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# HOMOGENIZATION OF NON-SYMMETRIC JUMP PROCESSES

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**Abstract**

We study homogenization for a class of non-symmetric pure jump Feller processes. The jump intensity involves periodic and aperiodic constituents, as well as oscillating and non-oscillating constituents. This means that the noise can come both from the underlying periodic medium and from external environments, and is allowed to have different scales. It turns out that the Feller process converges in distribution, as the scaling parameter goes to zero, to a Lévy process. As special cases of our result, some homogenization problems studied in previous works can be recovered. We also generalize the approach to the homogenization of symmetric stable-like processes with variable order. Moreover, we present some numerical experiments to demonstrate the usage of our homogenization results in the numerical approximation of first exit times.

*Keywords:* Feller processes; weak convergence; non-local operators; stable-like processes

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Secondary 60F17; 35R09

## 1. Introduction

As a subclass of Markov processes, Feller processes possess lots of nice properties, from both probabilistic and analytic perspectives [1, 8, 16, 26].

The generator of a Feller process is in general a non-local operator. It looks locally like the generator of a Lévy process, in the sense that it is given by a Lévy–Khintchine-type representation with an $x$-dependent Lévy triplet $(b(x), a(x), \eta(x, \cdot))$. For this reason, Feller processes are sometimes called Lévy-type processes or jump-diffusions, and their generators are called Lévy-type operators. Feller processes with no diffusion parts at all, i.e., with $a \equiv 0$, are called *(pure) jump processes*. If the generator of a Feller process is non-symmetric as an operator, the process is called *non-symmetric*.

Homogenization problems arise from the study of porous media, composite materials, and other physical and engineering systems [3, 12, 13]. Generally speaking, in a periodic

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structure, such as a medium or material, the heterogeneities are relatively small compared to its global dimension. Thus, two scales characterize the motion of particles in the structure: the microscopic one describing the heterogeneities, and the macroscopic one describing the global behavior of particles. The aim of homogenization is precisely to give the macroscopic properties of the particles while taking into account the properties of the microscopic structure.

In this paper, we focus on the homogenization of jump processes, periodic in space and locally periodic in noise. Consider a pure jump process in a periodic medium. The drift $b(x)$ and the jump kernel $\eta(x, \cdot)$ are periodic and of small scale in the spatial variable $x$, because of heterogeneities. In the mathematical formulation, the small scale is represented by a small parameter $\epsilon > 0$. From the realistic point of view, the noise may arise not only from the underlying periodic medium, but also from external environments. So we may assume that the jump kernel $\eta(x, dz)$ is of mixed scale in the noise variable $z$. This suggests that the generators of jump processes in a periodic medium will take the following form:

$$\begin{aligned}\mathcal{A}^{\epsilon} f(x) = &\int_{\mathbb{R}^d\setminus\{0\}} \left[f(x+z) - f(x) - z\cdot\nabla f(x)\mathbf{1}_{[1,2)}(\alpha)\mathbf{1}_B(z)\right]\kappa(x/\epsilon, z, z/\epsilon)J(z)dz \\ &+ \left(\tfrac{1}{\epsilon^{\alpha-1}}b(x/\epsilon) + c(x/\epsilon)\right)\cdot\nabla f(x)\mathbf{1}_{(1,2)}(\alpha).\end{aligned} \tag{1.1}$$

Here and after, we denote by $B$ the unit open ball in $\mathbb{R}^d$, and by $S := \partial B$ the unit sphere. Furthermore, the drift functions $b, c : \mathbb{R}^d \to \mathbb{R}^d$ are Borel measurable and periodic, and $J : \mathbb{R}^d \setminus \{0\} \to (0, \infty)$ is the density of a (not necessarily symmetric) $\alpha$-stable Lévy measure [35], with $\alpha \in (0, 2)$. More precise assumptions will be made in the next section.

The *jump coefficient* $\kappa : \mathbb{R}^d \times \left(\mathbb{R}^d \setminus \{0\}\right) \times \left(\mathbb{R}^d \setminus \{0\}\right) \to [0, \infty)$ is a Borel measurable function, periodic in its first variable. The normal scale of $\kappa$ in $z$ corresponds to the noise constituent coming from external environments. Furthermore, we assume that there exists a function $\kappa^* : \mathbb{R}^d \times \left(\mathbb{R}^d \setminus \{0\}\right) \times \left(\mathbb{R}^d \setminus \{0\}\right) \times \left(\mathbb{R}^d \setminus \{0\}\right) \to [0, \infty)$ that is periodic in its first and third variables, such that $\kappa(x, z, u) = \kappa^*(x, z, u, u)$. In this case, the jump coefficient $\kappa(x/\epsilon, z, z/\epsilon)$ in (1.1) is locally periodic in the noise variable $z$. This means that the small noise scale can be decomposed into two constituents, corresponding to the periodic medium and external environments, respectively.

Under some regularity assumptions (see the next section), each Lévy-type operator $\mathcal{A}^{\epsilon}$ can generate a Feller process on $\mathbb{R}^d$, say $X^{\epsilon}$. Our aim is to identify the limit of $X^{\epsilon}$ as the scaling parameter $\epsilon$ goes to zero. It turns out (see Theorem 1) that the limit process $X$, in the sense of convergence in distribution, is a Lévy process with the following generator:

$$\begin{aligned}\bar{\mathcal{A}} f(x) = &\int_{\mathbb{R}^d\setminus\{0\}} \left[f(x+z) - f(x) - z\cdot\nabla f(x)\mathbf{1}_{[1,2)}(\alpha)\mathbf{1}_{\{|z|<1\}}\right]\bar{\kappa}(z)J(z)dz \\ &+ \bar{c}\cdot\nabla f(x)\mathbf{1}_{(1,2)}(\alpha),\end{aligned}$$

where $\bar{\kappa}$ is a homogenized jump coefficient related to the function $\kappa$ and $\bar{c}$ is a homogenized constant.

Homogenization of Feller processes with jumps has been investigated by a number of authors. To name a few, the paper [22] considered the one-dimensional pure jump case, and [17] studied the homogenization of stable-like processes with variable order. See also [38] for a multi-dimensional generalization with diffusion terms involved. The paper [36] investigated the homogenization problem for a class of pure jump Lévy processes using a purely analytical approach—Mosco convergence. Recently, in [23], the authors of the present paper studied the

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periodic homogenization of stochastic differential equations (SDEs) with jump noise and corresponding non-local partial differential equations (PDEs). In the setting of the present paper, many of the homogenization problems in the literature mentioned above can be recovered. See Section 4 for comparisons.

The remainder of this paper is organized as follows. In the next section, we list our main assumptions and prove some preliminary results, such as the well-posedness of martingale problems, invariance and ergodicity, etc. Some technical results will be left to the appendices, with no effect on the smoothness of reading. In Section 3, we prove our main result to identify the homogenization limit. Some examples of resolving the homogenization problems in previous works are presented in Section 4. Section 5 is devoted to the stable-like case that is not covered by previous sections. Some numerical experiments for visualizing the homogenization result and approximating the first exit time are also provided in this section.

## 2. General assumptions and preliminary results

In the section, we collect general assumptions and some results we need. The most crucial results are Corollaries 1 and 2. The former allows us to obtain the functional convergence in the main theorem in the next section, while the latter gives the well-posedness of a Poisson equation that will be used to deal with the drift $\frac{1}{\epsilon^{\alpha-1}}b$. Most proofs in this section are quite short. We put other auxiliary but technical results into the appendix.

### 2.1. Assumptions

Firstly, we list our assumptions on the functions $b$, $c$, $\kappa$ (or $\kappa^*$), and $J$.

**Assumption 1.** *The functions b,c are in the Hölder class $C^\beta$ for some $\beta \in (0, 1)$, and they are periodic of period* 1.

**Assumption 2.** *The function $(x, z, u, v) \to \kappa^*(x, z, u, v)$ is periodic of period* 1 *in x and u. For the function $\kappa(x, z, u) := \kappa^*(x, z, u, u)$, there exist constants $\kappa_1, \kappa_2, \kappa_3 > 0$ such that for the same $\beta$ of* 1, *and all $x, x_1, x_2 \in \mathbb{R}^d$ and $z, u \in \mathbb{R}^d \setminus \{0\}$,*

$$\kappa_1 \le \kappa(x, z, u) \le \kappa_2, \tag{2.1}$$

$$|\kappa(x_1, z, u) - \kappa(x_2, z, u)| \le \kappa_3 |x_1 - x_2|^\beta. \tag{2.2}$$

*There exists a function $(x, z, u) \to \kappa_0(x, z, u)$, periodic of period 1 in x and u and also satisfying* (2.1) *and* (2.2)*, such that for all $x \in \mathbb{R}^d$ and $z \in \mathbb{R}^d \setminus \{0\}$,*

$$|\kappa^*(x, z, z/\epsilon, z/\epsilon) - \kappa_0(x, z, z/\epsilon)| \to 0, \quad \epsilon \to 0^+. \tag{2.3}$$

*We assume also that there exists a function $\tilde{\kappa} : \mathbb{R}^d \times \left(\mathbb{R}^d \setminus \{0\}\right) \to [0, \infty)$ such that for all $z \in \mathbb{R}^d \setminus \{0\}$,*

$$\sup_{x\in\mathbb{R}^d} |\kappa(x, \epsilon z, z) - \tilde{\kappa}(x, z)| \to 0, \quad \epsilon \to 0^+. \tag{2.4}$$

*In the case $\alpha \in (1, 2)$ and $b \neq 0$, we assume further that $\alpha + \beta \neq 2$ and for all $z \in \mathbb{R}^d \setminus \{0\}$,*

$$\frac{1}{\epsilon^{\alpha-1}} \sup_{x\in\mathbb{R}^d} |\kappa(x, \epsilon z, z) - \tilde{\kappa}(x, z)| \to 0, \quad \epsilon \to 0^+. \tag{2.5}$$

**Assumption 3.** *Assume that the function J is positive homogeneous of degree $-(d + \alpha)$ for some $\alpha \in (0, 2)$, i.e.*

$$J(rz) = r^{-(d+\alpha)} J(z), \quad r > 0, \ z \in \mathbb{R}^d \setminus \{0\}, \tag{2.6}$$

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*and that $J$ is bounded between two positive constants on the unit $(d-1)$-sphere $S$, i.e., there exist constants $j_1, j_2 > 0$ such that for all $\xi \in S$,*

$$j_1 \leq J(\xi) \leq j_2. \tag{2.7}$$

*In the case $\alpha = 1$, we assume additionally that for each $x \in \mathbb{R}^d$ and $r_1, r_2 \in (0, \infty)$,*

$$\int_S \xi \kappa(x, r_1\xi, r_2\xi) J(\xi) d\xi = 0. \tag{2.8}$$

We denote by $\mathcal{C}(\mathbb{T}^d)$ the space of continuous functions on the flat torus $\mathbb{T}^d := \mathbb{R}^d/\mathbb{Z}^d$ endowed with the supremum norm $\|f\|_\infty := \sup_{x\in\mathbb{T}^d} |f(x)|$. Denote by $\mathcal{C}^k(\mathbb{T}^d)$, with integer $k \geq 0$, the space of continuous functions on $\mathbb{T}^d$ possessing derivatives of orders not greater than $k$, endowed with the norm $\|f\|_{\mathcal{C}^k} := \|f\|_\infty + \sum_{|\alpha|\leq k} \sup_{x\in\mathbb{T}^d} |\nabla^\alpha f(x)|$. For a non-integer $\gamma > 0$, the Hölder space $\mathcal{C}^\gamma(\mathbb{T}^d)$ is defined as the subspace of $\mathcal{C}^{\lfloor\gamma\rfloor}$ consisting of functions whose $\lfloor\gamma\rfloor$th-order partial derivatives are locally Hölder continuous with exponent $\gamma - \lfloor\gamma\rfloor$. The space $\mathcal{C}^\gamma(\mathbb{T}^d)$ is a Banach space endowed with the norm

$$\|f\|_{\mathcal{C}^\gamma} := \|f\|_{\mathcal{C}^{\lfloor\gamma\rfloor}} + \sup_{x,y\in\mathbb{T}^d, x\neq y} \frac{|f(x) - f(y)|}{|x-y|^{\gamma-\lfloor\gamma\rfloor}}.$$

The space of infinitely differentiable functions on $\mathbb{T}^d$ is denoted by $\mathcal{C}^\infty(\mathbb{T}^d)$.

**Remark 1.**

(i) We shall always identify a periodic function on $\mathbb{R}^d$ of period 1 with its restriction to the compact space $\mathbb{T}^d$. Then Assumption 1 amounts to saying that $b, c \in \mathcal{C}^\beta(\mathbb{T}^d)$.

(ii) The Hölder exponent $\beta$ in Assumption 1 does not need to be the same as the one in (2.2). But in view of the embedding of Hölder spaces on compact spaces, we can assume them to be the same, without losing any generality. The assumption $\alpha + \beta \neq 2$ is due to [2], whose results will be used in Corollary 6.

(iii) The relation (2.4) or (2.5) ensures that the function $\tilde{\kappa}(x, z)$ is periodic in $x$ and also satisfies (2.1) and (2.2) with same constants $\beta, \kappa_1, \kappa_2, \kappa_3$. We also remark that only the assumptions (2.3)–(2.5) really contribute to the identification of the homogenization limit (see Lemma 2 and the main result Theorem 1), while all other assumptions are needed for constructing Feller processes and estimating heat kernels (see Proposition 1).

(iv) A typical example in which the assumptions (2.4) and (2.5) hold is the case where $\kappa(x, z, u)$ can be written as the quotient of two positive homogeneous functions in $z$. In the case where $\alpha \in (1, 2)$ and $b \neq 0$, the convergence (2.5) implies (2.4). In this case, there is a singularity in the drift coefficient $\frac{1}{\epsilon^{\alpha-1}} b$, and we need more regularities for $\kappa$ to cancel that singularity, as we will see in the proof of Theorem 1.

(v) The positive homogeneity assumption on $J$ is equivalent to saying that $J$ is the density of an $\alpha$-stable Lévy measure (cf. [35, Theorem 14.3]). By (2.6),

$$J(z) = J\left(|z| \cdot \frac{z}{|z|}\right) = |z|^{-(d+\alpha)} J\left(\frac{z}{|z|}\right).$$

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Then the assumption (2.7) implies

$$j_1|z|^{-(d+\alpha)} \le J(z) \le j_2|z|^{-(d+\alpha)}, \quad z \in \mathbb{R}^d \setminus \{0\}; \tag{2.9}$$

that is, $J$ is comparable with the density of the rotation-invariant $\alpha$-stable Lévy measure.

(vi) It is easy to verify that the assumptions (2.1) and (2.2) for $\kappa$ (and hence the same for $\tilde{\kappa}$ as we have seen in the third remark), together with (2.6)–(2.8) for $J$, ensure all assumptions in [20] for $\alpha \in (0, 1) \cup (1, 2)$ and in [37] for $\alpha = 1$. We will use the results therein in the sequel.

### 2.2. Feller processes

We need some auxiliary operators and processes. In fact, we will rescale the operator $\mathcal{A}^\epsilon$ and its canonical process in an effective fashion. For this purpose, we define the following non-local operators for $f \in \mathcal{C}^\infty(\mathbb{T}^d)$, the space of all smooth functions on the flat torus $\mathbb{T}^d$ (i.e., smooth periodic functions of period 1):

$$\begin{aligned}\tilde{\mathcal{A}}^\epsilon f(x) = \int_{\mathbb{R}^d\setminus\{0\}} &\left[f(x+z) - f(x) - z\cdot\nabla f(x)\left(\mathbf{1}_{\{1\}}(\alpha)\mathbf{1}_B(z) + \mathbf{1}_{(1,2)}(\alpha)\mathbf{1}_B(\epsilon z)\right)\right] \\ &\times \kappa(x, \epsilon z, z)J(z)dz + \left(b(x) + \epsilon^{\alpha-1}c(x)\right)\cdot\nabla f(x)\mathbf{1}_{(1,2)}(\alpha), \quad \epsilon > 0,\end{aligned} \tag{2.10}$$

$$\begin{aligned}\tilde{\mathcal{A}} f(x) = \int_{\mathbb{R}^d\setminus\{0\}} &\left[f(x+z) - f(x) - z\cdot\nabla f(x)\left(\mathbf{1}_{\{1\}}(\alpha)\mathbf{1}_B(z) + \mathbf{1}_{(1,2)}(\alpha)\right)\right] \\ &\times \tilde{\kappa}(x, z)J(z)dz + b(x)\cdot\nabla f(x)\mathbf{1}_{(1,2)}(\alpha).\end{aligned} \tag{2.11}$$

For notational simplicity, we shall allow the parameter $\epsilon$ to be zero so that $\tilde{\mathcal{A}}^0 := \tilde{\mathcal{A}}$. The periodicity and continuity of the function $x \to \kappa(x, z, u)$ and (2.1), (2.8), and (2.9) imply that $\tilde{\mathcal{A}}^\epsilon$, $\epsilon \ge 0$, are all well-defined unbounded operators on $(\mathcal{C}(\mathbb{T}^d), \|\cdot\|_\infty)$, whose domains contain $\mathcal{C}^\infty(\mathbb{T}^d)$. Moreover, it is easy to verify by (2.6) and (2.8) that for each $\epsilon > 0$, the operator $\tilde{\mathcal{A}}^\epsilon$ is a rescaling of $\mathcal{A}^\epsilon$ in the sense that

$$\tilde{\mathcal{A}}^\epsilon f(x) = \epsilon^\alpha(\mathcal{A}^\epsilon f_\epsilon)(\epsilon x), \quad f \in \mathcal{C}^\infty(\mathbb{T}^d). \tag{2.12}$$

Here and after, we write $f_\epsilon(x) := f(x/\epsilon)$.

