is a sheaf for the fppf (respectively Zariski) topology that we denote by $\widetilde{\mathfrak{T}}^{\text{rig}}$. We set $X_{T_0} = X \times_Y T_0$ and, for $\tau \in \{\text{ad}, \text{fppf}\}$, we denote by

$$g_{X/\mathfrak{T},\tau} : (X_{T_0}/\mathfrak{T})_{\text{rconv},\tau} \to (T_0/\mathfrak{T})_{\text{rconv},\tau},$$
$$\omega_{\mathfrak{T}^{\text{rig}}} : (X_{T_0}/\mathfrak{T})_{\text{rconv},\tau} \to (X/\mathfrak{S})_{\text{rconv},\tau}$$

the functorial morphisms of topoi (7.14.2).

By repeating arguments of § 4, we prove the following results in the rigid convergent topos.

Lemma 7.21 (Lemma 4.2). *Keep the notation of § 7.20. There exists a canonical equivalence of topoi:*

$$(X/\mathfrak{S})_{\text{rconv},\tau/g^*_{\text{rconv},\tau}(\widetilde{\mathfrak{T}}^{\text{rig}})} \xrightarrow{\sim} (X_{T_0}/\mathfrak{T})_{\text{rconv},\tau}, \qquad (7.21.1)$$

which identifies the localization morphism and $\omega_{\mathfrak{T}^{\text{rig}}}$.

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Lemma 7.22 (Lemma 4.3). *For any $\mathscr{O}^{\mathrm{rig}}_{X/\mathfrak{S}}$-module E of $(X/\mathfrak{S})_{\mathrm{rconv},\tau}$, there exists a canonical isomorphism in $\mathrm{D}^+(\mathfrak{T}^{\mathrm{rig}}_{\tau},\mathscr{O}_{\mathfrak{T}^{\mathrm{rig}}})$:*

$$(\mathrm{R}\,g_{\mathrm{rconv},\tau*}(E))_{\mathfrak{T}^{\mathrm{rig}}} \xrightarrow{\sim} (\mathrm{R}\,g_{X/\mathfrak{T},\tau*}(\omega^{*}_{\mathfrak{T}^{\mathrm{rig}}}(E)))_{\mathfrak{T}^{\mathrm{rig}}}. \tag{7.22.1}$$

Corollary 7.23 (Corollary 4.6). *Let $\mathfrak{S}'\to\mathfrak{S}$ be a morphism of $\mathbf{S}^{\diamond}$, Y' an S'-scheme and $h: Y'\to Y$ a morphism compatible with $S'\to S$. We set $X'=X\times_Y Y'$ and we denote by $g': X'\to Y'$ and $h': X'\to X$ the canonical morphisms:*

$$\begin{array}{ccc} X' & \xrightarrow{h'} & X \\ {\scriptstyle g'}\downarrow & \square & \downarrow{\scriptstyle g} \\ Y' & \xrightarrow{h} & Y \end{array}$$

Then, for any $\mathscr{O}_{X/\mathfrak{S}}[\frac{1}{p}]$-module E of $(X/\mathfrak{S})_{\mathrm{conv},\tau}$, the base-change morphism

$$h^{*}_{\mathrm{rconv},\tau}(\mathrm{R}\,g_{\mathrm{rconv},\tau*}(E)) \xrightarrow{\sim} \mathrm{R}\,g'_{\mathrm{rconv},\tau*}(h'^{*}_{\mathrm{rconv},\tau}(E)) \tag{7.23.1}$$

is an isomorphism.

Corollary 7.24 (Corollary 4.5). *Let $\mathscr{E}$ be a coherent crystal of $\mathscr{O}^{\mathrm{rig}}_{X/\mathfrak{S}}$-modules of $(X/\mathfrak{S})_{\mathrm{rconv},\mathrm{fppf}}$. Then we have (7.10.1)*

$$\mathrm{R}^i\,\alpha_{r*}(\mathscr{E})=0,\quad \forall i\geqslant 1. \tag{7.24.1}$$

Corollary 7.25. *Let $\mathscr{E}$ be a coherent crystal of $\mathscr{O}^{\mathrm{rig}}_{X/\mathfrak{S}}$-modules of $(X/\mathfrak{S})_{\mathrm{rconv},\mathrm{ad}}$. Then we have*

$$\mathrm{R}^i\,\rho_{X/\mathfrak{S}*}(\mathscr{E})=0,\quad \forall i\geqslant 1. \tag{7.25.1}$$

