The right-hand inequality, (1.10-rh), does follow by induction from (1.6) on replacing $M$ by $R$ in the integrand of (1.6). However, (1.7) merely implies that

$$M(a+2,b+2)-R(a,b) \geqq \frac{4}{(a-1)(b-1)}\int_0^{a-1}\int_0^{b-1}(M(\varepsilon,\eta)-R(\varepsilon,\eta))d\varepsilon d\eta$$

where the quantity on the right is negative by (1.10-rh). Consequently, even given Theorem 1, the left-hand inequality, (1.10-lh), would still be unproved.

*Conclusions.* The only parts of Weiner's edifice which remain intact are Lemma 1 and the following pair of inferences: (1.6) implies (1.10-rh), and (1.10) implies Palásti's conjecture. This is not to say that the other results are *necessarily* false, but rather that Weiner has failed to prove them. Indeed, (1.3) of [11] is assuredly true. Now, (1.10-rh) obviously implies that the limiting parking density is not greater than Palásti's conjecture of $\eta^2$. On the other hand, simulations indicate otherwise (cf. Akeda and Hori [1], Blaisdell and Solomon [2], Jodrey and Tory [5]). Therefore, it is most important to determine the status of (1.6) and (1.10-rh) of [11].

Recall that the LHS of (1.10) is the average number (say $N_1$) of cars parked in an $a \times b$ Rényi model, while the RHS is this average (say $N_2$) for the partitioned model discussed in Example 5 above. The obvious way to compare these averages, which are $O(ab)$, is to consider boundary effects. Assuming Palásti's conjecture actually holds, the correction terms (Dvoretzky and Robbins [3]) then imply that the difference between these averages (i.e. $N_1 - N_2$) is $3(1-\eta)^2 + o(1)$. That is, even if the conjecture is true, Weiner's critical inequalities (1.6) and (1.10-rh) cannot hold — indeed, they must be reversed. Finally if, as simulations suggest, the limiting parking density is actually greater than $\eta^2$, these critical inequalities must again be reversed.

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Dear Editor,

*Some comments on the letters by H. J. Weiner*

In this letter, I should like to comment on Weiner's reply [10] to my first letter [7] and to the letters of other authors, and on Weiner's recent alternative argument [11] for his paper [9]. The conclusion of [11] does not differ from that of [9]. I will discuss in this note the following two points:

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(a) Lemma 2 of [9], which states that '$\alpha \times \beta$ cars in an $a \times b$ rectangle with coordinates $(0,0)$, $(0,b)$, $(a,0)$, $(a,b)$ $(\alpha, \beta \leqslant a, b)$ intersect line segment $l$ which combines $(0, b-\beta)$ to $(a, b-\beta)$, in segments (of length $\alpha$) *in accord with* a one-dimensional law of Rényi parking of cars', cannot yet be thought to be established.

(b) The inequalities in [11] and [9] whose proofs are essentially based on Lemma 2 of [9], cannot, therefore, be said to be established.

First, to show that the statement of Lemma 2 of [9] is not yet established, let us refine and extend the argument in [7] and contrast it with the arguments by Rényi [6] and by Dvoretzky and Robbins [3] which resolved the one-dimensional car-parking problem. Although Weiner uses in [11] unit squares ($1 \times 1$ cars) in place of $\alpha \times \beta$ cars in [10] and [9], we follow his initial notation. As in [7], let us consider in the $a \times b$ rectangle a strip $L$ with coordinates $(\alpha/2, b-3\beta/2)$, $(a-\alpha/2, b-3\beta/2)$, $(\alpha/2, b-\beta/2)$, $(a-\alpha/2, b-\beta/2)$. The midparallel of the strip $L$ coincides with the segment $l$ (see Figure 1). It is clear that $\alpha \times \beta$ cars whose centres are inside $L$ certainly cross $l$. Therefore, the parking of cars which cross the segment $l$ becomes equivalent to the parking of cars whose centres are inside the strip $L$. We can assume that, just before the centre of a car is fixed inside $L$ for the first time, a certain number of cars is already parked inside another region of the $a \times b$ rectangle. The residual space (i.e. the space which is available for the centres of cars to be parked later) is the region which is bounded by the rectangle with coordinates $(\alpha/2, \beta/2)$, $(a-\alpha/2, \beta/2)$, $(\alpha/2, b-\beta/2)$, $(a-\alpha/2, b-\beta/2)$ and which is deleted by $(2\alpha) \times (2\beta)$ rectangles whose centres coincide respectively with those of already parked $\alpha \times \beta$ cars.