Denote by $\mathcal{D} = \mathcal{D}(\mathbb{R}_+; \mathbb{R}^d)$ (resp. $\mathcal{D}_{\text{per}} = \mathcal{D}(\mathbb{R}_+; \mathbb{T}^d)$) the space of all $\mathbb{R}^d$-valued (resp. $\mathbb{T}^d$-valued) càdlàg functions on $\mathbb{R}_+ := [0, \infty)$, equipped with the Skorokhod topology. The following proposition tells us that the operators $\mathcal{A}^\epsilon$, $\tilde{\mathcal{A}}^\epsilon$, and $\tilde{\mathcal{A}}$ can generate Feller processes, and the state space of the Feller processes associated to $\tilde{\mathcal{A}}^\epsilon$ or $\tilde{\mathcal{A}}$ can be taken as $\mathbb{T}^d$, which is a compact space. Meanwhile, the heat kernel estimates for $\tilde{\mathcal{A}}^\epsilon$ and $\tilde{\mathcal{A}}$ are crucial in proving the ergodicity of the associated processes. We also find the core for $\tilde{\mathcal{A}}^\epsilon$ and $\tilde{\mathcal{A}}$, which we will use to show the convergence of the associated invariant measures in the sequel.

**Proposition 1.** *Suppose that Assumption 1, Assumption 3, (2.1), and (2.2) hold with constants $\alpha \in (0, 2)$ and $\beta \in (0, 1)$.*

*(i) For every $\epsilon > 0$ and $x \in \mathbb{R}^d$, the martingale problem for $(\mathcal{A}^\epsilon, \delta_x)$ has a unique solution $\mathbb{P}^\epsilon_x$ on $(\mathcal{D}, \mathcal{B}(\mathcal{D}))$. The coordinate process $X^\epsilon$ is an $\mathbb{R}^d$-valued Feller process starting from $x$.*

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*(ii) For every $\epsilon \in [0, 1]$ and $x \in \mathbb{T}^d$, the martingale problem for $(\tilde{\mathcal{A}}^\epsilon, \delta_x)$ has a unique solution $\tilde{\mathbb{P}}^\epsilon_x$ on $(\mathcal{D}_{\text{per}}, \mathcal{B}(\mathcal{D}_{\text{per}}))$. The coordinate process $\tilde{X}^\epsilon$ is a $\mathbb{T}^d$-valued Feller process starting from $x$ with generator the closure of $\big(\tilde{\mathcal{A}}^\epsilon, C^\infty\big(\mathbb{T}^d\big)\big)$, and has a transition probability density $\tilde{p}^\epsilon(t; x, y)$, i.e., $\tilde{\mathbb{P}}^\epsilon_x(\tilde{X}^\epsilon_t \in A) = \int_A \tilde{p}^\epsilon(t; x, y)dy$, $A \in \mathcal{B}\big(\mathbb{T}^d\big)$. Moreover, the following hold:*

*(ii.1) The transition probability density $\tilde{p}^\epsilon(t; x, y)$ is jointly continuous on $(0, \infty) \times \mathbb{T}^d \times \mathbb{T}^d$.*

*(ii.2) For every $T > 0$, there exists a constant $0 < C < 1$, independent of $\epsilon \in [0, 1]$, such that for all $t \in (0, T]$ and $x, y \in \mathbb{T}^d$,*

$$\tilde{p}^\epsilon(t; x, y) \geq C \sum_{l \in \mathbb{Z}^d} \left[ t^{-d/\alpha} \wedge \left( t|x - y + l|^{-(d+\alpha)} \right) \right]. \tag{2.13}$$

*Proof.* All assertions for the case $\alpha \in (0, 1)$ follow from [20, Theorem 1.1, Theorem 1.3, Theorem 1.4, Remark 1.5]; the assertions for the case $\alpha = 1$ can be found in [37, Theorem 2.1, Theorem 2.3, Theorem 2.4]. In particular, for these two cases, the constant $C$ in the estimate (2.13) depends only on $(d, \alpha, \beta, \kappa_1, \kappa_2, \kappa_3, j_1, j_2)$ and hence is independent of $\epsilon \geq 0$, since $\kappa(x, \epsilon z, z)$ and $\tilde{\kappa}(x, z)$, the only quantities in $\{\tilde{\mathcal{A}}^\epsilon : \epsilon \geq 0\}$ that depend on $\epsilon$ when $\alpha \in (0, 1]$, satisfy (2.1) and (2.2) with uniform constants $\kappa_1, \kappa_2$. For the case $\alpha \in (1, 2)$, the properties of $\tilde{p}^\epsilon$ can be found in [11, Theorem 1.5]; for the reader's convenience, we have also included the proof in the appendix (see Proposition 5). In particular, by Proposition 5(iii), the constant $C$ of (2.13) depends on $(d, \alpha, \beta, \kappa_1, \kappa_2, \kappa_3, j_1, j_2)$ and the upper bound of the drift of each $\tilde{\mathcal{A}}^\epsilon$, $\epsilon \in [0, 1]$. The drift of $\tilde{\mathcal{A}}^0 = \tilde{\mathcal{A}}$ is $b$, while that of $\tilde{\mathcal{A}}^\epsilon$ with $\epsilon \in (0, 1]$ is

$$b(x) + \epsilon^{\alpha-1} c(x) - \int_{1 \leq |z| < \frac{1}{\epsilon}} z\kappa(x, \epsilon z, z) J(z) dz,$$

whose absolute value is bounded by $\|b\|_\infty + \|c\|_\infty + \frac{\kappa_2}{\alpha - 1}$ uniformly for $\epsilon \in (0, 1]$. Thus, $C$ in (2.13) can be chosen as independent of $\epsilon \in [0, 1]$. The proofs of the remaining parts are tedious, especially the proof that $C^\infty\big(\mathbb{T}^d\big)$ is the core of the generators, and we leave them to the appendix; see Proposition 6 and Corollary 5. $\square$

Of course, each of the processes $X^\epsilon$, $\tilde{X}$, and $\tilde{X}^\epsilon$ is defined on its own stochastic basis. However, by taking the product of the probability spaces, it is always possible to assume that

$X^\epsilon$, $\tilde{X}$ and $\tilde{X}^\epsilon$, $\epsilon > 0$, are all defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

We also assume for simplicity that

$$X^\epsilon_0 = \tilde{X}_0 = \tilde{X}^\epsilon_0 = 0.$$

If we identify a periodic function on $\mathbb{R}^d$ of period $\epsilon$ with its restriction to the $\epsilon$-torus $\mathbb{T}^d_\epsilon := \epsilon\mathbb{T}^d$, then each $\mathcal{A}^\epsilon$ maps $C^\infty\big(\mathbb{T}^d_\epsilon\big)$ into $C^\infty\big(\mathbb{T}^d_\epsilon\big)$ by virtue of the periodicity of $\kappa$ in $x$. In view of this, the canonical process $X^\epsilon$ can also be treated as a process taking values on $\mathbb{T}^d_\epsilon$, via the quotient map $\mathbb{R}^d \to \mathbb{T}^d_\epsilon$. We will use this treatment *only* in the rest of this section; the benefit is the relation below, which follows from the well-known fact that Feller semigroups and Feller

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processes are in one-to-one correspondence if we identify the processes that have the same finite-dimensional distributions (see, e.g., [8]).

**Lemma 1.** *We have the following identity in law:*

$$\{\tilde{X}_t^\epsilon\}_{t\geq 0} \overset{\mathrm{d}}{=} \left\{\tfrac{1}{\epsilon}X^\epsilon_{\epsilon^\alpha t}\right\}_{t\geq 0}, \quad \text{for each } \epsilon > 0.$$

*Proof.* We derive the generator for the Feller process $\left\{\frac{1}{\epsilon}X^\epsilon_{\epsilon^\alpha t}\right\}_{t\geq 0}$. For $f \in \mathcal{C}^\infty(\mathbb{T}^d)$, by (2.12),

$$\lim_{t\downarrow 0} \frac{\mathbb{E}^\epsilon_{\epsilon x}\left[f\left(\frac{1}{\epsilon}X^\epsilon_{\epsilon^\alpha t}\right)\right] - f(x)}{t} = \epsilon^\alpha \lim_{t\downarrow 0} \frac{\mathbb{E}^\epsilon_{\epsilon x}\left[f_\epsilon\left(X^\epsilon_{\epsilon^\alpha t}\right)\right] - f(x)}{\epsilon^\alpha t} = \epsilon^\alpha (\mathcal{A}^\epsilon f_\epsilon)(\epsilon x) = \tilde{\mathcal{A}}^\epsilon f(x).$$

Therefore, the Feller semigroup associated to $\left\{\frac{1}{\epsilon}X^\epsilon_{\epsilon^\alpha t}\right\}_{t\geq 0}$ is also generated by the closure of $\left(\tilde{\mathcal{A}}^\epsilon, \mathcal{C}^\infty(\mathbb{T}^d)\right)$. □

Denote by $\{\tilde{P}^\epsilon_t\}_{t\geq 0}$ (or $\{\tilde{P}_t\}_{t\geq 0}$) the Feller semigroup of $\tilde{X}^\epsilon$ (or $\tilde{X}$). Let $\tilde{X}^0 = \tilde{X}$ and $\tilde{P}^0_t = \tilde{P}_t$. Now, utilizing the lower bound of the transition probability density $\tilde{p}^\epsilon(t; x, y)$, we obtain the following exponential ergodicity.

**Proposition 2.** *For each $\epsilon \in [0, 1]$, the Feller process $\tilde{X}^\epsilon$ possesses a unique invariant probability distribution $\mu_\epsilon$ on $\mathbb{T}^d$. Moreover, there exist positive constants $C$ and $\rho$ which are independent of $\epsilon \in [0, 1]$, such that for each periodic bounded Borel function $f$ on $\mathbb{R}^d$ (i.e., $f$ is a bounded Borel function on $\mathbb{T}^d$),*

$$\sup_{x\in\mathbb{T}^d} \left|\tilde{P}^\epsilon_t f(x) - \int_{\mathbb{T}^d} f(y)\mu_\epsilon(dy)\right| \leq C\|f\|_\infty e^{-\rho t}$$

*for every $t \geq 0$.*

*Proof.* The proof is similar to that of [3, Theorem 3.3.2]; see also [23, Lemma 4.6]. The only problem is to show that the two constants $C$ and $\rho$ can be chosen to be independent of $\epsilon \in [0, 1]$. Thanks to the Doeblin-type result in [3, Theorem 3.3.1], it suffices to show that the map $\mathbb{T}^d \times \mathbb{T}^d \ni (x, y) \mapsto \tilde{p}^\epsilon(1; x, y)$ is bounded from below by a positive constant independent of $x$, $y$ and $\epsilon$. This follows immediately from the transition density estimate in (2.13) together with the compactness of the state space $\mathbb{T}^d$ and the joint continuity of $\tilde{p}^\epsilon$. □

Denote by $\mu = \mu_0$ the invariant probability measure of $\tilde{X}$.

**Lemma 2.** *As $\epsilon \to 0^+$, we have the weak convergence $\mu_\epsilon \Rightarrow \mu$.*

*Proof.* By the argument in [21, Lemma 2.4], we only need to show that $\tilde{P}^\epsilon_t f \to \tilde{P}_t f$ in $\mathcal{C}(\mathbb{T}^d)$ as $\epsilon \to 0^+$ for each $f \in \mathcal{C}(\mathbb{T}^d)$ and $t \geq 0$.

By Proposition 1, we know that $\mathcal{C}^\infty(\mathbb{T}^d)$ is a core for each $\tilde{\mathcal{A}}^\epsilon$, $\epsilon \geq 0$. Now fix an arbitrary $f \in \mathcal{C}^\infty(\mathbb{T}^d)$. If $\alpha \in (0, 1]$, then $\|\tilde{\mathcal{A}}^\epsilon f - \tilde{\mathcal{A}} f\|_\infty$ as $\epsilon \to 0^+$ by dominated convergence and (2.4). For the case $\alpha \in (1, 2)$, we use the fact that

$$|f(x+z) - f(x) - z\cdot\nabla f(x)| \leq \frac{1}{2}\|f\|_{\mathcal{C}^2}|z|^2\mathbf{1}_{\{|z|\leq 1\}} + 2\|f\|_{\mathcal{C}^1}|z|\mathbf{1}_{\{|z|>1\}} \leq 2\|f\|_{\mathcal{C}^2}\left(|z|^2 \wedge |z|\right),$$

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which follows from Taylor expansion, to derive

$$
\begin{aligned}
&\|\tilde{\mathcal{A}}^{\epsilon} f - \tilde{\mathcal{A}} f\|_{\infty} \\
&\le \|\epsilon^{\alpha-1} c \cdot \nabla f\|_{\infty} + \sup_{x\in\mathbb{T}^d} \left| \int_{\mathbb{R}^d\setminus\{0\}} z\cdot \nabla f(x)\,(1-\mathbf{1}_B(\epsilon z))\,\kappa(x,\epsilon z, z) J(z) dz \right| \\
&\quad + \sup_{x\in\mathbb{T}^d} \left| \int_{\mathbb{R}^d\setminus\{0\}} \big[f(x+z) - f(x) - z\cdot\nabla f(x)\big]\,(\kappa(x,\epsilon z,z) - \tilde{\kappa}(x,z)) J(z) dz \right| \\
&\le \epsilon^{\alpha-1}\|c\|_{\infty}\|f\|_{\mathcal{C}^1} + \kappa_2 j_2 \|f\|_{\mathcal{C}^1} \int_{|z|\ge 1/\epsilon} \frac{dz}{|z|^{d+\alpha-1}} \\
&\quad + 2 j_2 \|f\|_{\mathcal{C}^2} \int_{\mathbb{R}^d\setminus\{0\}} \left( \sup_{x\in\mathbb{T}^d} |\kappa(x,\epsilon z,z) - \tilde{\kappa}(x,z)| \right) (|z|^2\wedge |z|) \frac{dz}{|z|^{d+\alpha}},
\end{aligned}
\tag{2.14}
$$

which converges to zero as $\epsilon \to 0^+$, by (2.4) and dominated convergence. Now by the Trotter–Kato approximation theorem (see [15, Theorem III.4.8]), $\tilde{P}^{\epsilon}_t f \to \tilde{P}_t f$ in $\mathcal{C}(\mathbb{T}^d)$ as $\epsilon \to 0^+$ for all $f \in \mathcal{C}(\mathbb{T}^d)$, uniformly for $t$ in compact intervals. □

Now, using Proposition 2 and Lemma 2, we can obtain a useful functional convergence theorem.

**Corollary 1.** *Let $f$ be a bounded Borel function on $\mathbb{T}^d$. Then for every $t > 0$,*

$$
\mathbb{E}\left[ \left| \int_0^t f\left(\frac{X_s^{\epsilon}}{\epsilon}\right) ds - t\int_{\mathbb{T}^d} f(y)\mu(dy) \right|^2 \right] \to 0, \quad \textit{as } \epsilon \to 0^+.
$$

*For every $T > 0$,*

$$
\sup_{t\in[0,T]} \left| \int_0^t f\left(\frac{X_s^{\epsilon}}{\epsilon}\right) ds - t\int_{\mathbb{T}^d} f(y)\mu(dy) \right| \to 0, \quad \textit{in probability } \mathbb{P}, \ \textit{as } \epsilon \to 0^+. \tag{2.15}
$$

*Proof.* We follow the lines of [32, Proposition 2.4]. Fix $\epsilon > 0$. Let $\bar{f}_\epsilon := f - \int_{\mathbb{T}^d} f(y)\mu_\epsilon(dy)$. By virtue of Lemma 1 and Lemma 2, to prove the two limits, it suffices to prove that

$$
\epsilon^{\alpha} \int_0^{\epsilon^{-\alpha} t} \bar{f}_\epsilon(\tilde{X}_s^{\epsilon}) ds = \int_0^t \bar{f}_\epsilon(X_s^{\epsilon}/\epsilon) ds \to 0, \tag{2.16}
$$

in $L^2(\Omega, \mathbb{P})$ and also in probability uniformly in $t \in [0, T]$. By Proposition 2, for $0 \le s < t$ we have

$$
\mathbb{E}\Big(\bar{f}_\epsilon(\tilde{X}_t^{\epsilon}) \Big| \tilde{X}_s^{\epsilon}\Big) = \int_{\mathbb{T}^d} \bar{f}_\epsilon(y)\left[\tilde{p}^{\epsilon}\big(t-s, \tilde{X}_s^{\epsilon}, y\big)dy - \mu_\epsilon(dy)\right] \le C\|\bar{f}_\epsilon\|_{\infty} e^{-\rho(t-s)}, \tag{2.17}
$$

and then by the Markov property,

$$
\begin{aligned}
\mathbb{E}(\bar{f}_\epsilon(\tilde{X}_s^{\epsilon})\bar{f}_\epsilon(\tilde{X}_t^{\epsilon})) &= \mathbb{E}\Big[\bar{f}_\epsilon(\tilde{X}_s^{\epsilon})\mathbb{E}\Big(\bar{f}_\epsilon(\tilde{X}_t^{\epsilon})\Big|\tilde{X}_s^{\epsilon}\Big)\Big] \\
&\le C\|\bar{f}_\epsilon\|_{\infty}^2 e^{-\rho(t-s)} \le 4C\|f\|_{\infty}^2 e^{-\rho(t-s)}.
\end{aligned}
\tag{2.18}
$$

Hence, if we write $g_\epsilon(s) := \bar{f}_\epsilon(X^\epsilon_s/\epsilon)$, then as $\epsilon \to 0^+$,

$$\begin{aligned}
\mathbb{E}\left[\left|\int_0^t g_\epsilon(s)ds\right|^2\right] &= 2\epsilon^{2\alpha}\int_0^{\epsilon^{-\alpha}t}\int_0^r \mathbb{E}\big(\bar{f}_\epsilon\big(\tilde{X}^\epsilon_s\big)\bar{f}_\epsilon\big(\tilde{X}^\epsilon_r\big)\big)\,dsdr \\
&\le 8C\epsilon^{2\alpha}\|f\|^2_\infty \int_0^{\epsilon^{-\alpha}t}\int_0^r e^{-\rho(r-s)}dsdr \qquad (2.19)\\
&= 8C\epsilon^{2\alpha}\|f\|^2_\infty \rho^{-2}\left[-1+\rho\epsilon^{-\alpha}t + e^{-\rho\epsilon^{-\alpha}t}\right] \\
&\to 0.
\end{aligned}$$