Proof. By Lemma 7.21, the Zariski sheaf $\mathrm{R}^i\,\rho_{X/\mathfrak{S}*}(\mathscr{E})$ on $\mathrm{RConv}(X/\mathfrak{S})$ is associated to the presheaf

$$\mathfrak{T}^{\mathrm{rig}}\mapsto \mathrm{H}^i((T_0/\mathfrak{T})_{\mathrm{rconv},\mathrm{ad}},\mathscr{E}|_{\widetilde{\mathfrak{T}}^{\mathrm{rig}}}).$$

By [SGA4, V 4.3 and III 4.1], we can replace $\mathrm{RConv}(X/\mathfrak{S})$ by the full subcategory of objects whose underlying rigid space is affinoid, and it suffices to show that for such an object $\mathfrak{T}^{\mathrm{rig}}$,

$$\mathrm{H}^i((T_0/\mathfrak{T})_{\mathrm{conv},\mathrm{ad}},\mathscr{E}|_{\widetilde{\mathfrak{T}}}) \tag{7.25.2}$$

vanishes for $i\geqslant 1$. Let $\mathscr{U}=\{\mathfrak{Z}^{\mathrm{rig}}_i\to\mathfrak{Z}^{\mathrm{rig}}\}_{i=1}^{m}$ be an admissible covering by affinoids of an affinoid $\mathfrak{Z}^{\mathrm{rig}}$ in $\mathbf{R}_{/\mathfrak{T}^{\mathrm{rig}}}$. The Čech cohomology $\check{\mathrm{H}}^i(\mathscr{U},\mathscr{E}|_{\widetilde{\mathfrak{Z}}^{\mathrm{rig}}})$ is isomorphic to the cohomology $\mathrm{H}^i(\mathfrak{Z}^{\mathrm{rig}}_{\mathrm{ad}},\mathscr{E}_{\mathfrak{Z}^{\mathrm{rig}}})$, which vanishes by [Abb10, 4.8.26]. Since each admissible covering of $\mathfrak{Z}^{\mathrm{rig}}$ admits a refinement by finitely many affinoids, the vanishing of (7.25.2) follows from [Sta, 03F9]. □

8. Higher direct images of a convergent isocrystal

8.1 Let X be a k-scheme locally of finite type. For $\tau\in\{\mathrm{ad},\mathrm{fppf}\}$, the Frobenius homomorphism $\sigma:\mathrm{W}\to\mathrm{W}$ induces a morphism of topoi $(X'/\mathrm{W})_{\mathrm{rconv},\tau}\to(X/\mathrm{W})_{\mathrm{rconv},\tau}$ (7.14.2). For any sheaf $\mathscr{E}$ of $(X/\mathrm{W})_{\mathrm{rconv},\tau}$, we denote by $\mathscr{E}'$ the inverse image of $\mathscr{E}$ to $(X'/\mathrm{W})_{\mathrm{rconv},\tau}$.

As in § 5.14, a pair $(\mathscr{E},\varphi)$ consisting of a coherent crystal of $\mathscr{O}^{\mathrm{rig}}_{X/\mathfrak{S}}$-modules $\mathscr{E}$ of $(X/\mathrm{W})_{\mathrm{rconv},\tau}$ (Definition 7.18) and an isomorphism $\varphi: F^{*}_{X/k,\mathrm{rconv},\tau}(\mathscr{E}')\xrightarrow{\sim}\mathscr{E}$ is called a *convergent F-isocrystal of* $(X/\mathrm{W})_{\mathrm{rconv},\tau}$.

In this section, we prove the following result about the higher direct image of a convergent (F-)isocrystal of rigid convergent topos.

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Theorem 8.2. *Let $g : X \to Y$ be a smooth proper morphism of k-schemes locally of finite type and $\mathscr{E}$ (respectively $(\mathscr{E}, \varphi)$) a convergent isocrystal (respectively F-isocrystal) of $(X/\mathrm{W})_{\mathrm{rconv},\tau}$. Then $\mathrm{R}^i g_{\mathrm{rconv},\tau *}(\mathscr{E})$ (respectively $(\mathrm{R}^i g_{\mathrm{rconv},\tau *}(\mathscr{E}), \mathrm{R}^i g_{\mathrm{rconv},\tau *}(\varphi))$) is a convergent isocrystal (respectively F-isocrystal) of $(Y/\mathrm{W})_{\mathrm{rconv},\tau}$.*

By Corollary 7.25, Theorem 8.2 and arguments of § 5.13, we deduce the following variant for the convergent topos.

