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BOOK REVIEW

THE ERGODIC THEORY OF DISCRETE GROUPS

By P. J. NICHOLLS: £19.50, Paperback, 221pp, ISBN 0 521 37674 2.

The subject of the ergodic properties of discrete groups of isometries of hyperbolic space is of course not new. In fact the study of the geodesic flow on manifolds of constant negative curvature goes back even before the beginnings of ergodic theory in the thirties, being one of the earliest examples of what we now call chaotic motion. The attraction of these examples is that while they exhibit all the features of chaotic motion, they have at the same time an extremely rich underlying geometrical structure, which while simple enough to make their study tractable, nevertheless exhibits a wealth of interesting phenomena. Early work, with some notable exceptions, centred on two-dimensional finitely generated groups of the first kind, that is groups whose limit set is the whole circle at infinity and for which the quotient hyperbolic space is a surface of finite area. The proofs that for such groups both the geodesic flow and the action of the group on the sphere at infinity are ergodic were among the early triumphs of ergodic theory. As was already well understood in the thirties from the work of Hedlund, Hopf, Myrberg and others, the dynamics on the quotient surface and on the limit set are inextricably linked in a number of subtle ways.

The geometrical aspects of the theory received new stimulus with the construction in 1976 by S. J. Patterson of a measure on the limit set of a discrete hyperbolic group which transforms according to the rule

$$\gamma_*(\mu) = \gamma'(\xi)^\delta \mu \tag{1}$$

and which acts in many respects as a substitute for Lebesgue measure when the limit set does not have full measure. Here δ is the exponent of convergence of the group, that is, the exponent of convergence of the Poincaré series $\sum_{\gamma\in\Gamma} e^{-s\rho(0,\gamma 0)}$, where ρ is hyperbolic distance. In a paper which is now classic, 'The density at infinity of a discrete group of hyperbolic motions', *Publ. Math. IHES* **50** (1979) 171–202, Dennis Sullivan extended Patterson's construction to arbitrary dimensions, showing that by use of this measure one could redo much of the older ergodic theory, and that moreover ergodic phenomena had a crucial bearing on the finer geometrical properties of the groups concerned.

It was a very nice idea, and such is the aim of Nicholls' book, to give a systematic account of this body of material, including both a summary of the early work and a thorough reworking and filling in of many details from Sullivan's paper. For my own taste I should have liked to see rather more overview and motivation: there is not a particularly readable account of the history (rather this is to be gleaned from scattered references throughout the book), nor is there much attempt to point out applications and developments beyond the immediate topics included in the book. However it will certainly be very useful to have a detailed and organised account easily accessible.

Although there is of course some discussion of the standard ergodic theory results, particularly in the last chapter, the emphasis of Nicholls' book is strongly on the geometrical theory in higher dimensions. Its strength lies in a very careful and thorough exposition of the geometry of a discrete hyperbolic group acting on its limit set, and of the implications for the geodesic flow on the quotient manifold. Points on the limit set may be classified by the speed and manner in which they are approximated by elements of the group. The book begins with a detailed account of this classification. As becomes apparent later, the size of sets with different approximation properties is crucial in governing the ergodic behaviour. The central role played by the convergence or otherwise of the Poincaré series at the critical exponent is also discussed.

Chapters 3 and 4 contain the construction and detailed properties of the Patterson–Sullivan measure. This is closely connected with another notion introduced by Sullivan: a *conformal density of dimension* α. Roughly speaking, this is a family of measures which transform according to the rule (1) with α replacing δ. The situation is best in the case of convex cocompact groups, in which case there is a unique conformal density of dimension δ which turns out to be the Hausdorff dimension of the limit set. The orbital counting function, or number of orbit points within a ball of radius R of hyperbolic space, is well estimated in this case by $e^{\delta R}$.