The first result follows. On the other hand, for any $n \in \mathbb{N}_+$, since $(\lfloor \frac{nt}{T}\rfloor + 1)\frac{T}{n} \ge t$,

$$\begin{aligned}
&\mathbb{E}\left[\sup_{t\in[0,T]}\left|\int_0^t g_\epsilon(s)ds\right|^2\right] \\
&= \mathbb{E}\left[\sup_{k=0,\cdots,n}\left|\int_0^{k\frac{T}{n}} g_\epsilon(s)ds\right|^2 + \sup_{t\in[0,T]}\left(\left|\int_0^{\lfloor\frac{nt}{T}\rfloor\frac{T}{n}} g_\epsilon(s)ds\right| + \left|\int_{\lfloor\frac{nt}{T}\rfloor\frac{T}{n}}^t g_\epsilon(s)ds\right|\right)^2\right] \\
&\le 3\mathbb{E}\left[\sup_{k=0,\cdots,n}\left|\int_0^{k\frac{T}{n}} g_\epsilon(s)ds\right|^2\right] + 2\mathbb{E}\left[\sup_{t\in[0,T]}\left|\int_{\lfloor\frac{nt}{T}\rfloor\frac{T}{n}}^t g_\epsilon(s)ds\right|^2\right] \\
&\le 3\sup_{k=0,\cdots,n}\mathbb{E}\left[\left|\int_0^{k\frac{T}{n}} g_\epsilon(s)ds\right|^2\right] + 8\|f\|^2_\infty\frac{T^2}{n^2},
\end{aligned}$$

which goes to zero, by first letting $\epsilon \to 0^+$ and applying (2.19), and then letting $n \to \infty$. The second result follows by an application of Chebyshev's inequality. □

**Remark 2.**

(i) From (2.16), we have indeed proved the following ergodicity result: for every $\epsilon \in (0, 1]$ and bounded Borel function $f$ on $\mathbb{T}^d$,

$$\frac{1}{T}\int_0^T f\big(\tilde{X}^\epsilon_s\big)ds \to \int_{\mathbb{T}^d} f(y)\mu_\epsilon(dy), \quad \text{as } T\to\infty, \ \text{in } L^2(\Omega,\mathbb{P}).$$

This result also holds for $\epsilon = 0$, since Proposition 2 and thereby (2.17) and (2.18) are all valid for $\epsilon = 0$; similarly to (2.19), defining $\bar{f} := f - \int_{\mathbb{T}^d} f(y)\mu(dy)$, we have that

$$\mathbb{E}\left[\left|\frac{1}{T}\int_0^T \bar{f}\big(\bar{X}_s\big)ds\right|^2\right] = \frac{2}{T^2}\int_0^T\int_0^r \mathbb{E}\big(\bar{f}_\epsilon\big(\tilde{X}^\epsilon_s\big)\bar{f}_\epsilon\big(\tilde{X}^\epsilon_r\big)\big)\,dsdr \to 0, \quad \text{as } T\to\infty.$$

(ii) In the sequel, we shall use the following variant of (2.15). Let $f : \mathbb{T}^d \times \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$ be a bounded Borel function; then for every $T > 0$, as $\epsilon \to 0^+$,

$$\sup_{t\in[0,T]}\left|\int_0^t f\left(\frac{X^\epsilon_s}{\epsilon},\epsilon,z\right)ds - t\int_{\mathbb{T}^d} f(y,\epsilon,z)\mu(dy)\right| \to 0, \quad \text{in probability } \mathbb{P}. \qquad (2.20)$$

Clearly, this holds for the case where $f$ is separable as $f(y,\epsilon,z) = f_0(y)g(\epsilon,z)$. The general case follows by first making monotone approximations for the positive and negative

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parts of $f$ using simple functions (linear combinations of indicator functions), and then applying the monotone convergence theorem.

(iii) The proof of a similar result in another paper by the authors of the present paper, [23, Proposition 4.8], is partially incorrect, although the error there does not affect the main results of that paper. The correct proof needs to be carried out in the same way as here.

### 2.3. Non-local Poisson equation

Using the exponential ergodicity, we can also obtain the well-posedness of the non-local Poisson equation. Denote by $\mathcal{C}^\gamma_\mu(\mathbb{T}^d)$, $\gamma > 0$, the class of all $f \in \mathcal{C}^\gamma(\mathbb{T}^d)$ which are *centered* with respect to the invariant measure $\mu$, in the sense that $\int_{\mathbb{T}^d} f(x)\mu(dx) = 0$. It is easy to check that $\mathcal{C}_\mu(\mathbb{T}^d)$ is a closed subset, and hence a sub-Banach space, of $\mathcal{C}(\mathbb{T}^d)$ under the norm $\|\cdot\|_\infty$.

**Lemma 3.** *The restrictions* $\{\tilde{P}^\mu_t := \tilde{P}_t|_{\mathcal{C}_\mu(\mathbb{T}^d)}\}_{t\geq 0}$ *form a* $C_0$*-semigroup on the Banach space* $(\mathcal{C}_\mu(\mathbb{T}^d), \|\cdot\|_\infty)$*, with generator* $(\tilde{\mathcal{A}}_\mu, D(\tilde{\mathcal{A}}_\mu)) := \overline{(\tilde{\mathcal{A}}, \mathcal{C}^\infty_\mu(\mathbb{T}^d))}$*. Moreover, the set* $\{z \in \mathbb{C} \mid \mathrm{Re} z > -\rho\}$ *is contained in the resolvent set of* $\tilde{\mathcal{A}}_\mu$*.*

*Proof.* Since $\mu$ is invariant with respect to $\{\tilde{P}_t\}_{t\geq 0}$, it is easy to see that $\mathcal{C}_\mu(\mathbb{T}^d)$ is $\{\tilde{P}_t\}_{t\geq 0}$-invariant, in the sense that $\tilde{P}_t(\mathcal{C}_\mu(\mathbb{T}^d)) \subset \mathcal{C}_\mu(\mathbb{T}^d)$ for all $t \geq 0$. The first part of the lemma then follows from the corollary in [15, Subsection II.2.3]. By the exponential ergodicity result in Proposition 2, we have

$$\|\tilde{P}^\mu_t f\|_\infty \leq C\|f\|_\infty e^{-\rho t} \tag{2.21}$$

for all $f \in \mathcal{C}_\mu(\mathbb{T}^d)$ and $t \geq 0$. This yields the second part of the lemma, using [15, Theorem II.1.10(ii)]. □

**Corollary 2.** *Let* $\alpha \in (1, 2)$*. For every* $f \in \mathcal{C}^\beta_\mu(\mathbb{T}^d)$*, there exists a unique solution in* $\mathcal{C}^{\alpha+\beta}_\mu(\mathbb{T}^d)$ *to the Poisson equation*

$$\tilde{\mathcal{A}}u + f = 0. \tag{2.22}$$

*Proof.* If $u \in \mathcal{C}^{\alpha+\beta}_\mu(\mathbb{T}^d)$ is a solution, then by (2.21),

$$\int_0^\infty \tilde{P}^\mu_t f dt = -\int_0^\infty \tilde{P}^\mu_t \tilde{\mathcal{A}}_\mu u\, dt = -\int_0^\infty \frac{d}{dt}\tilde{P}^\mu_t u\, dt = u - \lim_{t\to\infty} \tilde{P}^\mu_t u = u.$$

This yields the uniqueness. Thanks to Corollary 6, the existence follows from a standard Fredholm alternative argument ([19, Section 5.3]). □

In accordance with the terminology of periodic homogenization, we will refer to equation (2.22) as the *cell problem*.

## 3. Homogenization result

In this section we will prove our homogenization result. Before that, some preparations are needed.

Firstly, we need a convergence lemma for locally periodic functions.

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**Lemma 4.** *Let $\phi : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$, $(x, y) \mapsto \phi(x, y)$ be a function periodic in $y$ with period 1.*

*(i) Let $1 < p < \infty$. Suppose that for each $x \in \mathbb{R}^d$, $\phi(x, \cdot) \in L^p\big([0, 1]^d\big)$, and for each $y \in \mathbb{R}^d$, $\phi(\cdot, y) \in L^{p'}_{loc}(\mathbb{R}^d)$, where $p'$ is the conjugate of $p$, i.e., $\frac{1}{p} + \frac{1}{p'} = 1$. Then for every compact set $K \subset \mathbb{R}^d$, we have*

$$\lim_{\epsilon \to 0^+} \int_K \phi\Big(x, \frac{x}{\epsilon}\Big)\, dx = \int_K \int_{\mathbb{T}^d} \phi(x, y) dy dx.$$

*(ii) Suppose that for each $x \in \mathbb{R}^d$, $\phi(x, \cdot) \in L^\infty\big([0, 1]^d\big)$, and for each $y \in \mathbb{R}^d$, $\phi(\cdot, y) \in L^1(\mathbb{R}^d)$. Then we have*

$$\lim_{\epsilon \to 0^+} \int_{\mathbb{R}^d} \phi\Big(x, \frac{x}{\epsilon}\Big)\, dx = \int_{\mathbb{R}^d} \int_{\mathbb{T}^d} \phi(x, y) dy dx.$$

In the case where the function $\phi$ is separable—that is, $\phi$ is of the form $\phi(x, y) = f(x)g(y)$ with $g$ periodic—the conclusions of the above lemma can be found in [12, Theorem 2.6]. The general case can be obtained via standard monotone approximations of the positive and negative parts of $\phi$ by simple functions and the monotone convergence theorem.

Now we are in a position to prove the homogenization result. To get rid of the singularity in the coefficient $\frac{1}{\epsilon^{\alpha-1}} b$ in the case $\alpha \in (1, 2)$, we need one more assumption on $b$.

**Assumption 4.** *The function $b$ satisfies the centering condition,*

$$\int_{\mathbb{T}^d} b(x)\mu(dx) = 0.$$

By virtue of Assumptions 1 and 4 and Corollary 2, when $\alpha \in (1, 2)$ there exists a function $\hat{b} \in \mathcal{C}^{\alpha+\beta}_\mu\big(\mathbb{T}^d\big)$ that uniquely solves the Poisson equation

$$\tilde{\mathcal{A}}\hat{b} + b = 0. \tag{3.1}$$

**Theorem 1.** *Suppose that Assumptions 1–4 hold. In the sense of weak convergence on the space $\mathcal{D}(\mathbb{R}_+; \mathbb{R}^d)$, we have*

$$X^\epsilon \Rightarrow \bar{X}, \quad \text{as } \epsilon \to 0^+.$$

*The limit process $\bar{X}$ is a Lévy process starting from 0 with Lévy triplet $(\bar{b}, 0, \bar{\nu})$ given by*

$$\begin{cases} \bar{b} = \mathbf{1}_{(0,1)}(\alpha) \displaystyle\int_{B\setminus\{0\}} \bar{\kappa}(z) z J(z) dz + \mathbf{1}_{(1,2)}(\alpha)\bar{c}, \\ \bar{\nu}(dz) = \bar{\kappa}(z) J(z) dz, \end{cases} \tag{3.2}$$

*with homogenized coefficients*

$$\bar{\kappa}(z) := \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \kappa_0(x, z, u) du \mu(dx),$$

$$\bar{c} := \int_{\mathbb{T}^d} \Big(I + \nabla\hat{b}(x)\Big) \cdot c(x)\mu(dx) + \int_{B^c} z \cdot \left(\int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \nabla\hat{b}(x)\kappa_0(x, z, u) du \mu(dx)\right) J(z) dz,$$

*where $\mu$ is the invariant probability measure of $\tilde{X}$ with generator (2.11), and $\hat{b}$ is uniquely determined by (3.1).*

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*Proof.* (i) We first prove the theorem for the case where $b \equiv 0$ or $\alpha \in (0, 1]$. By [8, Theorem 2.44], we know that the semimartingale characteristics of $X^\epsilon$ relative to the truncation function $\mathbf{1}_B$ are $(B^\epsilon, 0, \nu^\epsilon)$, where

$$\begin{cases} B_t^\epsilon = \mathbf{1}_{(0,1)}(\alpha) \displaystyle\int_0^t \int_{B\backslash\{0\}} z\kappa^* \left(\frac{X_s^\epsilon}{\epsilon}, z, \frac{z}{\epsilon}, \frac{z}{\epsilon}\right) J(z)dzds, +\mathbf{1}_{(1,2)}(\alpha) \int_0^t c\left(\frac{X_s^\epsilon}{\epsilon}\right) ds, \\ \nu^\epsilon(dz, dt) = \kappa^* \left(\dfrac{X_t^\epsilon}{\epsilon}, z, \dfrac{z}{\epsilon}, \dfrac{z}{\epsilon}\right) J(z)dzdt. \end{cases}$$

By applying the functional central limit theorem in [25, Theorem VIII.2.17], we only need to show that for all $t \in \mathbb{R}_+$ and every bounded continuous function $f : \mathbb{R}^d \to \mathbb{R}$ vanishing in a neighborhood of the origin, the following convergences hold in probability when $\epsilon \to 0^+$:

$$\sup_{0 \le s \le t} |B_s^\epsilon - \bar{b}s| \to 0, \tag{3.3}$$

$$\int_0^t \int_{\mathbb{R}^d\backslash\{0\}} f(z)\nu^\epsilon(dz, ds) \to t \int_{\mathbb{R}^d\backslash\{0\}} f(z)\bar{\nu}(dz). \tag{3.4}$$

Clearly, by Corollary 1 we have

$$\int_0^t c\left(\frac{X_s^\epsilon}{\epsilon}\right) ds \to t \int_{\mathbb{T}^d} c(x)\mu(dx), \quad \text{in probability}, \ \text{as } \epsilon \to 0^+. \tag{3.5}$$

When $\alpha \in (0, 1)$, we also have the following convergence in probability, uniformly with respect to $t$ in closed intervals:

$$\begin{aligned} &\int_0^t \int_{B\backslash\{0\}} z\kappa^* \left(\frac{X_s^\epsilon}{\epsilon}, z, \frac{z}{\epsilon}, \frac{z}{\epsilon}\right) J(z)dzds \\ &\overset{\epsilon \ll 1}{\sim} t \int_{B\backslash\{0\}} \left[\int_{\mathbb{T}^d} \kappa^* \left(x, z, \frac{z}{\epsilon}, \frac{z}{\epsilon}\right) \mu(dx)\right] zJ(z)dz \qquad \text{(by (2.20))} \\ &\overset{\epsilon \ll 1}{\sim} t \int_{B\backslash\{0\}} \left[\int_{\mathbb{T}^d} \kappa_0 \left(x, z, \frac{z}{\epsilon}\right) \mu(dx)\right] zJ(z)dz \quad \text{(by (2.3) and dominated convergence)} \\ &\xrightarrow{\epsilon \to 0^+} t \int_{B\backslash\{0\}} \left[\int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \kappa_0 \left(x, z, u\right) du\mu(dx)\right] zJ(z)dz \qquad \text{(by Lemma 4(i))}. \end{aligned}$$

In the second line, to apply (2.20) we take $f(y, \epsilon, z) = \kappa^*(y, z, z/\epsilon, z/\epsilon)$. In the last line, to apply Lemma 4(i) we take $K = B$, and $\phi(z, u) = \kappa_0(x, z, u)zJ(z)$ for fixed $x$. Choose $p' \in (1, \frac{d}{d+\alpha-1})$; then it is easy to verify from (2.1) and (2.9) that for each $u$, $\phi(\cdot, u) \in L^{p'}(K)$, and for each $z$, $\phi(z, \cdot) \in L^p([0, 1]^d)$. This proves the assertion (3.3). The assertion (3.4) follows in a similar fashion but with Lemma 4(ii) in place of Lemma 4(i) and letting $\phi(z, u) = \kappa_0(x, z, u)f(z)J(z)$.

