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THE FINITE BASIS PROBLEM FOR INVOLUTION SEMIGROUPS OF TRIANGULAR 2 × 2 MATRICES

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Abstract

Let $T_n(\mathbb{F})$ be the semigroup of all upper triangular $n \times n$ matrices over a field $\mathbb{F}$. Let $UT_n(\mathbb{F})$ and $UT_n^{\pm 1}(\mathbb{F})$ be subsemigroups of $T_n(\mathbb{F})$, respectively, having 0s and/or 1s on the main diagonal and 0s and/or ±1s on the main diagonal. We give some sufficient conditions under which an involution semigroup is nonfinitely based. As an application, we show that $UT_2(\mathbb{F})$, $UT_2^{\pm 1}(\mathbb{F})$ and $T_2(\mathbb{F})$ as involution semigroups under the skew transposition are nonfinitely based for any field $\mathbb{F}$.

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1. Introduction

An algebra is *finitely based* if the identities it satisfies are finitely axiomatisable. Otherwise it is *nonfinitely based*. One of the most fundamental and widely stated problems in universal algebra is the finite basis problem, which asks which algebras admit a finite basis for their identities. This problem has revealed a number of interesting and unexpected connections to other topics of theoretical and practical importance, for example feasible algorithms for membership in certain classes of formal languages [1] and classical number-theoretic conjectures such as the twin-prime conjecture, Goldbach's conjecture and the odd perfect number conjecture [19].

In the 1960s, Tarski asked if there is an algorithm to determine whether a finite algebra is finitely based or not. It is known that finite groups [17], finite associative rings [5], finite Lie algebras [10, 13] and finite lattices [14] are all finitely based. However, this is not true for all finite algebras. There exist groupoids with as few as three elements that are nonfinitely based [9, 16]. In general, the finite basis problem for finite algebras is undecidable [15], but the problem remains open when restricted to some important classes of algebraic structures, such as semigroups. In 1969,

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Perkins [18] published the first examples of nonfinitely based finite semigroups. For a survey of the many important and interesting results on the finite basis problem for semigroups obtained since then, see the survey by Volkov [21].

Recall that a unary operation * on a semigroup S is called an *involution* if S satisfies the identities

$$(x^*)^* \approx x, \quad (xy)^* \approx y^* x^*. \tag{1.1}$$

An *involution semigroup* is a pair $(S, ^*)$, where S is a semigroup with involution *. It is a routine consequence of (1.1) that $1^* = 1$ in an involution monoid. Important examples of involution semigroups include any group with inversion $^{-1}$ and the multiplicative $n \times n$ matrix semigroup over any field with transposition ^{T}. The motivation for investigating involution semigroups is to study natural generalisations of groups. Therefore, it is counterintuitive that an involution semigroup $(S, ^*)$ and its semigroup reduct S can behave independently with respect to the finite basis property. Many examples demonstrating this phenomenon have long been available (see [2–4, 11] for more information on the finite basis problem for involution semigroups).