Hopf's original proof of ergodicity of the geodesic flow on surfaces of constant negative curvature is based on the fact that there are no non-constant group-invariant bounded harmonic functions in the disc. This idea can be extended to higher dimensions by the use of hyperbolically harmonic functions. These functions are introduced and their basic properties established in Chapter 5. Study of harmonic functions in this context is crucial, as is seen from the following list of equivalent properties:

(i) The Poincaré series diverges at the exponent $n-1$.

(ii) $\mathbb{H}^n/\Gamma$ has no Green's function.

(iii) There are no non-constant bounded Γ-invariant functions on $\mathbb{H}^n \times \mathbb{H}^n$ harmonic in both variables.

(iv) The geodesic flow on $\mathbb{H}^n/\Gamma$ is ergodic.

In Chapter 6 these results are used to study ergodicity and transitivity properties of the action of Γ on the sphere at infinity S, on $S \times S$, and higher products.

Armed with the Sullivan–Patterson measure μ on the limit set it is natural to construct an invariant measure for the geodesic flow by the Hopf procedure of specifying a geodesic by its two endpoints at infinity and forming the product measure $\mu \times \mu \times dt$, where t is hyperbolic distance along the geodesic. It turns out to be possible to repeat a large part of the ergodic theory of groups of the first kind with respect to Lebesgue measure in this situation. For a geometrically finite group the new measure is finite. In general, there are some very interesting relations between ergodicity properties of the geodesic flow and the nature of the measure on the limit set, and one obtains a wealth of further informtion about the group in this way. These topics are the subject of Chapters 8 and 9.

The last chapter is mainly devoted to the specialization to Fuchsian groups and the derivation of the classic ergodic theory results, beginning with Hopf's proof that the geodesic flow on a surface of constant negative curvature is ergodic, followed by a nice exposition of the relations between the geodesic and horocycle flows and proofs that these flows are mixing and that the horocycle flow is uniquely ergodic. There is also a derivation of the asymptotic formula for the number of lattice points in a ball in hyperbolic space. This will be a convenient source of results that are somewhat scattered in the literature and could be a good reference for students. More subtle aspects of the ergodic theory, such as the Bernouilli property, Gibbs measures, or the remarkable results of Ratner on the rigidity of the horocycle flow, are not mentioned, and those looking for an account of the deeper ergodic properties will probably wish to turn elsewhere. In fact possibly the title of the book is rather misleading in that the word *ergodic* is not defined until Chapter 7, where we are presented with an account of the standard ergodic theorems, as modified to take care of infinite invariant measures.

The exposition of the book is in general clean and easy to follow, although I was disappointed at one or two rather crucial points in the argument to be referred elsewhere, without any hint of the methods involved, and not always to a particularly accessible reference at that: for example, in Chapter 5, the crucial fact of the equivalence of (i) and (ii) above was referred to a mimeographed set of notes of Ahlfors. The logic of these choices was not clear to me, especially since much of the material included on ergodic theory is readily available in many sources. In an expository book of this kind one feels that it would have been nice to have a sketch of all the main lines of argument in one place.

Besides the omissions in ergodic theory mentioned above, I should have expected to see at least some mention of the other parts of Sullivan's theory, for example the relation to Brownian motion on $\mathbb{H}/\Gamma$, and the very important aspect of applications to the rigidity theory of Kleinian groups. Some of the related material that is omitted here is to be found not only in the original sources but also in Patterson, '*Measures on limit sets of Kleinian groups*', in: *Analytical and Geometric Aspects of Hyperbolic Space*, LMS lecture notes 111, ed. D. B. A. Epstein (Cambridge University Press, Cambridge, 1984).

What Nicholls has produced is a clean and thorough reworking of the foundational parts of the theory. It will be useful to have a coherent account of this body of material in one place. For those who wish to study the detail with care, this will be a good starting point. Those who want more excitement in life will still prefer to start with Sullivan, but may well find themselves turning to Nicholls with some relief as the going gets rough.

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