(ii) We prove the general case where $b \neq 0$ and $\alpha \in (1, 2)$. Define $\hat{X}_t^\epsilon := X_t^\epsilon + \epsilon\hat{b}_\epsilon\left(X_t^\epsilon\right)$; the boundedness of $\hat{b}$ yields that $\hat{X}^\epsilon$ and $X^\epsilon$ have the same limit. Applying Corollary 5, Lemma 6, and (2.12), we have

$$\begin{aligned} \hat{X}_t^\epsilon = &\int_0^t c\left(\frac{X_s^\epsilon}{\epsilon}\right) ds + \int_0^t \frac{1}{\epsilon^{\alpha-1}} \left(\tilde{\mathcal{A}}^\epsilon\hat{b} - \tilde{\mathcal{A}}\hat{b}\right)\left(\frac{X_s^\epsilon}{\epsilon}\right) ds \\ &+ \int_0^t \int_0^\infty \int_{\mathbb{R}^d\backslash\{0\}} \epsilon\left[\hat{b}_\epsilon\left(X_{s-}^\epsilon + \mathbf{1}_{[0,\kappa(X_{s-}^\epsilon/\epsilon, z, z/\epsilon))}(r)z\right) - \hat{b}_\epsilon\left(X_{s-}^\epsilon\right)\right]\tilde{N}(dz, dr, ds) \end{aligned}$$

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$$
\begin{aligned}
&+\int_0^t\int_0^\infty\int_{B\setminus\{0\}}\mathbf{1}_{[0,\kappa(X^\epsilon_{s-}/\epsilon,z,z/\epsilon))}(r)z\tilde{N}(dz,dr,ds)\\
&+\int_0^t\int_0^\infty\int_{B^c}\mathbf{1}_{[0,\kappa(X^\epsilon_{s-}/\epsilon,z,z/\epsilon))}(r)zN(dz,dr,ds)\\
=:\ & I^\epsilon_1(t)+I^\epsilon_2(t)+I^\epsilon_3(t)+I^\epsilon_4(t)+I^\epsilon_5(t),
\end{aligned}
$$

where $N$ is a Poisson random measure on $\mathbb{R}^d\times[0,\infty)\times[0,\infty)$ with intensity measure $J(z)dz\times m\times m$ and $\tilde{N}$ is the associated compensated Poisson random measure. The convergence of $I^\epsilon_1$ is shown in (3.5). For $I^\epsilon_2$ we derive, similarly to (2.14),

$$
\begin{aligned}
&\frac{1}{\epsilon^{\alpha-1}}\left(\tilde{\mathcal{A}}^\epsilon\hat{b}-\tilde{\mathcal{A}}\hat{b}\right)(x/\epsilon)\\
&=\frac{1}{\epsilon^{\alpha-1}}\int_{\mathbb{R}^d\setminus\{0\}}\left[\hat{b}(x/\epsilon+z)-\hat{b}(x/\epsilon)-z\cdot\nabla\hat{b}(x/\epsilon)\right](\kappa(x/\epsilon,\epsilon z,z)-\tilde{\kappa}(x/\epsilon,z))J(z)dz\\
&\quad+\left(c(x/\epsilon)+\int_{B^c}z\kappa(x/\epsilon,z,z/\epsilon)J(z)dz\right)\cdot\nabla\hat{b}(x/\epsilon)\\
&=:\ II_1(x/\epsilon)+II_2(x/\epsilon).
\end{aligned}
$$

Define $\gamma=(\alpha+\beta)\wedge 2>\alpha$. Since $\hat{b}\in\mathcal{C}^{\alpha+\beta}_\mu(\mathbb{T}^d)$, we apply Taylor expansion to get that for all $x\in\mathbb{T}^d$,

$$
\begin{aligned}
&\left|\hat{b}(x+z)-\hat{b}(x)-z\cdot\nabla\hat{b}(x)\right|\\
&\le|z|\int_0^1\left|\nabla\hat{b}(x+rz)-\nabla\hat{b}(x)\right|dr\mathbf{1}_{\{|z|\le1\}}+2\|\hat{b}\|_{\mathcal{C}^1}|z|\mathbf{1}_{\{|z|>1\}}\\
&\le\frac{1}{\gamma}\|\hat{b}\|_{\mathcal{C}^\gamma}|z|^\gamma\mathbf{1}_{\{|z|\le1\}}+2\|\hat{b}\|_{\mathcal{C}^1}|z|\mathbf{1}_{\{|z|>1\}}\\
&\le 2\|\hat{b}\|_{\mathcal{C}^\gamma}(|z|^\gamma\wedge|z|).
\end{aligned}
$$

Then, applying the assumption (2.5) and dominated convergence, we estimate $II_1$ as follows:

$$
\begin{aligned}
|II_1(X^\epsilon_s/\epsilon)|&\le\sup_{x\in\mathbb{T}^d}|II_1(x/\epsilon)|\\
&\le 2j_2\|\hat{b}\|_{\mathcal{C}^\gamma}\int_{\mathbb{R}^d\setminus\{0\}}\left(\frac{1}{\epsilon^{\alpha-1}}\sup_{x\in\mathbb{T}^d}|\kappa(x,\epsilon z,z)-\tilde{\kappa}(x,z)|\right)(|z|^\gamma\wedge|z|)\frac{dz}{|z|^{d+\alpha}}\\
&\xrightarrow{\epsilon\to0^+}0.
\end{aligned}
$$

Using the same argument as the proof of (3.3), we have the following locally uniform convergence in $t$ in probability, as $\epsilon\to0^+$:

$$
\begin{aligned}
I^\epsilon_2(t)&\sim\int_0^t II_2\left(\frac{X^\epsilon_s}{\epsilon}\right)ds\\
&=\int_0^t\left\{\left[c\left(\frac{X^\epsilon_s}{\epsilon}\right)+\int_{B^c}z\kappa\left(\frac{X^\epsilon_s}{\epsilon},z,\frac{z}{\epsilon}\right)J(z)dz\right]\cdot\nabla\hat{b}\left(\frac{X^\epsilon_s}{\epsilon}\right)\right\}ds\\
&\to t\left[\int_{\mathbb{T}^d}c(x)\cdot\nabla\hat{b}(x)\mu(dx)+\int_{B^c}z\cdot\left(\int_{\mathbb{T}^d}\int_{\mathbb{T}^d}\nabla\hat{b}(x)\kappa_0(x,z,u)du\mu(dx)\right)J(z)dz\right].
\end{aligned}
$$

For $I_3^\epsilon$, we use Itô's isometry to get

$$\begin{aligned}\mathbb{E}\left(|I_3^\epsilon(t)|^2\right) &= \mathbb{E}\int_0^t\int_0^\infty\int_{\mathbb{R}^d\setminus\{0\}}\left|\epsilon\left[\hat{b}_\epsilon\left(X_{s-}^\epsilon+\mathbf{1}_{[0,\kappa(X_{s-}^\epsilon/\epsilon,z,z/\epsilon))}(r)z\right)-\hat{b}_\epsilon\left(X_{s-}^\epsilon\right)\right]\right|^2 J(dz)drds\\ &= \mathbb{E}\int_0^t\int_{\mathbb{R}^d\setminus\{0\}}\epsilon^2\left|\hat{b}_\epsilon\left(X_{s-}^\epsilon+z\right)-\hat{b}_\epsilon\left(X_{s-}^\epsilon\right)\right|^2\kappa\left(\frac{X_{s-}^\epsilon}{\epsilon},z,\frac{z}{\epsilon}\right)J(dz)ds\\ &\le \kappa_2 j_2 t\left(4\|\hat{b}\|_\infty^2\epsilon^2\int_{B_\epsilon^c}\frac{dz}{|z|^{d+\alpha}}+\|\hat{b}\|_{C^1}^2\int_{B_\epsilon\setminus\{0\}}|z|^2\frac{dz}{|z|^{d+\alpha}}\right)\\ &= \kappa_2 j_2 t\omega_{d-1}\left(\frac{4\|\hat{b}\|_\infty^2}{\alpha}+\frac{\|\hat{b}\|_{C^1}^2}{2-\alpha}\right)\epsilon^{2-\alpha},\end{aligned}$$

which goes to zero as $\epsilon\to 0^+$, where $\omega_{d-1}$ is the surface area of the unit sphere in $\mathbb{R}^d$. This implies that $I_3^\epsilon(t)$ converges to 0 locally uniformly in $t$ in probability. Since the local uniform topology is stronger than the Skorokhod topology in the space $\mathcal{D}$ (see, for instance, [25, Proposition VI.1.17]), $I_3^\epsilon$ converges to 0 in the Skorokhod topology in probability and thereby in distribution. Furthermore, it is easy to verify that the semimartingale characteristics of $I_4^\epsilon+I_5^\epsilon$ are $(0, 0, \nu^\epsilon)$, whose convergence is proved in (3.4). Combining these convergences and using the functional central limit theorem again, we get the results. □

**Remark 3.**

(i) Note that $\kappa_1\le\bar{\kappa}(z)\le\kappa_2$ for all $z$, so the homogenized measure $\bar{\nu}$ is an $\alpha$-stable Lévy measure.

(ii) The generator of the limit process $\bar{X}$, restricted to $C^\infty(\mathbb{T}^d)$, is

$$\begin{aligned}\bar{\mathcal{A}}f(x) = &\int_{\mathbb{R}^d\setminus\{0\}}\left[f(x+z)-f(x)-z\cdot\nabla f(x)\mathbf{1}_{[1,2)}(\alpha)\mathbf{1}_{\{|z|<1\}}\right]\bar{\kappa}(z)J(z)dz\\ &+\bar{c}\cdot\nabla f(x)\mathbf{1}_{(1,2)}(\alpha).\end{aligned}$$

(iii) Note that the homogenized coefficients $\bar{\kappa}$ and $\bar{c}$ both depend on the invariant distribution $\mu$ of the auxiliary process $\tilde{X}$. Proposition 2 tells that $\mu$ can be approximated by large-time distributions of $\tilde{X}$, with exponentially small error. But in practice this scheme is not efficient, since one needs to generate an enormously large number of samples at a large time in order to compute the measure $\mu$ by Monte Carlo methods. However, by Remark 2(i), we can approximate $\mu$ by the long-time average of a single path of $\tilde{X}$, owing to the ergodicity. Indeed, taking $f=\mathbf{1}_A$ for some $A\in\mathcal{B}(\mathbb{T}^d)$, we have

$$\frac{1}{T}\int_0^T\mathbf{1}_A(\tilde{X}_s)ds\to\mu(A),\quad \text{as } T\to\infty,\ \text{in } L^2(\Omega,\mathbb{P}).$$

## 4. Examples and comparisons

In this section, we present some examples that cover several results in earlier papers.

**Example 1.** (*Pure jump Lévy processes.*) In the special case that $b=c\equiv 0$ and $\kappa^*(x,z,u,v)\equiv\kappa^*(u)$ which is a periodic function of period 1 and satisfies $\kappa_1\le\kappa^*(u)\le\kappa_2$ for all $u$, the homogenized constant is $\bar{\kappa}=\int_{\mathbb{T}^d}\kappa^*(u)du$ and $\bar{b}=0$. This is the case presented in [36, Remark 5].

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Note that in that paper, the authors use a purely analytical approach—Mosco convergence—to identify the limit process.

**Example 2.** (*SDEs with jump noise.*) Let $L^\alpha = \{L_t^\alpha\}_{t\geq 0}$ be a $d$-dimensional isotropic $\alpha$-stable Lévy process on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0})$ given by

$$L_t^\alpha = \int_0^t \int_{B\setminus\{0\}} y\tilde{N}^\alpha(dy, ds) + \int_0^t \int_{B^c} yN^\alpha(dy, ds),$$

where $1<\alpha<2$, $N^\alpha$ is a Poisson random measure on $\big(\mathbb{R}^d \setminus \{0\}\big) \times \mathbb{R}_+$ with jump intensity measure $\nu^\alpha(dy) = \frac{dy}{|y|^{d+\alpha}}$, and $\tilde{N}^\alpha$ is the associated compensated Poisson random measure; that is, $\tilde{N}^\alpha(dy, ds) := N^\alpha(dy, ds) - \nu^\alpha(dy)ds$. Consider the following SDE:

$$\begin{aligned} X_t^{x,\epsilon} &= x + \int_0^t \left(\frac{1}{\epsilon^{\alpha-1}} b\left(\frac{X_{s-}^{x,\epsilon}}{\epsilon}\right) + c\left(\frac{X_{s-}^{x,\epsilon}}{\epsilon}\right)\right) ds \\ &\quad + \int_0^t \int_{B\setminus\{0\}} \sigma\left(\frac{X_{s-}^{x,\epsilon}}{\epsilon}, y\right) \tilde{N}^\alpha(dy, ds) + \int_0^t \int_{B^c} \sigma\left(\frac{X_{s-}^{x,\epsilon}}{\epsilon}, y\right) N^\alpha(dy, ds), \end{aligned} \tag{4.1}$$

where the functions $b$, $c$ are both periodic of period 1, while the function $\sigma(x, y)$ is periodic in $x$ of period 1, and odd in $y$ in the sense that $\sigma(x, -y) = -\sigma(x, y)$ for all $x, y \in \mathbb{R}^d$. We assume that $\sigma \in \mathcal{C}^{1,2}(\mathbb{R}^d \times \mathbb{R}^d)$ and that there exist constants $C_1 > 0$, $C_2 > 1$ such that for all $x_1, x_2, x, y \in \mathbb{R}^d$,

$$|\sigma(x_1, y) - \sigma(x_2, y)| \leq C_1|x_1 - x_2||y|, \quad C_2^{-1}|y| \leq |\sigma(x, y)| \leq C_2|y|.$$

Assume in addition that for every $x$, $\sigma(x, \cdot)$ is uniformly continuous and is a $\mathcal{C}^2$-diffeomorphism with inverse $\tau(x, \cdot) := \sigma(x, \cdot)^{-1}$. Then we know that (4.1) possesses a unique strong solution which is a Feller process, for each $\epsilon > 0$; see [23, Theorem 4.2, Corollary 4.3].

Now the generator of the solution process $X^{x,\epsilon}$ restricted to $\mathcal{C}^\infty\big(\mathbb{T}^d\big)$ is

$$\begin{aligned} \mathcal{A}_\alpha^\epsilon f(x) := &\int_{\mathbb{R}^d\setminus\{0\}} \left[f\left(x + \sigma\left(\frac{x}{\epsilon}, y\right)\right) - f(x) - \sigma\left(\frac{x}{\epsilon}, y\right) \cdot \nabla f(x)\mathbf{1}_B(y)\right] \nu^\alpha(dy) \\ &+ \left[\frac{1}{\epsilon^{\alpha-1}} b\left(\frac{x}{\epsilon}\right) + c\left(\frac{x}{\epsilon}\right)\right] \cdot \nabla f(x). \end{aligned}$$

Through a change of variables and using the oddness of $y \to \sigma(x, y)$, we can rewrite this in the form in (1.1) with

$$\kappa(x, z, u) \equiv \kappa(x, z) := |\det \nabla_z \tau(x, z)| \frac{|z|^{d+\alpha}}{|\tau(x, z)|^{d+\alpha}}; \tag{4.2}$$

that is,

$$\int_A \kappa(x, z) \frac{dz}{|z|^{d+\alpha}} = \int_{\mathbb{R}^d\setminus\{0\}} \mathbf{1}_A(\sigma(x, y))\nu^\alpha(dy), \quad A \in \mathcal{B}\big(\mathbb{R}^d \setminus \{0\}\big). \tag{4.3}$$

Then the function $\kappa$ satisfies the assumptions (2.1) and (2.2) (see [23, Assumption H3, Lemma 2.3, Proposition 2.5]), as well as the assumption (2.3) with $\kappa_0(x, z, u) \equiv \kappa(x, z)$. Note that for each $x$, the oddness of $\sigma(x, \cdot)$ implies the oddness of $\tau(x, \cdot)$, and further the symmetry of $\kappa(x, \cdot)$, in the sense that

$$\kappa(x, z) = \kappa(x, -z) \quad \text{for all } x, z.$$

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We assume further that

$$\tfrac{1}{\epsilon}\sigma(x, \epsilon y) \to \nabla_y \sigma(x, 0) \cdot y, \quad \text{uniformly in } x \text{ and } y, \quad \text{as } \epsilon \to 0^+$$

(cf. [23, Assumption H5]). Then we can easily prove (e.g., by [34, Theorem 7.17]) that for each $z$,

$$\tfrac{1}{\epsilon}\tau(x, \epsilon z) \to \nabla_z \tau(x, 0) \cdot z \quad \text{and} \quad \nabla_z \tau(x, \epsilon z) \to \nabla_z \tau(x, 0), \quad \text{uniformly in } x, y, \quad \text{as } \epsilon \to 0^+.$$

Hence, we conclude that the function $\kappa$ defined in (4.2) satisfies the assumption (2.5) with

$$\tilde{\kappa}(x, z) \equiv \tilde{\kappa}(x) := |\det \nabla_z \tau(x, 0)| \frac{1}{|\nabla_z \tau(x, 0)|^{d+\alpha}}.$$

Applying Theorem 1, we know that the sequence of solutions $X^{x,\epsilon}$ converges in distribution to a Lévy process $\bar{X}^x$ starting from $x$ with Lévy triplet $(\bar{b}, 0, \bar{\nu})$ given in (3.2). By the symmetry of $\kappa$ and $\nu^\alpha$, the homogenized constant $\bar{b} = \int_{\mathbb{T}^d} (I + \nabla \hat{b}(x)) \cdot c(x)\mu(dx)$, where $\mu$ is the invariant measure of the Feller process generated by

$$\begin{aligned}\tilde{\mathcal{A}}_\alpha f(x) := &\int_{\mathbb{R}^d \setminus \{0\}} \left[ f\big(x + \nabla_y \sigma(x, 0) \cdot y\big) - f(x) - y \cdot \nabla_y \sigma(x, 0) \cdot \nabla f(x) \mathbf{1}_B(y) \right] \nu^\alpha(dy) \\ &+ b(x) \cdot \nabla f(x),\end{aligned}$$

and $\hat{b}$ is the unique solution to the Poisson equation $\tilde{\mathcal{A}}_\alpha \hat{b} = b$. Moreover, the homogenized function is $\bar{\kappa}(z) = \int_{\mathbb{T}^d} \kappa(x, z)\mu(dx)$. This coincides with the result in [23, Theorem 5.2]. To see this, we derive $\bar{\nu}(A)$ for $A \in \mathcal{B}\big(\mathbb{R}^d \setminus \{0\}\big)$: by (4.3),

$$\bar{\nu}(A) = \int_A \int_{\mathbb{T}^d} \kappa(x, z)\mu(dx) \frac{dz}{|z|^{d+\alpha}} = \int_{\mathbb{R}^d \setminus \{0\}} \int_{\mathbb{T}^d} \mathbf{1}_A(\sigma(x, y))\mu(dx)\nu^\alpha(dy).$$

In particular, this also generalizes the result in [18], where the author considers the special case $\sigma(x, y) = \sigma_0(x)y$.

**Example 3.** (*One-dimensional jump processes.*) Consider the one-dimensional case with $\alpha \in (1, 2)$, $c \equiv 0$, and $\kappa^*(x, z, u, v) \equiv \kappa^*(x, v)$, that is,

$$\mathcal{A}^\epsilon_{1\mathrm{d}} f(x) = \int_{-\infty}^{+\infty} \left[ f(x+z) - f(x) - zf'(x)\mathbf{1}_{\{|z|<1\}}(z) \right] \kappa^*\big(\tfrac{x}{\epsilon}, \tfrac{z}{\epsilon}\big) J(z)dz + \tfrac{1}{\epsilon^{\alpha-1}} b\big(\tfrac{x}{\epsilon}\big) f'(x).$$

Here $J$ is the density of an $\alpha$-stable Lévy measure on $\mathbb{R} \setminus \{0\}$ (see [35, Remark 14.4]); that is,

$$J(z) = j^+ z^{-(1+\alpha)} \mathbf{1}_{(0,+\infty)}(z) + j^- |z|^{-(1+\alpha)} \mathbf{1}_{(-\infty,0)},$$

with constants $j^+, j^- > 0$, so that the assumption (2.7) is fulfilled.