Matrices and matrix operations constitute basic tools for many branches of mathematics. The finite basis problem for (involution) semigroups formed by matrices was originally stated in [20]. The more recent review [21] repeated the question and stimulated much attention, with a number of important results having been subsequently established [3, 12, 23]. Given a field $\mathbb{F}$, let $T_n(\mathbb{F})$ be the semigroup of all upper triangular $n \times n$ matrices. Let $UT_n(\mathbb{F})$ and $UT_n^{\pm 1}(\mathbb{F})$ be the subsemigroups of $T_n(\mathbb{F})$, respectively, having 0s and/or 1s on the main diagonal and 0s and/or ± 1s on the main diagonal. These three semigroups admit a natural unary operation: reflection with respect to the secondary diagonal (from the top right to the bottom left corner). We denote by A^* the result of applying this operation to the matrix A. Clearly, the unary operation $A \mapsto A^*$, called *the skew transposition*, is an involution anti-automorphism of $T_n(\mathbb{F})$, $UT_n(\mathbb{F})$ and $UT_n^{\pm 1}(\mathbb{F})$, which makes $(T_n(\mathbb{F}), ^*)$, $(UT_n(\mathbb{F}), ^*)$ and $(UT_n^{\pm 1}(\mathbb{F}), ^*)$ involution semigroups. Since $T_n(\mathbb{Z})$ contains a subsemigroup isomorphic to the free semigroup with two generators, $T_2(\mathbb{Z})$ as well as the semigroup containing $T_2(\mathbb{Z})$ as subsemigroup satisfies only the trivial identity and so is finitely based. Therefore, $T_n(\mathbb{F})$ is finitely based whenever $\mathsf{char}(\mathbb{F}) = 0$. In [23], Volkov and Goldberg proved that $T_n(\mathbb{F})$ over a finite field is inherently nonfinitely based (that is, it is not contained in any finitely based locally finite variety) whenever $n \geq 4$ and $\mathsf{char}(\mathbb{F}) > 2$. Zhang *et al.* [25] have shown that $T_2(\mathbb{F})$ with $\mathsf{char}(\mathbb{F}) = 2$ is finitely based. Further, they proved that $T_n(\mathbb{F})$ with $\mathsf{char}(\mathbb{F}) = 2$ is hereditarily finitely based (that is, it generates a variety all semigroups of which are finitely based) if and only if $n \leq 2$ in [26]. Chen *et al.* [7] have shown that $UT_2(\mathbb{F})$ is finitely based for any field $\mathbb{F}$. Zhang *et al.* [24] have shown that the involution semigroups $(T_2(\mathbb{F}), ^*)$ and $(UT_2(\mathbb{F}), ^*)$ with $\mathsf{char}(\mathbb{F}) = 0$ are nonfinitely based. In [22], Volkov proved that $UT_3(\mathbb{F})$ as well as $(UT_3(\mathbb{F}), ^*)$ with $\mathsf{char}(\mathbb{F}) = 0$ are nonfinitely based.

In this paper, we consider the finite basis problem for the involution semigroups $(T_2(\mathbb{F}), ^*)$, $(UT_2(\mathbb{F}), ^*)$ and $(UT_2^{\pm 1}(\mathbb{F}), ^*)$. We establish two sufficient conditions under

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which an involution monoid is nonfinitely based. By applying one of the sufficient conditions to the involution semigroups $(T_2(\mathbb{F}), {}^*), (UT_2(\mathbb{F}), {}^*)$ and $(UT_2^{\pm1}(\mathbb{F}), {}^*)$, we show that $(T_2(\mathbb{F}), {}^*)$ and $(UT_2(\mathbb{F}), {}^*)$ are nonfinitely based when $\mathsf{char}(\mathbb{F}) \neq 0$, and $(UT_2^{\pm1}(\mathbb{F}), {}^*)$ is nonfinitely based for any field $\mathbb{F}$. Together with the results of [24], this shows that $(T_2(\mathbb{F}), {}^*), (UT_2(\mathbb{F}), {}^*)$ and $(UT_2^{\pm1}(\mathbb{F}), {}^*)$ are nonfinitely based for any field $\mathbb{F}$.

The paper is organised as follows. We recall some preliminary knowledge and notation in Section 2. In Section 3, some sufficient conditions under which an involution semigroup is nonfinitely based will be given. As an application, in Section 4, it is shown that $(T_2(\mathbb{F}), {}^*)$ and $(UT_2(\mathbb{F}), {}^*)$ are nonfinitely based when $\mathsf{char}(\mathbb{F}) \neq 0$, and $(UT_2^{\pm1}(\mathbb{F}), {}^*)$ is nonfinitely based for any field $\mathbb{F}$.

2. Preliminaries

Most of the notation and background material of this article are given in this section. Refer to the monograph of Burris and Sankappanavar [6] for any undefined notation and terminology on universal algebra in general.