Let $C_1, C_2, \cdots$ denote the cars and their centres which are parked on $L$ for the

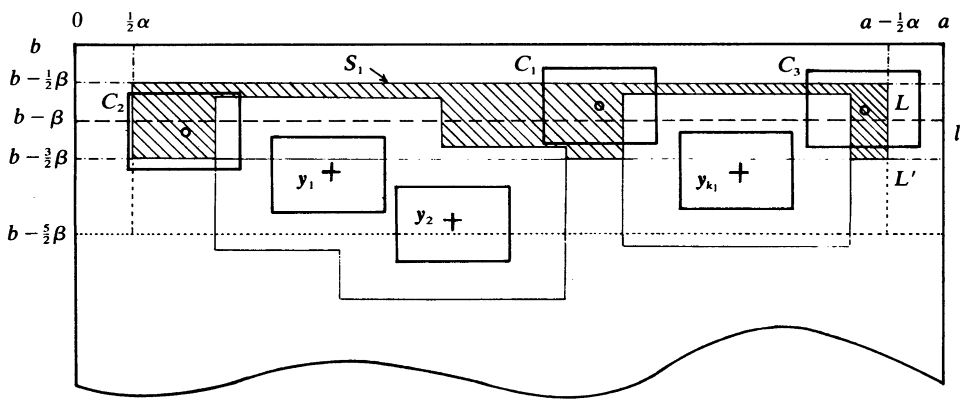


Figure 1

first time, for the second time and so on. Let us assume that, just before $C_1$ is parked, there are $k_1$ cars inside a strip $L'$ which is defined by the rectangle with coordinates $(\alpha/2, b-5\beta/2)$, $(a-\alpha/2, b-5\beta/2)$, $(\alpha/2, b-3\beta/2)$, $(a-\alpha/2, b-3\beta/2)$. Let $\mathbf{y}_1, \mathbf{y}_2, \cdots, \mathbf{y}_{k_1}$ be their positions. It is clear that the strip $L$ is deleted by $(2\alpha)\times(2\beta)$ rectangles with respective coordinates $\mathbf{y}_1, \cdots, \mathbf{y}_{k_1}$ (see Figure 1). According to the above assumption, the residual space for $C_1$ is the shaded region in Figure 1. Let $S_1$ indicate this region and also its area. Generally, $S_1$ depends on $\mathbf{y}_1, \cdots, \mathbf{y}_{k_1}$; in that case we write it as $S_1(\mathbf{y}_1, \cdots, \mathbf{y}_{k_1})$. As $C_1$ is sampled uniformly at random inside $S_1$, the probability density for the position $\mathbf{X}_1 = (x, y)$ of $C_1$ can be written as follows:

$$f_1(x, y \mid \mathbf{y}_1, \cdots, \mathbf{y}_{k_1}) = \frac{1}{S_1(\mathbf{y}_1, \cdots, \mathbf{y}_{k_1})}, \qquad \text{for } (x, y) \in S_1.$$

Therefore, the probability density of the $x$ coordinate of $C_1$ becomes

$$g_1(x \mid \mathbf{y}_1, \cdots, \mathbf{y}_{k_1}) = \frac{1}{S_1(\mathbf{y}_1, \cdots, \mathbf{y}_{k_1})} \int_{(x,y)\in S_1} dy.$$

As can be seen from Figure 1, it is obvious that this is a step function of $x$ and is not generally a constant. This is the substance of the statement in [7], namely, 'the probability of car placement is smaller in the region where the width is narrower than in the region where it is broader and vice versa'. Thus, the first car $C_1$ which crosses the segment $l$ is not sampled uniformly with respect to the $x$ coordinate but is dependent on other cars $\mathbf{y}_1, \cdots, \mathbf{y}_{k_1}$ which are previously parked inside the strip $L'$

This should be contrasted with one-dimensional Rényi parking where the $x$ coordinate of the first car is uniformly distributed on the interval $[\alpha/2, a-\alpha/2]$. We note here that one-dimensional car parking corresponds to the parking of $\alpha\times\beta$ cars in the $a \times b$ rectangle where $a \geqq \alpha$ and $\beta \leqq b < 2\beta$. In this case, therefore, it holds that $S_1 = (a-\alpha)(b-\beta)$ and $g_1(x) = (b-\beta)/(a-\alpha)(b-\beta) = 1/(a-\alpha)$ in accord with the above argument.