Besides (2.1), (2.2), and Assumptions 1 and 4, we assume further that there exist two functions $\kappa_0^+, \kappa_0^- : \mathbb{R}^d \setminus \{0\} \to [0, \infty)$ such that for each $x$,

$$\lim_{y \to \pm\infty} y^{-1} \int_0^y \kappa^*(x, v)dv = \kappa_0^\pm(x).$$

Note that this is the type of assumption in [22]. Then, by L'Hôpital's rule, we have

$$\lim_{v \to \pm\infty} \kappa^*(x, v) = \kappa_0^\pm(x).$$

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Thus, our assumption (2.3) is fulfilled by letting

$$\kappa_0(x, z, u) \equiv \kappa_0(x, z) := \kappa_0^+(x)\mathbf{1}_{(0,+\infty)}(z) + \kappa_0^-(x)\mathbf{1}_{(-\infty,0)}(z).$$

And the assumption (2.4) holds trivially. Now, by Theorem 1, the Feller process generated by $\mathcal{A}_{1d}^\epsilon$ converges in distribution, as $\epsilon \to 0^+$, to a one-dimensional $\alpha$-stable Lévy process $\bar{X}$ with Lévy triplet $(\bar{b}, 0, \bar{\nu})$ as in (3.2). Let $\mu$ be the invariant measure of the Feller process generated by

$$\tilde{\mathcal{A}}_{1d}f(x) = \int_{-\infty}^{+\infty} \left[f(x+z) - f(x) - zf'(x)\right] \kappa^*(x, z)J(z)dz + b(x)f'(x).$$

Then the homogenized drift $\bar{b}$ is

$$\bar{b} = \frac{1}{\alpha - 1} \int_0^1 \left(j^+\kappa^+(x) + j^-\kappa^-(x)\right) \hat{b}'(x)\mu(dx),$$

where $\hat{b}$ is the unique solution to the Poisson equation $\tilde{\mathcal{A}}_{1d}\hat{b} = b$. Define two constants $\bar{\kappa}^\pm := \int_{\mathbb{T}^d} \kappa_0^\pm(x)\mu(dx)$; then

$$\bar{\kappa}(z) = \bar{\kappa}^+\mathbf{1}_{(0,+\infty)}(z) + \bar{\kappa}^-\mathbf{1}_{(-\infty,0)}(z).$$

Note that the authors of [22] consider the operators of the form $\tilde{\mathcal{A}}_{1d}$ with $\kappa^*(\frac{x}{\epsilon}, \frac{z}{\epsilon})$ and $\frac{1}{\epsilon^{\alpha-1}}b(\frac{x}{\epsilon})$ in place of $\kappa^*(x, z)$ and $b(x)$, which are slightly different from $\mathcal{A}_{1d}^\epsilon$, but the homogenized jump measure in [22] coincides with $\bar{\nu}$.

## 5. Generalization to symmetric stable-like processes with variable order

One class of pure jump processes that is of great interest is the class of *stable-like* processes (see the survey [24] and references therein). Locally, a stable-like process looks like a stable process, so that for every $x$, its jump measure $\eta(x, \cdot)$ is $\boldsymbol{\alpha}(x)$-stable [35, Theorem 14.3], i.e.,

$$\eta(x, A) = \int_0^\infty \int_S \mathbf{1}_A(r\xi)\rho(x, d\xi)\frac{dr}{r^{1+\boldsymbol{\alpha}(x)}}, \quad A \in \mathcal{B}\big(\mathbb{R}^d \setminus \{0\}\big), \tag{5.1}$$

where $\rho$ is a map from $\mathbb{R}^d$ to the space $\mathcal{M}(S)$ of finite measures on $S$, called the *spherical part* of $\eta$ or the *spectral measure* of the process; the stability index $\boldsymbol{\alpha}$ is now a function taking values in $(0, 2)$. Because of the variety of $\boldsymbol{\alpha}$, such a jump kernel $\eta$ cannot be written as the product of a bounded function $\kappa$ with a reference Lévy measure with constant stability index (cf. (2.1) and (2.6)), so the homogenization framework in previous sections cannot be applied to such jump processes. However, we can slightly modify the assumptions for the coefficient $\kappa$ to deal with such a case. Note that some authors also use the term 'stable-like' to refer to the case (1.1) (or (A.1), e.g., [10]). But we shall reserve it for the case (5.1).

### 5.1. Homogenization result

For there to exist a jump process with jump kernel (5.1), we need some assumptions [27]:

- The function $\boldsymbol{\alpha} : \mathbb{R}^d \to (0, 2)$ is of class $\mathcal{C}^1$, periodic of period 1, and satisfies that for all $x \in \mathbb{R}^d$,

$$0 < \alpha := \min_{x\in\mathbb{R}^d} \boldsymbol{\alpha}(x) \le \bar{\alpha} := \max_{x\in\mathbb{R}^d} \boldsymbol{\alpha}(x) < 2, \tag{5.2}$$

  where the minimum and maximum of $\boldsymbol{\alpha}$ are attainable since $\boldsymbol{\alpha}$ is continuous and periodic.

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- The function $\rho : \mathbb{R}^d \to \mathcal{M}(S)$ is periodic of period 1 and symmetric, i.e., $\rho(x, \xi) = \rho(x, -\xi)$ for all $x \in \mathbb{R}^d$ and $\xi \in S$; it has a density, again denoted by $\rho$, i.e, $\rho(x, d\xi) = \rho(x, \xi)d\xi$; and it satisfies the following conditions:
  - $\inf_{x\in\mathbb{R}^d} \left(\rho(x, S) \wedge \inf_{\theta\in S} \int_S (\theta \cdot \xi)^2 \rho(x, d\xi)\right) > 0$;
  - $\rho$ is Lipschitz in the sense that there exists $C > 0$ such that for all $x, y \in \mathbb{R}^d$,

$$|\rho(x, S) - \rho(y, S)| + \mathcal{W}_1(\hat{\rho}(x, \cdot), \hat{\rho}(y, \cdot)) \le C|x - y|,$$

    where $\hat{\rho} = (\rho(\cdot, S))^{-1}\rho$ is the normalized probability measure of $\rho$ and $\mathcal{W}_1$ is the Wasserstein-1 distance of probability measures (e.g., [39]);
  - $\rho$ is dominated by a probability function $\rho_0$ on $S$; that is, there exists a constant $C > 0$ such that $\rho(x, \xi) \le C\rho_0(\xi)$ for all $x \in \mathbb{R}^d$, $\xi \in S$.

We still consider the operator $\mathcal{A}^\epsilon$ in (1.1), with coefficients as follows:

- $b = c \equiv 0$; $J(z) = |z|^{-(d+\alpha)}$ is the density of a rotation-invariant $\alpha$-stable Lévy measure with $\alpha$ given in (5.2);
- $\kappa^*$ is given by

$$\kappa^*(x, z, u, v) \equiv \kappa^*(x, v) := \rho(x, v/|v|)|v|^{\alpha-\boldsymbol{\alpha}(x)}.$$

The resulting function $\kappa(x, z, u) \equiv \kappa^*(x, u)$ does not satisfy either (2.1) or (2.2) in general. But (2.3) still holds with

$$\kappa_0(x, z, u) \equiv \kappa_0(x, z) := \rho(x, z/|z|)\mathbf{1}_{\{\boldsymbol{\alpha}(x)=\alpha\}},$$

and (2.4) holds trivially with $\tilde{\kappa} = \kappa = \kappa^*$.

Note that because of the symmetry of $\rho$, the indicator function $\mathbf{1}_{[1,2)}(\alpha)$ in (1.1) has no effect. The jump measure of $\mathcal{A}^\epsilon$ is of the form (5.1) with $\boldsymbol{\alpha}$ and $\rho$ replaced by

$$\boldsymbol{\alpha}_\epsilon(x) := \boldsymbol{\alpha}(x/\epsilon), \quad \rho^\epsilon(x, \xi) := \epsilon^{\boldsymbol{\alpha}(x/\epsilon)-\alpha}\rho(x/\epsilon, \xi). \tag{5.3}$$

Since $\kappa$ and $\tilde{\kappa}$ coincide and the jump kernel $\rho$ is symmetric, we see that $\tilde{\mathcal{A}}^\epsilon \equiv \tilde{\mathcal{A}}$ for all $\epsilon$ (cf. (2.10) and (2.11)). Their jump measure is given by (5.1) with $\rho(x, d\xi) = \rho(x, \xi)d\xi$.

The counterpart of Proposition 1 is the following, where the well-posedness is taken from [27, Theorem 3.1] and the heat kernel estimate is adapted from [28, Proposition 3.1, Theorem 5.1].

**Proposition 3.** *Under the conditions listed above, for every $x \in \mathbb{R}^d$, the martingale problems for $(\mathcal{A}^\epsilon, \delta_x)$, $\epsilon > 0$, and $(\tilde{\mathcal{A}}, \delta_x)$ have unique solutions $\mathbb{P}^\epsilon_x$ on $(\mathcal{D}, \mathcal{B}(\mathcal{D}))$ and $\tilde{\mathbb{P}}_x$ on $(\mathcal{D}_{\text{per}}, \mathcal{B}(\mathcal{D}_{\text{per}}))$, respectively. The coordinate processes $X^\epsilon$ and $\tilde{X}$ are respectively $\mathbb{R}^d$- and $\mathbb{T}^d$-valued Feller processes, starting from x. Moreover, $\tilde{X}$ has a jointly continuous transition probability density $\tilde{p}(t; x, y)$ satisfying that for every $T > 0$, there exist constants $0 < C_1 < 1$, $C_2, C_3 > 0$, and $\delta \in (0, 1)$ such that for all $t \in (0, T]$ and $x, y \in \mathbb{T}^d$,*

$$\begin{aligned}\tilde{p}(t; x, y) \ge \sum_{l\in\mathbb{Z}^d} \Big\{ & C_1 \left[t^{-d/\boldsymbol{\alpha}(x)} \wedge \left(t|x - y + l|^{-(d+\boldsymbol{\alpha}(x))}\right)\right] \left(1 - C_2 t^\gamma\right) \\ & - C_3 t^\delta \left[1 \wedge |x - y + l|^{-(d+\boldsymbol{\alpha}(x))}\right] \Big\} \vee 0,\end{aligned} \tag{5.4}$$

*with any $0 < \gamma < 1/(d + \boldsymbol{\alpha}(x))$ and $0 < \delta < 1 - \beta(d + \boldsymbol{\alpha}(x))$.*

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By (5.3), we have for all $\epsilon > 0$ and 'good' test functions $f : \mathbb{R}^d \to \mathbb{R}$ that

$$\mathcal{A}^\epsilon f(x) = \epsilon^{-\alpha}\big(\tilde{\mathcal{A}} f_{1/\epsilon}\big)(x/\epsilon), \tag{5.5}$$

so that Lemma 1 holds with $\{X_t^\epsilon\}_{t\geq 0} \overset{\mathrm{d}}{=} \big\{\epsilon \tilde{X}_{t/\epsilon^\alpha}\big\}_{t\geq 0}$ in this case. Proposition 2 and Lemma 2 hold trivially with $\tilde{X}^\epsilon \equiv \tilde{X}$, $\tilde{P}_t^\epsilon \equiv \tilde{P}_t$, and $\mu_\epsilon \equiv \mu$. In particular, to prove the counterpart of Proposition 2, as indicated in its own proof, it suffices to show that there exists a $t_0 > 0$ such that $\tilde{p}(t_0; x, y)$ is bounded from below by a positive constant independent of $x, y \in \mathbb{T}^d$. To this end, we choose $\gamma_0 > 0$ and $t_0 < 1$ such that

$$\begin{cases} \gamma_0 < 1/(d+\bar{\alpha}), & t_0^{1/\alpha} \geq 1/\sqrt{2}, \\ 1 - C_2 t_0^{\gamma_0} > 0, & C_1 t_0^{-d/\bar{\alpha}}\big(1 - C_2 t_0^{\gamma_0}\big) - C_3 > 0. \end{cases}$$

Since, for any $x, y \in \mathbb{T}^d = [0, 1]^d$, there is always an $l \in \mathbb{Z}^d$ such that $|x - y + l| \leq 1/\sqrt{2} \leq t_0^{1/\alpha} \leq t_0^{1/\boldsymbol{\alpha}(x)} < 1$, we obtain from (5.4) that

$$\tilde{p}(t; x, y) \geq C_1 t_0^{-d/\boldsymbol{\alpha}(x)}\left(1 - C_2 t^{\gamma_0}\right) - C_3 t_0^\delta \geq C_1 t_0^{-d/\bar{\alpha}}\big(1 - C_2 t_0^{\gamma_0}\big) - C_3,$$

where the last quantity is positive and independent of $x, y$. This proves that Proposition 2 holds true for the case here. As a consequence, Corollary 1 also holds. Therefore, Part (i) of the proof of Theorem 1 can still proceed with no obstacles. In conclusion, we get the following homogenization result for stable-like processes, which recovers the result of [17, Theorem 1].

**Theorem 2.** *Under the same assumptions as Proposition 3, we have the following weak convergence on the space* $\mathcal{D}(\mathbb{R}_+; \mathbb{R}^d)$*:*

$$X^\epsilon \Rightarrow \bar{X}, \quad \text{as } \epsilon \to 0^+,$$

*where the limit process* $\bar{X}$ *is a Lévy process with Lévy triplet* $(0, 0, \bar{\nu})$ *given by*

$$\bar{\nu}(A) = \int_0^\infty \int_S \mathbf{1}_A(r\xi)\left(\int_{\mathbb{T}^d} \rho(x, \xi)\mathbf{1}_{\{\boldsymbol{\alpha}(x)=\alpha\}}\mu(dx)\right) d\xi \frac{dr}{r^{1+\alpha}}, \quad A \in \mathcal{B}\big(\mathbb{R}^d \setminus \{0\}\big),$$

*with* $\mu$ *being the invariant probability measure of* $\tilde{X}$ *with jump measure* (5.1).

**Remark 4.** In the special case that $\mu(\{x \in \mathbb{T}^d : \boldsymbol{\alpha}(x) = \alpha\}) = 0$, the homogenized measure $\bar{\nu}$ is trivially zero, i.e., $X^\epsilon$ converges weakly to the constant zero process. In this sense the scaling (5.5) is strong. One might expect that it would give a non-trivial limit if we changed the scaling (5.5) to

$$\mathcal{A}^\epsilon f(x) = \big[\epsilon^{-\boldsymbol{\alpha}}\big(\tilde{\mathcal{A}} f_{1/\epsilon}\big)\big](x/\epsilon).$$

But the latter cannot yield a scaling for the generated processes $X^\epsilon$ and $\tilde{X}$ like Lemma 1. Therefore, the functional convergence in Corollary 1 and thus the final homogenization result are unknown in this case.

### 5.2. Numerical simulations

In this subsection, we will present a numerical experiment to help visualize the homogenization result. Furthermore, for a jump particle in a periodic structure, a typical topic of interest in practical applications is the distribution of the first exit time at which the particle escapes a given domain. We will also give some visualizations for the empirical mean of the first exit time.

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5.2.1. *Numerical scheme.* Set the dimension $d=2$, let

$$\boldsymbol{\alpha}(x)=1+\frac{1}{4}\Big[\mathbf{1}_{\left[0,\frac{3}{8}\right)}(x_1)\cos\left(\tfrac{8\pi}{3}x_1\right)+\mathbf{1}_{\left(\frac{5}{8},1\right]}(x_1)\cos\left(\tfrac{8\pi}{3}(1-x_1)\right)-\mathbf{1}_{\left[\frac{3}{8},\frac{5}{8}\right]}(x_1)$$
$$+\mathbf{1}_{\left[0,\frac{3}{8}\right)}(x_2)\cos\left(\tfrac{8\pi}{3}x_2\right)+\mathbf{1}_{\left(\frac{5}{8},1\right]}(x_2)\cos\left(\tfrac{8\pi}{3}(1-x_2)\right)-\mathbf{1}_{\left[\frac{3}{8},\frac{5}{8}\right]}(x_2)\Big]$$

for $x=(x_1,x_2)\in[0,1]^2$, and let

$$\rho(x,d\xi)\equiv\rho(d\xi):=\sum_{i=1}^{4}\delta_{e_i}(d\xi),\qquad \xi\in\mathbb{S}^1,$$

where $e_1=(1,0)$, $e_2=(0,1)$, $e_3=(-1,0)$, $e_4=(0,-1)$ form canonical orthonormal bases for $\mathbb{R}^2$. It is easy to verify that such $\boldsymbol{\alpha}$ and $\rho$ satisfy all conditions listed at the beginning of the last subsection. The minimum of $\boldsymbol{\alpha}$ is $\alpha=\frac{1}{2}$. The spectral measure of the process $X^\epsilon$ is $\rho^\epsilon(x,d\xi)=\epsilon^{\alpha(x/\epsilon)-\frac{1}{2}}\rho(d\xi)$, while the spectral measure of the limit process $\bar{X}$ is $\bar{\rho}(d\xi)=\mu\left(\boldsymbol{\alpha}^{-1}\left(\frac{1}{2}\right)\right)\rho(d\xi)$.

We use the method in [31] to simulate the 'one-step' stable random vectors, and then use the scheme developed in [6] to simulate the stable-like processes $X^\epsilon$ and $\tilde{X}$ by gluing all one-step stable random vectors together. The convergence of the latter scheme is proved in [7]. Note that the distribution of each one-step stable random vector depends on the position of the previous step. As for the limit Lévy process $\bar{X}$, the one-step vectors are independent of the previous positions.