Corollary 8.3. *Keep the assumption of Theorem 8.2. The higher direct image of a convergent isocrystal (respectively F-isocrystal) of $(X/\mathrm{W})_{\mathrm{conv,zar}}$ (§ 5.14) is a convergent isocrystal (respectively F-isocrystal) of $(Y/\mathrm{W})_{\mathrm{conv,zar}}$.*

We first show that the smooth case of Theorem 8.2 can be deduced from the corresponding statement in convergent topos (Theorem 5.10).

Proposition 8.4. *Keep the notation and assumption of Theorem 8.2. If Y is moreover smooth over k, then $\mathrm{R}^i g_{\mathrm{rconv},\tau *}(\mathscr{E})$ is a coherent crystal of $\mathscr{O}^{\mathrm{rig}}_{Y/\mathrm{W}}$-modules.*

Proof. We first prove the assertion for the admissible topology. The sheaf $\mathscr{F} = \rho_{X/\mathrm{W}*}(\mathscr{E})$ is a coherent crystal of $\mathscr{O}^{\mathrm{rig}}_{X/\mathrm{W}}$-modules of $(X/\mathrm{W})_{\mathrm{conv,zar}}$ and $\rho^*_{X/\mathrm{W}}(\mathscr{F}) \xrightarrow{\sim} \mathscr{E}$ (Proposition 7.19). By Theorem 5.10, $\mathrm{R}^i g_{\mathrm{conv,zar}*}(\mathscr{F})$ is a coherent crystal of $\mathscr{O}_{Y/\mathrm{W}}[\frac{1}{p}]$-modules of $(Y/\mathrm{W})_{\mathrm{conv,zar}}$. We consider the composition

$$\mathrm{R}^i g_{\mathrm{conv,zar}*}(\mathscr{F}) \xrightarrow{\sim} \rho_{Y/\mathrm{W}*}\rho^*_{Y/\mathrm{W}}(\mathrm{R}^i g_{\mathrm{conv,zar}*}(\mathscr{F})) \to \rho_{Y/\mathrm{W}*}(\mathrm{R}^i g_{\mathrm{rconv,ad}*}(\mathscr{E})), \tag{8.4.1}$$

where the first arrow is an isomorphism by Proposition 7.19 and the second arrow is induced by the base-change morphism. By Lemmas 4.3 and 7.22, $\mathrm{R}^i g_{\mathrm{conv,zar}*}(\mathscr{F})$ (respectively $\rho_{Y/\mathrm{W}*}(\mathrm{R}^i g_{\mathrm{rconv,ad}*}(\mathscr{E}))$) is the sheaf associated to the presheaf on $\mathrm{Conv}(Y/\mathrm{W})$:

$$\mathfrak{T} \mapsto \mathrm{H}^i((X_{T_0}/\mathfrak{T})_{\mathrm{conv,zar}}, \omega^*_{\mathfrak{T}}(\mathscr{F})) \quad (\text{respectively } \mathfrak{T} \mapsto \mathrm{H}^i((X_{T_0}/\mathfrak{T})_{\mathrm{rconv,ad}}, \omega^*_{\mathfrak{T}^{\mathrm{rig}}}(\mathscr{E}))).$$

By Corollary 7.25, the canonical morphism

$$\mathrm{H}^i((X_{T_0}/\mathfrak{T})_{\mathrm{conv,zar}}, \omega^*_{\mathfrak{T}}(\mathscr{F})) \xrightarrow{\sim} \mathrm{H}^i((X_{T_0}/\mathfrak{T})_{\mathrm{rconv,ad}}, \omega^*_{\mathfrak{T}^{\mathrm{rig}}}(\mathscr{E}))$$

is an isomorphism. The composition (8.4.1) is induced by the above morphisms and hence is an isomorphism. In view of the definition of $\rho_{Y/\mathrm{W}*}$ (§ 7.11), we deduce that $\rho^*_{Y/\mathrm{W}}(\mathrm{R}^i g_{\mathrm{conv,zar}*}(\mathscr{F})) \xrightarrow{\sim} \mathrm{R}^i g_{\mathrm{rconv,ad}*}(\mathscr{E})$ by (8.4.1). Then the assertion for admissible topology follows from Proposition 7.19.