Furthermore, there are differences between parking on the segment $l$ and one-dimensional car parking. The latter is also characterized by the facts that the number of cars which will eventually be parked to the left of the first car is independent of the number of cars which will be parked to the right of it and that these two numbers have the same distribution which depends only on the length of the segment [3]. We show that this is not the case for parking on the segment $l$ if $b > 2\beta$ holds. Let us assume that, after $C_1$ is parked, residual space is left on both sides of it (see Figure 1). We suppose that $C_2$ and $C_3$ will be parked respectively to the left and to the right of $C_1$ (see Figure 1). That the presence of $C_2$ might affect the position of $C_3$ is shown as follows. At the first step, the presence of $C_2$ affects the later parking of a car inside the strip $L'$ and nearby $C_2$.

The parking of the latter may give an influence to the later parking of other cars. Such an effect is transmitted to one car after another and may eventually influence the position of $C_3$. As a result, the positions of $C_2$ and $C_3$ are not independent of each other. Consequently, the number of cars which will be parked to the left of $C_1$ is not independent of the number of cars to be parked to the right of it. Hence we cannot readily conclude that these two numbers have the same distribution.

Thus, the argument parallel to that of the one-dimensional car parking problem [3], [6] seems to be difficult to carry further. The statement of Lemma 2 of [9] therefore loses its theoretical basis or at least needs more profound argument to overcome the difficulties discussed above.

Secondly, Weiner proves a number of inequalities in his alternative argument [11]. In the lemma of [11], he shows the following inequalities (Weiner's equation numbers are retained):

$$M(a+\alpha, b) \geqq M(a, b) \tag{1.3}$$

$$M(a+\alpha, b) \leqq M(a, b) + M_\beta(b) \tag{1.4}$$

$$M(a+2\alpha, b) \geqq M(a, b) + M_\beta(b) \tag{1.5}$$

where $M(a, b)$ is the expected number of $\alpha \times \beta$ cars which can be parked on an $a \times b$ rectangle $(a > \alpha, b > \beta)$ and where $M_\beta(b)$ is the expected number of segments of length $\beta$ which can be parked on a segment of length $b(>\beta)$. Here, the notation of [9] is restored. These inequalities correspond respectively to (2.5a), (2.6a) and (2.6b) of [9]. The proof for (1.3) of [11] seems to be correct. The proofs for (1.4) and (1.5) of [11], however, are based on Lemma 2 of [9] which is not yet established, as discussed above. Moreover, in the theorem of [11], Weiner shows the following inequalities:

$$M(a, b) \leqq M_\alpha(a) + M_\beta(b) - 1 + \frac{4}{(a-\alpha)(b-\beta)} \int_0^{a-\alpha} d\xi \int_0^{b-\beta} d\eta M(\xi, \eta) \tag{1.6}$$

$$\begin{aligned} &M(a+2\alpha, b+2\beta) \geqq \\ &\quad M_\alpha(a) + M_\beta(b) - 1 + \frac{4}{(a-\alpha)(b-\beta)} \int_0^{a-\alpha} d\xi \int_0^{b-\beta} d\eta M(\xi, \eta) \end{aligned} \tag{1.7}$$

which correspond to (2.9) and (2.10) of [9], respectively. For the proof of these inequalities, Weiner uses inequalities (1.4) and (1.5) of [11] for which the theoretical basis is insecure. Therefore, the conclusion of [9] and [11], i.e., $\lim_{a,b\to\infty}(ab)^{-1}M(a, b) = (\lim_{a\to\infty} a^{-1}M_\alpha(a))^2 \equiv \eta^2$, is still open to question.

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