Let $\mathcal{X}$ be a countably infinite alphabet throughout. For any subset $\mathcal{A}$ of $\mathcal{X}$, let $\mathcal{A}^* = \{x^* \mid x \in \mathcal{A}\}$ be a disjoint copy of $\mathcal{A}$. Elements of $\mathcal{A} \cup \mathcal{A}^*$ are called *variables*. The *free $*$-semigroup* over $\mathcal{A}$ is the free semigroup $(\mathcal{A} \cup \mathcal{A}^*)^+$ with unary operation * given by $(x^*)^* = x$ for all $x \in \mathcal{A}$ and $(x_1x_2\cdots x_m)^* = x_m^*x_{m-1}^*\cdots x_1^*$ for all $x_1, x_2, \ldots, x_m \in \mathcal{A} \cup \mathcal{A}^*$. Let $(\mathcal{A} \cup \mathcal{A}^*)^\times = (\mathcal{A} \cup \mathcal{A}^*)^+ \cup \{\emptyset\}$. Elements of $(\mathcal{A} \cup \mathcal{A}^*)^\times$ are called *words*. A word $\mathbf{w}'$ is a *factor* of a word $\mathbf{w}$ if $\mathbf{w} = \mathbf{a}\mathbf{w}'\mathbf{b}$ for some $\mathbf{a}, \mathbf{b} \in (\mathcal{A} \cup \mathcal{A}^*)^\times$.

Let $\mathbf{w} \in (\mathcal{A} \cup \mathcal{A}^*)^+$ be any word. The *content* of $\mathbf{w}$, denoted by $\mathsf{con}(\mathbf{w})$, is the set of variables occurring in $\mathbf{w}$. The number of times that a variable $x \in \mathcal{A} \cup \mathcal{A}^*$ occurs in a word $\mathbf{w}$ is denoted by $\mathsf{occ}(x, \mathbf{w})$. For example, let $\mathbf{w} = x^*zxy^*xyzyx^*$. Then $\mathsf{con}(\mathbf{w}) = \{x, y, z, x^*, y^*\}$, $\mathsf{occ}(x, \mathbf{w}) = \mathsf{occ}(x^*, \mathbf{w}) = 2, \mathsf{occ}(y, \mathbf{w}) = \mathsf{occ}(z, \mathbf{w}) = 2$ and $\mathsf{occ}(y^*, \mathbf{w}) = 1$.

For any variables x and y of a word $\mathbf{w}$, the expression ${}_fx$ (respectively ${}_lx$) means the first (respectively last) occurrence of x in the word $\mathbf{w}$. We write ${}_ix \prec {}_jy$ to indicate that within $\mathbf{w}$, the ith occurrence of x precedes the jth occurrence of y in $\mathbf{w}$ for $i, j \in \{f, l\}$.

For any word $\mathbf{w} \in (\mathcal{A} \cup \mathcal{A}^*)^+$ and variables $x_1, x_2, \ldots, x_n \in \mathcal{A}$, let $\mathbf{w}(x_1, x_2, \ldots, x_n)$ be the word obtained from $\mathbf{w}$ by retaining only the variables $x_1, x_1^*, x_2, x_2^*, \ldots, x_n, x_n^*$. For instance, if $\mathbf{w} = x^*zxy^*xyzyx^*$, then $\mathbf{w}(x) = x^*x^2x^*$, $\mathbf{w}(x, y) = x^*xy^*xy^2x^*$ and $\mathbf{w}(y, z) = zy^*yzy$.

The set $\mathsf{T}(\mathcal{A})$ of *terms* over the alphabet $\mathcal{A}$ is the smallest set containing $\mathcal{A}$ that is closed under concatenation and *. The *subterms* of a term $\mathbf{t}$ are defined as follows: $\mathbf{t}$ is a subterm of $\mathbf{t}$; if $\mathbf{s}_1\mathbf{s}_2$ is a subterm of $\mathbf{t}$, where $\mathbf{s}_1, \mathbf{s}_2 \in \mathsf{T}(\mathcal{A})$, then $\mathbf{s}_1$ and $\mathbf{s}_2$ are subterms of $\mathbf{t}$; if $\mathbf{s}^*$ is a subterm of $\mathbf{t}$, where $\mathbf{s} \in \mathsf{T}(\mathcal{A})$, then $\mathbf{s}$ is a subterm of $\mathbf{t}$. The proper inclusion $(\mathcal{A} \cup \mathcal{A}^*)^\times \subsetneq \mathsf{T}(\mathcal{A})$ holds and the identities (1.1) can be used to convert any nonempty term $\mathbf{t} \in \mathsf{T}(\mathcal{A})$ into some unique word $\lfloor \mathbf{t} \rfloor \in (\mathcal{A} \cup \mathcal{A}^*)^+$. For instance, $\lfloor x(x^2(yx^*)^*)^*zy^* \rfloor = xy(x^*)^3zy^*$.