In all path sampling, we always use time-step size $\Delta t=0.01$. There are two ways to approximately compute $\mu\left(\boldsymbol{\alpha}^{-1}\left(\frac{1}{2}\right)\right)$, as mentioned in Remark 3(iii). We first generate 1000 sample paths of the process $\tilde{X}$ with 100 steps by the above-mentioned scheme, and count the number of samples at the final time step inside the set $\boldsymbol{\alpha}^{-1}\left(\frac{1}{2}\right)=\left[\frac{3}{8},\frac{5}{8}\right]^2$. Then we use a sample path with time length $T=100$, and calculate the time-average $\frac{1}{t}\int_0^t\mathbf{1}_{\boldsymbol{\alpha}^{-1}\left(\frac{1}{2}\right)}\left(\tilde{X}_s\right)ds$ for varying $t$. Figure 1 shows the results from using these two methods. In particular, it shows that when $t$ is large, the two results are very close.

Figure 2 shows the sample paths on the plane for the processes $X^\epsilon$ with $\epsilon=10^{-1}, 10^{-2}, 10^{-3}, 10^{-4}, 10^{-5}$ and the limit process $\bar{X}$. As we can see, when the scaling parameter $\epsilon$ gets smaller and smaller, the path of $X^\epsilon$ becomes more and more concentrated into small clusters.

5.2.2. *Simulations of first exit time.* For $x\in\mathcal{D}$ and $r>0$, define

$$\begin{aligned}S_r(x)&:=\inf\{t\geq 0:|x(t)|\geq r\text{ or }|x(t-)|\geq r\},\\ S_{r+}(x)&:=\inf\{t\geq 0:|x(t)|>r\text{ or }|x(t-)|>r\},\\ V(x)&:=\{r>0:S_r(x)<S_{r+}(x)\}.\end{aligned}$$

It is easy to see that

$$S_r(x)=\inf\left\{t\geq 0:\sup_{0\leq s\leq t}|x(s)|\geq r\right\},$$

which is exactly the *first exit time* for the path $x$ to escape the ball of radius $r$. In order to simplify the notation as before, we define $X^0:=\bar{X}$. Using [25, Lemma VI.2.10], we know

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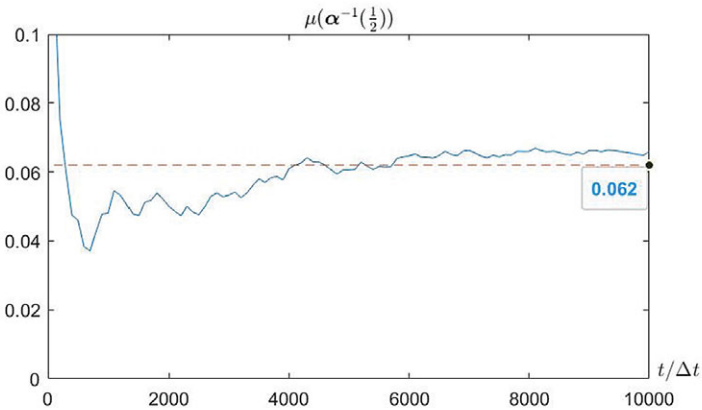


FIGURE 1. Computations of $\mu\left(\boldsymbol{\alpha}^{-1}\left(\frac{1}{2}\right)\right)$. The horizontal coordinate indicates the number of steps $t/\Delta t$, and the vertical coordinate indicates the time-average of times of a single path inside $\boldsymbol{\alpha}^{-1}\left(\frac{1}{2}\right)$, with time-step size $\Delta t = 0.01$. The dashed line shows the results of the Monte Carlo method, implemented by generating 1000 samples with 100 steps.

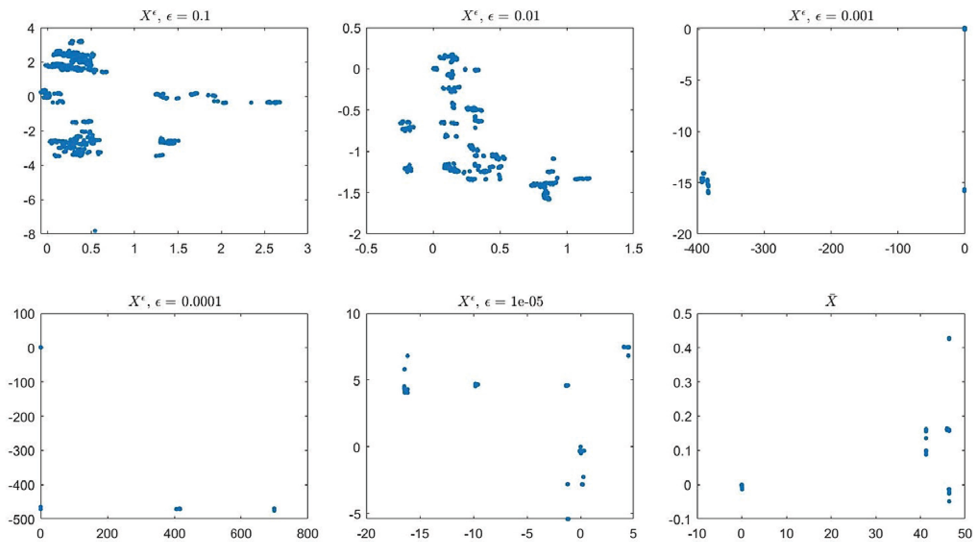


FIGURE 2. Sample paths for $X^\epsilon$ and $\bar{X}$ on the time interval [0, 10] with time-step size 0.01. The coordinates represent the particle positions in $\mathbb{R}^2$.

that for all $\epsilon \geq 0$ and $\omega \in \Omega$, $V(X^\epsilon(\omega))$ is an at most countable subset of $\mathbb{R}_+$. It follows that each set

$$U^\epsilon = \{r > 0 : \mathbb{P}(r \in V(X^\epsilon)) = 0\}$$

has full measure in $\mathbb{R}_+$, and thus we have the following result.

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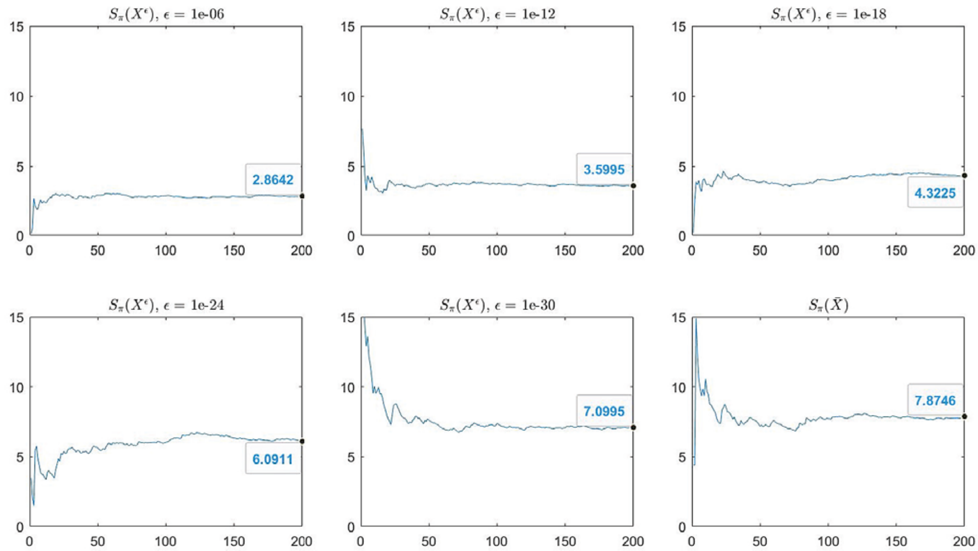


FIGURE 3. Empirical mean of the first exit time for $X^\epsilon$ and $\bar{X}$. The horizontal and vertical coordinates indicate the number of test samples and the average to the present, respectively. The labeled points give the values of the empirical mean of 200 samples with time-step size 0.01.

**Lemma 5.** *The set* $\cap_{\epsilon>0} U^\epsilon$ *also has full measure in* $\mathbb{R}_+$.

Now for each $r \in \cap_{\epsilon>0} U^\epsilon$, the mapping $X^\epsilon \mapsto S_r(X^\epsilon)$ is continuous for all $\epsilon \geq 0$, by virtue of [25, Proposition VI.2.11]. Hence, by the continuous mapping theorem (see, e.g., [16, Corollary 3.1.9]), we have the following corollary, of which the second statement follows from [4, Theorem 25.12].

**Corollary 3.** *For each* $r \in \cap_{\epsilon>0} U^\epsilon$, $S_r(X^\epsilon) \Rightarrow S_r(\bar{X})$ *as* $\epsilon \to 0^+$. *If in addition the family* $\{S_r(X^\epsilon)\}_{\epsilon>0}$ *is uniformly integrable, then* $\mathbb{E}(S_r(X^\epsilon)) \to \mathbb{E}(S_r(\bar{X}))$.

We choose $r = \pi$. Figure 3 shows the empirical mean of the first exit time to escape the ball of radius $\pi$ for the processes $X^\epsilon$, $\epsilon = 10^{-6}, 10^{-12}, 10^{-18}, 10^{-24}, 10^{-30}$, and the limit process $\bar{X}$. From this figure, we can see that as $\epsilon$ gets smaller, the convergence rate of the empirical mean with respect to the number of samples decreases.

Figure 4 shows the trend of the empirical mean of $S_\pi(X^\epsilon)$, $\epsilon = 10^{-n}$, with respect to $n$. It follows that the difference between the empirical mean of $S_\pi(X^\epsilon)$ and that of $S_\pi(\bar{X})$ is almost inversely proportional to $n$, so as to be proportional to $\frac{1}{\log(\epsilon^{-1})}$. Even though this rate is not strict, we can still conclude from the figure that the convergence rate of the mean first exit time with respect to $\epsilon$ is very slow.

Therefore, the method of simulating the first exit time of a particle in a periodic structure by choosing a very small $\epsilon$ is quite expensive (in computational time) and not precise in general. The advantage of our homogenization result is that we can directly use the limit process we have just identified to study its distribution properties, instead of using approximations.

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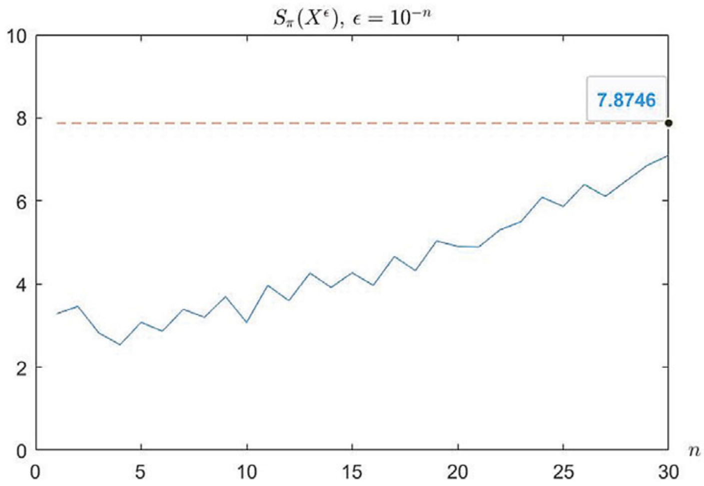


FIGURE 4. Empirical mean of the first exit time for $X^\epsilon$ with $\epsilon = 10^{-n}$, $n = 1, 2, \cdots, 30$. The horizontal coordinate indicates the parameter $n$, and the vertical coordinate indicates the empirical mean of $S_\pi(X^\epsilon)$. The dashed line, provided for reference, gives the empirical mean of $S_\pi(\bar{X})$. All empirical means are simulated with 200 samples with time-step size 0.01.

## Appendix A. Properties of the semigroups and generators

As we have seen in Section 2, we need the Feller nature of the semigroups and the properties of the generators in order to study the ergodicity of the canonical Feller processes; see Propositions 1 and 2. We devote this section to investigating the semigroups and generators. As corollaries, we also obtain the solvability of the Poisson equations with zeroth-order terms and the generalized Itô formula, which are used in Corollary 2 and in the proof of our main result, Theorem 1.

We consider the following operator:

$$\mathcal{L}^{b,\eta} f(x) := b(x) \cdot \nabla f(x) + \int_{\mathbb{R}^d \setminus \{0\}} \left[ f(x+z) - f(x) - z \cdot \nabla f(x) \mathbf{1}_B(z) \right] \kappa^\sharp(x, z) J(z) dz, \tag{A.1}$$

with $\eta(x, dz) := \kappa^\sharp(x, z)J(z)dz$. We suppose that the vector field $b : \mathbb{R}^d \to \mathbb{R}^d$ is in the Hölder class $\mathcal{C}^\beta$ with some $\beta \in (0, 1)$ and periodic of period 1, satisfying that for all $x \in \mathbb{R}^d$,

$$|b(x)| \le b_0$$

for some constant $b_0 > 0$. Suppose that $J$ satisfies (2.9), that is,

$$j_1 |z|^{-(d+\alpha)} \le J(z) \le j_2 |z|^{-(d+\alpha)}, \quad z \in \mathbb{R}^d \setminus \{0\},$$

with $\alpha \in (1, 2)$, and suppose that $\kappa^\sharp(x, z)$ is periodic in $x$ of period 1 and satisfies similar conditions as (2.1) and (2.2); that is, for all $x, x_1, x_2, z \in \mathbb{R}^d$,

$$\kappa_1 \le \kappa^\sharp(x, z) \le \kappa_2,$$
$$|\kappa^\sharp(x_1, z) - \kappa^\sharp(x_2, z)| \le \kappa_3 |x_1 - x_2|^\beta.$$

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Note that the operators $\tilde{\mathcal{A}}$, $\mathcal{A}^\epsilon$, and $\tilde{\mathcal{A}}^\epsilon$ are all of the form (A.1), for appropriate choices of $\kappa^\sharp$. It is easy to verify that $\mathcal{L}^{b,\eta} f \in \mathcal{C}\left(\mathbb{T}^d\right)$ for each $f \in \mathcal{C}^{1+\gamma}\left(\mathbb{T}^d\right)$ with $1+\gamma > \alpha$. Now we treat $\mathcal{L}^{b,\eta}$ as a perturbation of $\mathcal{L}^\eta := \mathcal{L}^{0,\eta}$ by the gradient operator $\mathcal{L}^b := \mathcal{L}^{b,0} = b \cdot \nabla$, and follow [5, 9, 20] to investigate the heat kernel for $\mathcal{L}^{b,\eta}$.

We introduce the following functions on $(0, \infty) \times \mathbb{R}^d$ for later use:

$$\varrho_\gamma(t; x) := t^{\gamma/\alpha} \left( t^{-(d+\alpha)/\alpha} \wedge |x|^{-(d+\alpha)} \right), \quad \gamma \in \mathbb{R}.$$

For brevity, we write $c_0$ for the set of constants $(d, \alpha, \beta, \kappa_1, \kappa_2, \kappa_3, j_1, j_2)$. Before investigating the semigroups generated by $\mathcal{L}^{b,\eta}$, we need some facts about the heat kernels of $\mathcal{L}^\eta$ and $\mathcal{L}^{b,\eta}$.

By virtue of the periodicity assumptions on the coefficients, we can choose the underlying space to be $\mathbb{T}^d$ instead of $\mathbb{R}^d$ (cf. [3, Section 3.3.2]). Indeed, if $\mathfrak{q}^\eta(t; x, y) : [0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is the fundamental solution of $\mathcal{L}^\eta$, then for any test function $f \in \mathcal{C}^\infty(\mathbb{R}^d)$ that is periodic of period 1, the function $u(t, x) := \int_{\mathbb{R}^d} f(y)\mathfrak{q}^\eta(t; x, y)dy$ must be periodic in $x$, thanks to the Kolmogorov backward equation $\frac{\partial u}{\partial t} + \mathcal{L}^\eta u = 0$ and the periodicity of its initial value $u(0, x) = f(x)$ and of all coefficients in $\mathcal{L}^\eta$. Now we define $q^\eta(t; x, y) := \sum_{l\in\mathbb{Z}^d} \mathfrak{q}^\eta(t; x, y + l)$; then $q^\eta$ is periodic in $y$ and $u(t, x) = \int_{\mathbb{T}^d} f(y)q^\eta(t; x, y)dy$. Therefore, we can restrict $q^\eta$ to a function from $[0, \infty) \times \mathbb{T}^d \times \mathbb{T}^d$ to $\mathbb{R}$, which is exactly the fundamental solution of $\mathcal{L}^\eta$ on the state space $\mathbb{T}^d$. The same arguments hold for the operator $\mathcal{L}^{b,\eta}$. Keeping these in mind, the following facts about the operator $\mathcal{L}^\eta$ are adapted from Theorem 1.1, Theorem 1.2, Theorem 1.3, Theorem 1.4, Remark 1.5, and Lemma 3.17 in [20].