Using Corollary 7.24, one verifies the proposition for fppf topology by comparing $\mathrm{R}^i g_{\mathrm{rconv,ad}}(\mathscr{E})$ and $\mathrm{R}^i g_{\mathrm{rconv,fppf}}(\mathscr{E})$ in a similar way as above. □

8.5 To prove Theorem 8.2, we use a construction of Ogus in his proof of proper surjective descent for convergent isocrystals [Ogu84]. Let $\mathfrak{T}$ be an adic formal W-scheme of finite type and $f : Z \to T_0$ a projective and surjective k-morphism. Then f factors through a closed immersion $Z \to \mathbb{P}^N_{T_0}$ for some integer $N \geqslant 1$. Let $\mathbb{P}^N_{\mathfrak{T}}$ be the formal W-scheme associated to the inductive system $(\mathbb{P}^N_{\mathfrak{T}_n})_{n\geqslant 1}$. By § 3.18, we can construct a family of adic formal $\mathbb{P}^N_{\mathfrak{T}}$-schemes $\{\mathfrak{T}_{Z,n}(\mathbb{P}^N_{\mathfrak{T}})\}_{n\geqslant 0}$. Based on the following result, Ogus showed the proper surjective descent for convergent isocrystals [Ogu84, 4.6].

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Theorem 8.6 [Ogu84, 4.7 and 4.8]. *For n large enough, the morphism $\pi : \mathfrak{T}_{Z,n}(\mathbb{P}^N_{\mathfrak{T}}) \to \mathfrak{T}$ is faithfully rig-flat (i.e. π^{rig} is faithfully flat) [Abb10, 5.5.9].*

A variant of Theorem 5.2 holds for rigid convergent topos.

Proposition 8.7. *For every locally of finite type k-scheme X, the morphism*

$$F_{X/k,\mathrm{rconv,fppf}} : (X/\mathrm{W})_{\mathrm{rconv,fppf}} \to (X'/\mathrm{W})_{\mathrm{rconv,fppf}} \tag{8.7.1}$$

is an equivalence of topoi.

Proof. By Lemma 5.4 and (7.3.1), the canonical functor $\mathrm{RConv}(X/\mathrm{W}) \to \mathrm{RConv}(X'/\mathrm{W})$ induced by $F_{X/k}$ (7.14.1) is fully faithful. In view of Proposition 2.5, Lemma 5.5 and § 7.11, the assertion follows. □

8.8 *Proof of Theorem 8.2.* We prove the assertion for convergent isocrystals. Then the assertion for convergent F-isocrystals follows from Proposition 8.7 and a similar argument as in Corollary 5.15. The question being local (Corollary 7.23), we may assume that Y is separated and of finite type by Corollary 7.23. Moreover, we may assume that Y is reduced.

By applying alteration to each irreducible component of Y [dJon96, 4.1], there exist a smooth k-scheme $\widetilde{Y}$ and a proper surjective k-morphism $\widetilde{Y} \to Y$. By Chow's lemma [EGAII, 5.6.1], there exists a surjective k-morphism $Z \to \widetilde{Y}$ such that the composition $f : Z \to \widetilde{Y} \to Y$ is projective and surjective. We set $\mathscr{F} = \mathrm{R}^i g_{\mathrm{rconv,fppf}\,*}(\mathscr{E})$. In view of Corollary 7.23 and Proposition 8.4, the inverse image of $\mathscr{F}$ to $(\widetilde{Y}/\mathrm{W})_{\mathrm{rconv,fppf}}$ is a coherent crystal. Then so is $f^*_{\mathrm{rconv,fppf}}(\mathscr{F})$.

Let $(\mathfrak{T}, u)$ be an object of $\mathrm{Conv}(Y/\mathrm{W})$. The morphism f factors through a closed immersion $Z \to \mathbb{P}^N_Y$ for some integer $N \geqslant 1$. We set $T_Z = T_0 \times_Y Z$. We take again the notation of § 8.5 for the projective and surjective k-morphism $T_Z \to T_0$. We choose an integer n such that the morphism $\pi : \mathfrak{T}_{T_Z,n}(\mathbb{P}^N_{\mathfrak{T}}) \to \mathfrak{T}$ is faithfully rig-flat (Theorem 8.6). We set $\mathfrak{R} = \mathfrak{T}_{T_Z,n}(\mathbb{P}^N_{\mathfrak{T}})$, $\mathfrak{R}^{(1)} = \mathfrak{R} \times_{\mathfrak{T}} \mathfrak{R}$ and denote by $p_1, p_2 : \mathfrak{R}^{(1)} \to \mathfrak{R}$ two projections.