**Proposition 4.**

*(i) The fundamental solution $q^\eta(t; x, y) : [0, \infty) \times \mathbb{T}^d \times \mathbb{T}^d \to \mathbb{R}$ of $\mathcal{L}^\eta$ has the following properties: for all $(t, y) \in (0, \infty) \times \mathbb{T}^d$, the function $x \to q^\eta(t; x, y)$ is differentiable and the derivative $\nabla_x q^\eta(t; x, y)$ is jointly continuous on $(0, \infty) \times \mathbb{T}^d \times \mathbb{T}^d$; the integral in $\mathcal{L}_x^\eta q^\eta(t; x, y)$ is absolutely integrable and the function $\mathcal{L}_x^\eta q^\eta(t; x, y)$ is jointly continuous on $(0, \infty) \times \mathbb{T}^d \times \mathbb{T}^d$. For every $T > 0$, there exists a constant $C_1 = C_1(c_0, T) > 0$ such that for all $t \in (0, T]$ and $x, y \in \mathbb{T}^d$,*

$$q^\eta(t; x, y) \le C_1 \sum_{l\in\mathbb{Z}^d} \varrho_\alpha(t; x - y + l), \tag{A.2}$$

$$|\nabla_x q^\eta(t; x, y)| \le C_1 \sum_{l\in\mathbb{Z}^d} \varrho_{\alpha-1}(t; x - y + l), \tag{A.3}$$

$$|\mathcal{L}_x^\eta q^\eta(t; x, y)| \le C_1 \sum_{l\in\mathbb{Z}^d} \varrho_0(t; x - y + l); \tag{A.4}$$

*there exist $T_0 = T_0(c_0) > 0$ and $C_2 = C_2(c_0) > 0$ such that for all $t \in (0, T_0]$ and $x, y \in \mathbb{T}^d$,*

$$q^\eta(t; x, y) \ge C_2 \sum_{l\in\mathbb{Z}^d} \varrho_\alpha(t; x - y + l). \tag{A.5}$$

*(ii) Define a family of operators by*

$$T_t^\eta f(x) = \int_{\mathbb{T}^d} f(y)q^\eta(t; x, y)dy, \quad f \in \mathcal{C}\left(\mathbb{T}^d\right); \tag{A.6}$$

*then $\{T_t^\eta\}_{t\ge0}$ forms a Feller semigroup on the Banach space $(\mathcal{C}\left(\mathbb{T}^d\right), \|\cdot\|_\infty)$ with generator the closure of $\left(\mathcal{L}^\eta, \mathcal{C}^\infty\left(\mathbb{T}^d\right)\right)$. The domain of the generator contains $\mathcal{C}^{1+\gamma}\left(\mathbb{T}^d\right)$ with $1 + \gamma > \alpha$, on which the restriction of the generator is $\mathcal{L}^\eta$.*

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Note that the joint continuity of $\nabla_x q^\eta(t; x, y)$ is not mentioned explicitly in the previous references, but it is a consequence of [20, Lemma 3.1, Lemma 3.5, Theorem 3.7, Lemma 3.10, Equation (59)]. In addition, the above reference shows only that $\mathcal{C}^2(\mathbb{T}^d)$ is contained in the domain of the generator, but we can easily generalize to our case, using the same argument as the proofs of [20, Theorem 1.3(3a), Proposition 4.9] and the fact that $\mathcal{L}^\eta f \in \mathcal{C}(\mathbb{T}^d)$ for each $f \in \mathcal{C}^{1+\gamma}(\mathbb{T}^d)$ with $1+\gamma > \alpha$.

For notational simplicity, the summation over the lattice $\mathbb{Z}^d$ will be omitted in all subsequent results. Keep in mind that there will be a summation over $\mathbb{Z}^d$ whenever the letter $l$ is involved in the expression without ambiguity.

The following facts about the heat kernel of $\mathcal{L}^{b,\eta}$ are adapted from [11, Theorem 1.5], where the authors omitted the proofs, pointing out that they are similar to the proofs in [5]. Since the two-sided estimates of the heat kernel of $\mathcal{L}^{b,\eta}$ are important for later use and also for the main part of the paper, we will only elaborate on their proof.

**Proposition 5.** *There is a unique function $q^{b,\eta}(t; x, y)$ which is jointly continuous on $(0, \infty) \times \mathbb{T}^d \times \mathbb{T}^d$ and solves the following variation of parameters formula (or Duhamel's formula)*

$$q^{b,\eta}(t; x, y) = q^\eta(t; x, y) + \int_0^t \int_{\mathbb{T}^d} q^{b,\eta}(t-s; x, z) b(z) \cdot \nabla_z q^\eta(s; z, y) dz ds, \tag{A.7}$$

*and satisfying that for every $T > 0$, there is a constant $C = C(c_0, T, b_0) > 0$ such that on $(0, T] \times \mathbb{T}^d \times \mathbb{T}^d$,*

$$|q^{b,\eta}(t; x, y)| \le C \varrho_\alpha(t; x-y+l).$$

*Moreover, $q^{b,\eta}$ enjoys the following properties:*

*(i) (Conservativeness.) For all $t > 0$, $x \in \mathbb{T}^d$, $\int_{\mathbb{T}^d} q^{b,\eta}(t; x, y) dy = 1$.*

*(ii) (Chapman–Kolmogorov equation.) For all $s, t > 0$, $x, y \in \mathbb{T}^d$,*

$$\int_{\mathbb{T}^d} q^{b,\eta}(t; x, z) q^{b,\eta}(s; z, y) dz = q^{b,\eta}(t+s; x, y).$$

*(iii) (Two-sided estimates.) For every $T > 0$, there is a constant $C_3 = C_3(c_0, T, b_0) > 1$ such that on $(0, T] \times \mathbb{T}^d \times \mathbb{T}^d$,*

$$C_3^{-1} \varrho_\alpha(t; x-y+l) \le q^{b,\eta}(t; x, y) \le C_3 \varrho_\alpha(t; x-y+l).$$

*(iv) (Gradient estimate.) The function $\nabla_x q^{b,\eta}(t; x, y)$ is jointly continuous on $(0, \infty) \times \mathbb{T}^d \times \mathbb{T}^d$. For every $T > 0$, there is a constant $C_4 = C_4(c_0, T, b_0) > 0$ such that on $(0, T] \times \mathbb{T}^d \times \mathbb{T}^d$,*

$$|\nabla_x q^{b,\eta}(t; x, y)| \le C_4 \varrho_{\alpha-1}(t; x-y+l). \tag{A.8}$$

*Proof.* We follow the lines of [5, Theorem 2, Lemma 15] to prove (iii). We define a sequence of functions $\{q_n^\eta : (0, \infty) \times \mathbb{T}^d \times \mathbb{T}^d \mid n \in \mathbb{N}\}$ recursively by

$$q^{(0)}(t; x, y) := q^\eta(t; x, y),$$
$$q^{(n+1)}(t; x, y) := \int_0^t \int_{\mathbb{T}^d} q^{(n)}(t-s; x, z) b(z) \cdot \nabla_z q^\eta(s; z, y) dz ds, \quad n \in \mathbb{N}.$$

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By (A.2), (A.3), and [20, Equation (92), Lemma 5.17(c)], we have

$$\begin{aligned}
|q^{(1)}(t;x,y)| &\le \int_0^t \int_{\mathbb{T}^d} q^{\eta}(t-s;x,z)\left|b(z)\cdot\nabla_z q^{\eta}(s;z,y)\right| dzds \\
&\le C_1^2\|b\|_\infty \int_0^t \int_{\mathbb{R}^d} \varrho_\alpha(t-s;x-z+l)\varrho_{\alpha-1}(t;z-y)dzds \\
&\le C_1^2\|b\|_\infty B\big(\tfrac{\alpha}{2},\tfrac{\alpha-1}{2}\big)\varrho_{2\alpha-1}(s;x-y+l) \\
&\le C_1^2\|b\|_\infty B\big(\tfrac{\alpha}{2},\tfrac{\alpha-1}{2}\big)t^{\frac{\alpha-1}{\alpha}}\varrho_\alpha(t;x-y+l) \\
&=: c_1(c_0,T,t,b_0)\varrho_\alpha(t;x-y+l).
\end{aligned}$$

Note that the positive constant $c_1$ is increasing in $t$ since $\alpha \in (1,2)$. By iteration and (A.5), we obtain

$$|q^{(n)}(t;x,y)| \le [c_1(c_0,T,t,b_0)]^n\,\varrho_\alpha(t;x-y+l) \le [c_1(c_0,T,t,b_0)]^n\,C_2^{-1}q^{\eta}(t;x,y).$$

Choose $t_0 \le T_0$ small enough so that $c_1(c_0,T,t_0,b_0) \le \frac{C_2}{1+C_2}$ and $T = n_0 t_0$ for some $n_0 \in \mathbb{N}_+$. Then for all $(t,x,y) \in (0,t_0]\times\mathbb{T}^d\times\mathbb{T}^d$,

$$\begin{aligned}
\left(1-\frac{C_2^{-1}c_1(c_0,T,t_0,b_0)}{1-c_1(c_0,T,t_0,b_0)}\right)q^{\eta}(t;x,y) &\le q^{(0)}(t;x,y) - \sum_{n=1}^{\infty}|q^{(n)}(t;x,y)| \\
&\le \sum_{n=0}^{\infty} q^{(n)}(t;x,y) \\
&\le \sum_{n=0}^{\infty}|q^{(n)}(t;x,y)| \le \frac{C_2^{-1}}{1-c_1(c_0,T,t_0,b_0)}q^{\eta}(t;x,y).
\end{aligned}$$

Set

$$c_2(c_0,T,t_0,b_0) := \left(1-\frac{C_2^{-1}c_1(c_0,T,t_0,b_0)}{1-c_1(c_0,T,t_0,b_0)}\right)^{-1} \vee \frac{C_2^{-1}}{1-c_1(c_0,T,t_0,b_0)}.$$

An argument similar to that of [5, Section 3] yields that the series $\sum_{n=0}^{\infty} q^{(n)}$ converges on $(0,t_0]\times\mathbb{T}^d\times\mathbb{T}^d$ to $q^{b,\eta}$. So we get that for all $(t,x,y)\in(0,t_0]\times\mathbb{T}^d\times\mathbb{T}^d$,

$$(c_2(c_0,T,t_0,b_0))^{-1}q^{\eta}(t;x,y) \le q^{b,\eta}(t;x,y) \le c_2(c_0,T,t_0,b_0)q^{\eta}(t;x,y).$$

Now we apply (ii) and (A.5) to deduce that for any $t\in(0,T]$ and $x,y\in\mathbb{T}^d$,

$$\begin{aligned}
q^{b,\eta}(t;x,y) &= \int_{\mathbb{T}^d}\cdots\int_{\mathbb{T}^d}\int_{\mathbb{T}^d} q^{b,\eta}\big(\tfrac{t}{n_0};x,x_1\big)q^{b,\eta}\big(\tfrac{t}{n_0};x_1,x_2\big)\cdots q^{b,\eta}\big(\tfrac{t}{n_0};x_{n_0-1},y\big)dx^{(n_0-1)} \\
&\ge c_2^{-n_0}\int_{\mathbb{T}^d}\cdots\int_{\mathbb{T}^d}\int_{\mathbb{T}^d} q^{\eta}\big(\tfrac{t}{n_0};x,x_1\big)q^{\eta}\big(\tfrac{t}{n_0};x_1,x_2\big)\cdots q^{\eta}\big(\tfrac{t}{n_0};x_{n_0-1},y\big)dx^{(n_0-1)} \\
&= c_2^{-n_0}q^{\eta}(t;x,y) \ge c_2^{-T/t_0}C_2\varrho_\alpha(t;x-y+l),
\end{aligned}$$

where $dx^{(n_0-1)} := dx_1 dx_2 \cdots dx_{n_0-1}$, and similarly,

$$q^{b,\eta}(t;x,y) \le c_2^{T/t_0} C_1 \varrho_\alpha(t;x-y+l).$$

The result (iii) follows by taking $C_3(c_0,T,b_0) = (c_2(c_0,T,t_0,b_0))^{T/t_0}[C_1(c_0,T) \vee (C_2(c_0))^{-1}] > 1$. □

**Corollary 4.** *The following version of the variation-of-parameters formula holds:*

$$q^{b,\eta}(t;x,y) = q^\eta(t;x,y) + \int_0^t \int_{\mathbb{T}^d} q^\eta(t-s;x,z)b(z)\cdot\nabla_z q^{b,\eta}(s;z,y)dzds. \tag{A.9}$$

*The function $\mathcal{L}_x^{b,\eta} q^{b,\eta}(t;x,y)$ is jointly continuous on $(0,\infty)\times\mathbb{T}^d\times\mathbb{T}^d$. For every $T>0$, there is a constant $C_5 = C_5(c_0,T,b_0) > 0$ such that on $(0,T]\times\mathbb{T}^d\times\mathbb{T}^d$,*

$$|\mathcal{L}_x^{b,\eta} q^{b,\eta}(t;x,y)| \le C_5 \varrho_0(t;x-y+l). \tag{A.10}$$

*Proof.* The formula (A.9) follows from an argument similar to the proof of (A.7); cf. [9, Theorem 4.2]. We prove (A.10). Recall that $\mathcal{L}^{b,\eta} = \mathcal{L}^\eta + b\cdot\nabla$ and $\alpha > 1$. By (A.3), (A.4), and [20, Equation (92)], for all $(t,x,y) \in (0,T]\times\mathbb{T}^d\times\mathbb{T}^d$ we have

$$|\mathcal{L}_x^{b,\eta} q^\eta(t;x,y)| \le C(c_0,T,b_0)\varrho_0(t;x-y+l).$$

It follows from (A.8) and [20, Equation (92), Lemma 5.17(c)] that

$$\begin{aligned}
&\int_0^t \int_{\mathbb{T}^d} \left| \mathcal{L}_x^{b,\eta} q^\eta(t-s;x,z)b(z)\cdot\nabla_z q^{b,\eta}(s;z,y)\right| dzds \\
&\le C(c_0,T,b_0)\int_0^t\int_{\mathbb{R}^d} (t-s)\varrho_{-\alpha}(t-s;x-z+l)s\varrho_{-1}(s;z-y)dzds \\
&\le C(c_0,T,b_0)B\big(1-\tfrac{\alpha}{2},\tfrac{1}{2}\big)\varrho_{\alpha-1}(t;x-y+l) \\
&\le C(c_0,T,b_0)\varrho_0(t;x-y+l),
\end{aligned}$$

where $B$ is the beta function. Combining these estimates with (A.9), we get (A.10). The joint continuity of $\mathcal{L}_x^{b,\eta} q^{b,\eta}(t;x,y)$ follows from the joint continuity of $\mathcal{L}_x^\eta q^\eta(t;x,y)$, $\nabla_x q^\eta(t;x,y)$, and $\nabla_x q^{b,\eta}(t;x,y)$ and (A.9). □

Define a family of operators

$$T_t^{b,\eta} f = \int_{\mathbb{T}^d} q^{b,\eta}(t;\cdot,y)f(y)dy, \quad f \in \mathcal{C}\big(\mathbb{T}^d\big). \tag{A.11}$$

By Proposition 5, $\big\{T_t^{b,\eta}\big\}_{t\ge0}$ forms a (one-parameter operator) semigroup which is Markovian (positivity-preserving, conservative, and sub-Markovian) and Feller (each $T_t^{b,\eta}$ maps $\mathcal{C}\big(\mathbb{T}^d\big)$ to $\mathcal{C}\big(\mathbb{T}^d\big)$). We can also prove strong continuity. Hence we have the following result.

**Proposition 6.** *The family of operators $\big\{T_t^{b,\eta}\big\}_{t\ge0}$ forms a Feller semigroup on $\mathcal{C}\big(\mathbb{T}^d\big)$. Let $\big(\hat{\mathcal{L}}^{b,\eta}, D\big(\hat{\mathcal{L}}^{b,\eta}\big)\big)$ be the generator; then for all $\gamma > \alpha - 1$, $\mathcal{C}^{1+\gamma}\big(\mathbb{T}^d\big) \subset D(\hat{\mathcal{L}}^{b,\eta})$ and $\hat{\mathcal{L}}^{b,\eta} = \mathcal{L}^{b,\eta}$ on $\mathcal{C}^{1+\gamma}\big(\mathbb{T}^d\big)$. Moreover, $\mathcal{C}^\infty\big(\mathbb{T}^d\big)$ is a core of $\hat{\mathcal{L}}^{b,\eta}$.*

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*Proof.* (i) Fix $f \in \mathcal{C}(\mathbb{T}^d)$. For every $\epsilon > 0$, there is a constant $\delta > 0$ such that $|f(x) - f(y)| < \epsilon$ with $|x - y| < \delta$, $x, y \in \mathbb{T}^d$. Then by Parts (i) and (iii) of Proposition 5,

$$\begin{aligned}
&\sup_x \left| T_t^{b,\eta} f(x) - f(x) \right| \\
&\le \sup_x \int_{\mathbb{T}^d} q^{b,\eta}(t; x, y) |f(y) - f(x)| dy \\
&\le \epsilon \sup_x \int_{\substack{|x-y|<\delta \\ y \in \mathbb{T}^d}} q^{b,\eta}(t; x, y) dy + 2\|f\|_\infty \sup_x \int_{\substack{|x-y|\ge\delta \\ y \in \mathbb{T}^d}} \varrho_\alpha(t; x - y + l) dy \\
&\le \epsilon + 2\|f\|_\infty t \int_{|z|\ge\delta} \left( t^{-(d+\alpha)/\alpha} \wedge |z|^{-(d+\alpha)} \right) dz.
\end{aligned}$$

When $t \to 0^+$,

$$\int_{|z|\ge\delta} \left( t^{-(d+\alpha)/\alpha} \wedge |z|^{-(d+\alpha)} \right) dz \le \int_{|z|\ge\delta} |z|^{-(d+\alpha)} dz < \infty,$$

and then $\|T_t^{b,\eta} f - f\|_\infty \to 0$. This proves that $\{T_t^{b,\eta}\}_{t\ge0}$ is strongly continuous on $\mathcal{C}(\mathbb{T}^d)$. Thus, $\{T_t^{b,\eta}\}_{t\ge0}$ is a Feller semigroup.

(ii) To identify the generator of $\{T_t^{b,\eta}\}_{t\ge0}$, we fix $f \in \mathcal{C}^{1+\gamma}(\mathbb{T}^d)$ with $1 + \gamma > \alpha$. We claim that for every $g \in \mathcal{C}^\infty(\mathbb{T}^d)$,

$$\lim_{t\to0} \int_{\mathbb{T}^d} \frac{1}{t} \left( T_t^{b,\eta} f(x) - f(x) \right) g(x) dx = \int_{\mathbb{T}^d} \mathcal{L}^{b,\eta} f(x) g(x) dx. \tag{A.12}$$

Then, using [14, Theorem 1.24] and the fact that $\mathcal{C}^\infty(\mathbb{T}^d)$ is vaguely (i.e., weak-$*$) dense in the space $\mathcal{M}_b(\mathbb{T}^d)$ of all bounded signed Radon measures on $\mathbb{T}^d$, which is the topological dual of $\mathcal{C}(\mathbb{T}^d)$, we get that $\mathcal{C}^{1+\gamma}(\mathbb{T}^d)$ is contained in the domain of $\hat{\mathcal{L}}^{b,\eta}$, and the restriction of $\hat{\mathcal{L}}^{b,\eta}$ on $\mathcal{C}^{1+\gamma}(\mathbb{T}^d)$ equals $\mathcal{L}^{b,\eta}$.

Now we prove the claim (A.12). By (A.6), (A.11), and (A.7) we have

$$\begin{aligned}
&\int_{\mathbb{T}^d} \frac{1}{t} \left( T_t^{b,\eta} f(x) - f(x) \right) g(x) dx - \int_{\mathbb{T}^d} \mathcal{L}^{b,\eta} f(x) g(x) dx \\
&= \int_{\mathbb{T}^d} \left[ \frac{1}{t} \left( T_t^{\eta} f(x) - f(x) \right) - \mathcal{L}^{\eta} f(x) \right] g(x) dx \\
&\quad + \frac{1}{t} \int_{\mathbb{T}^d} \left( \int_{\mathbb{T}^d} \int_0^t \int_{\mathbb{T}^d} q^{b,\eta}(t - s; x, z) b(z) \cdot \nabla_z q^{\eta}(s; z, y) f(y) dz ds dy - b(x) \cdot \nabla f(x) \right) g(x) dx \\
&=: I + II.
\end{aligned}$$

The term $I$ goes to zero, by Proposition 4(ii), as $t \to 0$. For the term $II$, we use Fubini's theorem and integration by parts, which we can do by the periodicity of $b$, $f$, $g$, $x \to q^{\eta}(t; x, y)$, and $x \to q^{b,\eta}(t; x, y)$; then we get

$$\begin{aligned}
II &= \frac{1}{t} \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} q^{b,\eta}(t - s; x, z) g(x) \left[ b(z) \cdot \left( \int_{\mathbb{T}^d} \nabla_z q^{\eta}(s; z, y) f(y) dy \right) - b(x) \cdot \nabla f(x) \right] dx dz ds \\
&= \frac{1}{t} \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} q^{b,\eta}(t - s; x, z) g(x) \int_{\mathbb{T}^d} \left[ b(z) \cdot \nabla_z q^{\eta}(s; z, y) - b(x) \cdot \nabla_x q^{\eta}(s; x, y) \right] f(y) dy dx dz ds
\end{aligned}$$

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$$
\begin{aligned}
&+\frac{1}{t}\int_0^t\int_{\mathbb{T}^d}\int_{\mathbb{T}^d} q^{b,\eta}(t-s;x,z)g(x)b(x)\cdot\nabla_x\left[\int_{\mathbb{T}^d} q^{\eta}(s;x,y)f(y)dy-f(x)\right]dxdzds\\
=:&\ II_1+\frac{1}{t}\int_0^t\int_{\mathbb{T}^d}\int_{\mathbb{T}^d}\nabla_x\left(q^{b,\eta}(t-s;x,z)g(x)b(x)\right)\cdot\left[\int_{\mathbb{T}^d} q^{\eta}(s;x,y)f(y)dy-f(x)\right]dxdzds\\
=:&\ II_1+II_2.
\end{aligned}
$$

Since the function $(s,x,y)\to b(x)\cdot\nabla_x q^\eta(s;x,y)$ is uniformly continuous on $[0,t]\times\mathbb{T}^d\times\mathbb{T}^d$, there exists a constant $C>0$ such that $|b(x)\cdot\nabla_x q^\eta(s;x,y)|<C$ for all $(s,x,y)\in[0,t]\times\mathbb{T}^d\times\mathbb{T}^d$; and for every $\epsilon>0$, there is $\delta>0$ such that $|b(z)\cdot\nabla_x q^\eta(s;z,y)-b(x)\cdot\nabla_x q^\eta(s;x,y)|<\epsilon$ for $|x-z|<\delta$. Then by Proposition 5(iii), for $t\to 0$,

$$
\begin{aligned}
|II_1|&\le\|f\|_\infty\|g\|_\infty\Bigg(\epsilon\frac{1}{t}\int_0^t\iint_{\substack{|x-z|<\delta\\ x,z\in\mathbb{T}^d}} q^{b,\eta}(t-s;x,z)dxdzds\\
&\qquad\qquad+2c\frac{1}{t}\int_0^t\iint_{\substack{|x-z|\ge\delta\\ x,z\in\mathbb{T}^d}} q^{b,\eta}(t-s;x,z)dxdzds\Bigg)\\
&\le\|f\|_\infty\|g\|_\infty\left(\epsilon+2C\frac{1}{t}\int_0^t\int_{\mathbb{T}^d}\int_{|y|\ge\delta}\rho_\alpha(t;y)dydzds\right)\\
&\le\|f\|_\infty\|g\|_\infty\left(\epsilon+2Ct\int_{|y|\ge\delta}|y|^{-(d+\alpha)}dy\right)\\
&\to\epsilon\|f\|_\infty\|g\|_\infty.
\end{aligned}
$$

Since $\epsilon>0$ is arbitrary, $II_1\to 0$ as $t\to 0$. Moreover, the strong continuity of the semigroup $\{T_t\}_{t\ge 0}$ and dominated convergence imply that $II_2\to 0$ as $t\to 0$. Thus, we get (A.12). For more general results of domains and representations of generators of Feller processes on $\mathbb{R}^d$, we refer the readers to [29] and references therein.

(iii) Finally, we prove that $\mathcal{C}^\infty(\mathbb{T}^d)$ is a core of the generator. We divide this proof into three steps.

*Step 1:* We prove that for every $f\in\mathcal{C}(\mathbb{T}^d)$ and all $t>0$, $T_t^{b,\eta}f$ is differentiable and the integral in $\mathcal{L}^{b,\eta}T_t^{b,\eta}f\in\mathcal{C}(\mathbb{T}^d)$ is absolutely integrable, and for all $x\in\mathbb{T}^d$,

$$
\nabla T_t^{b,\eta}f(x)=\int_{\mathbb{T}^d}\nabla_x q^{b,\eta}(t;x,y)f(y)dy, \tag{A.13}
$$

$$
\mathcal{L}^{b,\eta}T_t^{b,\eta}f(x)=\int_{\mathbb{T}^d}\mathcal{L}_x^{b,\eta}q^{b,\eta}(t;x,y)f(y)dy. \tag{A.14}
$$

Using the estimate (A.8) and writing the derivative as the limit of a difference quotient, we obtain (A.13) by dominated convergence. Furthermore, (A.14) follows from (A.13) and Fubini's theorem. The continuity of the function $\mathcal{L}^{b,\eta}T_t^{b,\eta}f$ follows from the joint continuity of $\mathcal{L}_x^{b,\eta}q^{b,\eta}(t;x,y)$ and (A.14).

*Step 2:* Since the semigroup $\{T_t^{b,\eta}\}$ is Feller, its generator $\left(\hat{\mathcal{L}}^{b,\eta},D(\hat{\mathcal{L}}^{b,\eta})\right)$ is closed in $\mathcal{C}(\mathbb{T}^d)$ (see [8, Definition 1.24]). By (i), we see that $\left(\mathcal{L}^{b,\eta},\mathcal{C}^\infty(\mathbb{T}^d)\right)\subset\left(\hat{\mathcal{L}}^{b,\eta},D(\hat{\mathcal{L}}^{b,\eta})\right)$, whence the former is closable in $\mathcal{C}(\mathbb{T}^d)$. Define $\left(\bar{\mathcal{L}}^{b,\eta},\bar{D}\right):=\overline{\left(\mathcal{L}^{b,\eta},\mathcal{C}^\infty(\mathbb{T}^d)\right)}$. In this step, we show that for every $f\in\mathcal{C}(\mathbb{T}^d)$ and all $t>0$, $T_t^{b,\eta}f\in\bar{D}$ and $\bar{\mathcal{L}}^{b,\eta}T_t^{b,\eta}f=\mathcal{L}^{b,\eta}T_t^{b,\eta}f$.

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Let $\{\phi_n\}_{n\in\mathbb{N}}$ be a standard mollifier such that $\mathrm{supp}(\phi_n)\subset B(0, 1/n)$. Then $T_t^{b,\eta}f*\phi_n\in\mathcal{C}^\infty(\mathbb{T}^d)$ and $\|T_t^{b,\eta}f*\phi_n-T_t^{b,\eta}f\|_\infty\to 0$ as $n\to\infty$. By the definition of the closure $(\bar{\mathcal{L}}^{b,\eta},\bar{D})$, it suffices to show that $\|\mathcal{L}^{b,\eta}(T_t^{b,\eta}f*\phi_n)-\mathcal{L}^{b,\eta}T_t^{b,\eta}f\|_\infty\to 0$ as $n\to\infty$. Using (A.13), (A.14), (A.8), (A.10), [20, Lemma 5.17(a)], and Fubini's theorem, we have

$$
\begin{aligned}
&\left|\mathcal{L}^{b,\eta}\big(T_t^{b,\eta}f*\phi_n\big)(x)-\big(\mathcal{L}^{b,\eta}T_t^{b,\eta}f\big)*\phi_n(x)\right|\\
&\le\left|\int_{\mathbb{R}^d}(b(x)-b(x-y))\cdot\nabla T_t^{b,\eta}f(x-y)\phi_n(y)dy\right|\\
&\quad+\left|\int_{\mathbb{R}^d}\int_{\mathbb{R}^d\setminus\{0\}}\Big[T_t^{b,\eta}f(x-y+z)-T_t^{b,\eta}f(x-y)-z\cdot\nabla T_t^{b,\eta}f(x-y)\mathbf{1}_B(z)\Big]\right.\\
&\qquad\qquad\left.\times\big(\kappa^\sharp(x,z)-\kappa^\sharp(x-y,z)\big)J(z)\phi_n(y)dydz\right|\\
&\le\frac{1}{n^\beta}\|b\|_{\mathcal{C}^\beta}\|\nabla T_t^{b,\eta}f\|_\infty+\frac{1}{n^\beta}\frac{\kappa_3}{\kappa_1}\|\mathcal{L}^{b,\eta}T_t^{b,\eta}f\|_\infty\\
&\le\frac{1}{n^\beta}C(c_0,T,b_0)\|f\|_\infty\left(\|b\|_{\mathcal{C}^\beta}t^{-\frac{1}{\alpha}}+\frac{\kappa_3}{\kappa_1}t^{-1}\right).
\end{aligned}
$$

Let $n\to\infty$; we get $\|\mathcal{L}^{b,\eta}(T_t^{b,\eta}f*\phi_n)-(\mathcal{L}^{b,\eta}T_t^{b,\eta}f)*\phi_n\|_\infty\to 0$. Since $\mathcal{L}^{b,\eta}T_t^{b,\eta}f\in\mathcal{C}(\mathbb{T}^d)$ by Step 1, $(\mathcal{L}^{b,\eta}T_t^{b,\eta}f)*\phi_n\to\mathcal{L}^{b,\eta}T_t^{b,\eta}f$ in $\mathcal{C}(\mathbb{T}^d)$ as $n\to\infty$. Thus, we have $\|\mathcal{L}^{b,\eta}(T_t^{b,\eta}f*\phi_n)-\mathcal{L}^{b,\eta}T_t^{b,\eta}f\|_\infty\to 0$ as $n\to\infty$, which completes this step.

*Step 3:* By construction, we have $(\bar{\mathcal{L}}^{b,\eta},\bar{D})\subset(\hat{\mathcal{L}}^{b,\eta},D(\hat{\mathcal{L}}^{b,\eta}))$. Now we show the converse; that is, for an arbitrary $f\in D(\hat{\mathcal{L}}^{b,\eta})$, we show that $f\in\bar{D}$ and $\bar{\mathcal{L}}^{b,\eta}f=\hat{\mathcal{L}}^{b,\eta}f$. Let $f_n=T_{1/n}^{b,\eta}f$. Since $\|f_n-f\|_\infty\to 0$ as $n\to\infty$, by the definition of the closure $(\bar{\mathcal{L}}^{b,\eta},\bar{D})$, we only need to show $\|\bar{\mathcal{L}}^{b,\eta}f_n-\hat{\mathcal{L}}^{b,\eta}f\|_\infty\to 0$. From Step 2, we have $f_n\in\bar{D}$ and $\bar{\mathcal{L}}^{b,\eta}f_n=\hat{\mathcal{L}}^{b,\eta}f_n$. It follows that

$$\|\bar{\mathcal{L}}^{b,\eta}f_n-\hat{\mathcal{L}}^{b,\eta}f\|_\infty=\|\hat{\mathcal{L}}^{b,\eta}f_n-\hat{\mathcal{L}}^{b,\eta}f\|_\infty=\|T_{1/n}^{b,\eta}\hat{\mathcal{L}}^{b,\eta}f-\hat{\mathcal{L}}^{b,\eta}f\|_\infty\to 0,\qquad n\to\infty.$$

This gives $(\hat{\mathcal{L}}^{b,\eta},D(\hat{\mathcal{L}}^{b,\eta}))\subset(\bar{\mathcal{L}}^{b,\eta},\bar{D})$ and thus $\bar{\mathcal{L}}^{b,\eta}=\hat{\mathcal{L}}^{b,\eta}$, which completes the whole proof. □

## Appendix B. SDEs and non-local PDEs

The following result is a consequence of the nature of Feller semigroups (see [30, Theorem 2.3, Corollary 2.5] and [16, Theorem 4.4.1, Proposition 4.1.7]).

**Corollary 5.** *The canonical Feller process $(X^{b,\eta};(\Omega,\mathcal{F},\mathbb{P}))$ corresponding to $\{T_t^{b,\eta}\}_{t\ge 0}$ with càdlàg trajectories is the unique solution to the martingale problem for $(\mathcal{L}^{b,\eta},\mathbb{P}\circ(X_0^{b,\eta})^{-1})$, and also the unique weak solution to the following SDE:*

$$
\begin{aligned}
dX_t=b(X_t)dt&+\int_0^\infty\int_{B\setminus\{0\}}\mathbf{1}_{[0,\kappa^\sharp(X_{t-},z)]}(r)z\tilde{N}(dz,dr,dt)\\
&+\int_0^\infty\int_{B^c}\mathbf{1}_{[0,\kappa^\sharp(X_{t-},z)]}(r)zN(dz,dr,dt),
\end{aligned}
\tag{B.1}
$$

*where $N$ is a Poisson random measure on $\mathbb{R}^d\times[0,\infty)\times[0,\infty)$ with intensity measure $J(z)dz\times m\times m$ and $\tilde{N}$ is the associated compensated Poisson random measure.*

We have a generalized version of Itô's formula, as follows. The proof is similar to that of [40, Lemma 3.4] and is therefore omitted.

**Lemma 6.** *Let $f \in \mathcal{C}^{1+\gamma}(\mathbb{T}^d)$ with $1+\gamma > \alpha$. If $X$ satisfies the SDE (B.1), then*

$$f(X_t) - f(X_0) = \int_0^t \mathcal{L}^{b,\eta} f(X_s) ds \\ + \int_0^t \int_0^\infty \int_{\mathbb{R}^d \setminus \{0\}} \left[ f(X_{s-} + \mathbf{1}_{[0,\kappa^\sharp(X_{s-},z)]}(r) z) - f(X_{s-}) \right] \tilde{N}(dz, dr, ds).$$

We can solve the non-local Poisson equation with zeroth-order term using the semigroup representation.

**Corollary 6.** *For every $f \in \mathcal{C}^\beta(\mathbb{T}^d)$ and $\lambda > 0$, there exists a unique classical solution $u \in \mathcal{C}^{\alpha+\beta}(\mathbb{T}^d)$ to the Poisson equation*

$$\lambda u - \mathcal{L}^{b,\eta} u = f. \tag{B.2}$$

*Proof.* We first prove that if $u_\lambda \in \mathcal{C}^{\alpha+\beta}(\mathbb{T}^d)$ is a solution of (B.2), then $u_\lambda$ must have the representation

$$u_\lambda(x) = \int_0^\infty e^{-\lambda t} T_t^{b,\eta} f(x) dt, \tag{B.3}$$

and there exists a constant $C = C(c_0, b_0, \lambda) > 0$ not depending on $f$ such that

$$\|u_\lambda\|_{\mathcal{C}^{\alpha+\beta}} \le C \|f\|_{\mathcal{C}^\beta}. \tag{B.4}$$

Since the restriction of the generator $\hat{\mathcal{L}}^{b,\eta}$ on $\mathcal{C}^{\alpha+\beta}(\mathbb{T}^d)$ is $\mathcal{L}^{b,\eta}$, we have

$$\int_0^\infty e^{-\lambda t} T_t^{b,\eta} f dt = \int_0^\infty e^{-\lambda t} T_t^{b,\eta} (\lambda u_\lambda - \mathcal{L}^{b,\eta} u_\lambda) dt = -\int_0^\infty \frac{d}{dt} \left( e^{-\lambda t} T_t^{b,\eta} u_\lambda \right) dt \\ = u_\lambda - \lim_{t \to \infty} e^{-\lambda t} T_t^{b,\eta} u_\lambda = u_\lambda,$$

where we have used the fact that $\|e^{-\lambda t} T_t^{b,\eta} u_\lambda\|_\infty \le e^{-\lambda t} \|u_\lambda\|_\infty \to 0$ as $t \to \infty$. This gives (B.3) and the uniqueness follows. Further, using the Schauder-type estimates in [2, Theorem 7.1, Theorem 7.2], there exists a constant $C = C(c_0, b_0, \lambda) > 0$ such that

$$\|u_\lambda\|_{\mathcal{C}^{\alpha+\beta}} \le C(\|u_\lambda\|_\infty + \|f\|_{\mathcal{C}^\beta}).$$

The representation (B.3) yields that

$$\|u_\lambda\|_\infty \le \|f\|_\infty \int_0^\infty e^{-\lambda t} dt = \frac{1}{\lambda} \|f\|_\infty.$$

The estimate (B.4) follows.

Moreover, it is shown in [33, Theorem 3.4] that when the function $\kappa^\sharp$ is a constant, the existence and uniqueness hold in $\mathcal{C}^{\alpha+\beta}(\mathbb{T}^d)$. We can now obtain the existence of (B.2) via the energy estimate (B.4) and the method of continuity (see [19, Section 5.2]; also cf. [23, Theorem 3.2]). □

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## Competing interests

There were no competing interests to declare which arose during the preparation or publication process of this article.

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