Note that $\mathfrak{R}$ and $\mathfrak{R}^{(1)}$ define objects of $\mathrm{Conv}(Z/\mathrm{W})$ by (3.18.2) and then of $\mathrm{Conv}(Y/\mathrm{W})$. Moreover, $\{\pi^{\mathrm{rig}} : \mathfrak{R}^{\mathrm{rig}} \to \mathfrak{T}^{\mathrm{rig}}\}$ defines an fppf covering of $\mathrm{RConv}(Y/\mathrm{W})$. Since $f^*_{\mathrm{rconv,fppf}}(\mathscr{F})$ is a coherent crystal of $\mathscr{O}^{\mathrm{rig}}_{Z/\mathrm{W}}$-modules, the following modules are coherent:

$$\mathscr{F}_{\mathfrak{R}^{\mathrm{rig}}} = (f^*_{\mathrm{rconv,fppf}}(\mathscr{F}))_{\mathfrak{R}^{\mathrm{rig}}}, \quad \mathscr{F}_{\mathfrak{R}^{(1),\mathrm{rig}}} = (f^*_{\mathrm{rconv,fppf}}(\mathscr{F}))_{\mathfrak{R}^{(1),\mathrm{rig}}} \tag{8.8.1}$$

and we have isomorphisms

$$p_2^{\mathrm{rig}*}(\mathscr{F}_{\mathfrak{R}^{\mathrm{rig}}}) \xrightarrow{\sim} \mathscr{F}_{\mathfrak{R}^{(1),\mathrm{rig}}} \xleftarrow{\sim} p_1^{\mathrm{rig}*}(\mathscr{F}_{\mathfrak{R}^{\mathrm{rig}}}). \tag{8.8.2}$$

Then we obtain a descent datum on $\mathscr{F}_{\mathfrak{R}^{\mathrm{rig}}}$ for the fppf covering $\{\pi^{\mathrm{rig}} : \mathfrak{R}^{\mathrm{rig}} \to \mathfrak{T}^{\mathrm{rig}}\}$. There exist a coherent $\mathscr{O}_{\mathfrak{T}^{\mathrm{rig}}}$-module $\mathscr{M}$ and an isomorphism [Abb10, 5.11.11]

$$\pi^{\mathrm{rig}*}(\mathscr{M}) \xrightarrow{\sim} \mathscr{F}_{\mathfrak{R}^{\mathrm{rig}}}. \tag{8.8.3}$$

On the other hand, since $\mathscr{F}$ is a sheaf for the fppf topology, for any $\mathcal{U} \in \mathbf{Ob}(\mathbf{Rf}/_{\mathfrak{T}^{\mathrm{rig}}})$, we have an exact sequence

$$0 \to \mathscr{F}(\mathcal{U}) \to \mathscr{F}(\mathcal{U} \times_{\mathfrak{T}^{\mathrm{rig}}} \mathfrak{R}^{\mathrm{rig}}) \to \mathscr{F}(\mathcal{U} \times_{\mathfrak{T}^{\mathrm{rig}}} \mathfrak{R}^{(1),\mathrm{rig}}). \tag{8.8.4}$$

By comparing (8.8.3) and (8.8.4), we deduce that $\mathscr{F}_{\mathfrak{T}^{\mathrm{rig}}}$ is isomorphic to $\mathscr{M}$ and hence is coherent.

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Let $g : \mathfrak{T}' \to \mathfrak{T}$ be a morphism of $\mathrm{Conv}(Y/\mathrm{W})$. Choose an integer n large enough such that $\mathfrak{R}' = \mathfrak{T}_{T'_Z,n}(\mathbb{P}^N_{\mathfrak{T}'}) \to \mathfrak{T}'$ and $\mathfrak{R} = \mathfrak{T}_{T_Z,n}(\mathbb{P}^N_{\mathfrak{T}}) \to \mathfrak{T}$ are faithfully rig-flat. Since the construction of $\mathfrak{R}$ is functorial, we have a W-morphism $h : \mathfrak{R}' \to \mathfrak{R}$ compatible with g. Moreover, h induces a morphism of $\mathrm{Conv}(Z/\mathrm{W})$. The transition morphism of $f^*_{\mathrm{rconv,fppf}}(\mathscr{F})$ associated to h^{rig} is an isomorphism. Since h dominates g in $\mathrm{Conv}(Y/\mathrm{W})$, we deduce that the transition morphism $c_{g^{\mathrm{rig}}}$ of $\mathscr{F}$ associated to g^{rig} is an isomorphism by fppf descent (cf. [Xu19, proof of 9.13]). Then $\mathscr{F}$ is a crystal and the theorem follows. □

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