Remark 2.1. For any subterm $\mathbf{s}$ of a term $\mathbf{t}$, either $\lfloor \mathbf{s} \rfloor$ or $\lfloor \mathbf{s}^* \rfloor$ is a factor of $\lfloor \mathbf{t} \rfloor$.

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An *identity* is an expression $\mathbf{s} \approx \mathbf{t}$ formed by nonempty terms $\mathbf{s}, \mathbf{t} \in \mathsf{T}(\mathcal{A})$. A *word identity* is an identity $\mathbf{u} \approx \mathbf{v}$ formed by words $\mathbf{u}, \mathbf{v} \in (\mathcal{A} \cup \mathcal{A}^*)^+$. An identity $\mathbf{s} \approx \mathbf{t}$ is *directly deducible* from an identity $\mathbf{s}' \approx \mathbf{t}'$ if there exists some substitution $\varphi : \mathcal{A} \to \mathsf{T}(\mathcal{A})$ such that $\mathbf{s}'\varphi$ is a subterm of $\mathbf{s}$, and replacing this particular subterm $\mathbf{s}'\varphi$ of $\mathbf{s}$ with $\mathbf{t}'\varphi$ results in the term $\mathbf{t}$. An identity $\mathbf{s} \approx \mathbf{t}$ is *deducible* from some set Σ of identities if there exists a sequence $\mathbf{s} = \mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_r = \mathbf{t}$ of terms such that each identity $\mathbf{s}_i \approx \mathbf{s}_{i+1}$ is directly deducible from some identity in Σ.

Remark 2.2 [8, Sublemma 2.2]. An identity $\mathbf{s} \approx \mathbf{t}$ is deducible from (1.1) if and only if $\lfloor \mathbf{s} \rfloor = \lfloor \mathbf{t} \rfloor$.

An involution semigroup $(S, {}^*)$ *satisfies* an identity $\mathbf{s} \approx \mathbf{t}$, or $\mathbf{s} \approx \mathbf{t}$ is *satisfied by* $(S, {}^*)$, if for any substitution $\varphi : \mathcal{A} \to S$, the elements $\mathbf{s}\varphi$ and $\mathbf{t}\varphi$ of S coincide; in this case, $\mathbf{s} \approx \mathbf{t}$ is also said to be an *identity* of $(S, {}^*)$. An identity is satisfied by an involution semigroup $(S, {}^*)$ if and only if it is deducible from the identities of $(S, {}^*)$.

Clearly, any involution monoid that satisfies a word identity $\mathbf{s} \approx \mathbf{t}$ also satisfies the word identity $\mathbf{s}(x_1, x_2, \ldots, x_n) \approx \mathbf{t}(x_1, x_2, \ldots, x_n)$ for any $x_1, x_2, \ldots, x_n \in \mathcal{A}$, since assigning the element 1 to a variable in a word identity is effectively the same as deleting that variable.

For any involution semigroup $(S, {}^*)$, let $\mathsf{id}(S, {}^*)$ denote the set of all identities satisfied by $(S, {}^*)$. A set $\Sigma \subseteq \mathsf{id}(S, {}^*)$ is an *identity basis* for $(S, {}^*)$ if every identity in $\mathsf{id}(S, {}^*)$ is deducible from Σ. An involution semigroup $(S, {}^*)$ is *finitely based* if it has some finite identity basis; otherwise, it is said to be *nonfinitely based*.

Lemma 2.3. *Let (1.1) $\cup\, \Sigma$ be any basis for a finitely based involution semigroup $(S, {}^*)$. Then there exists a finite subset Σ_{fin} of Σ such that (1.1) $\cup\, \Sigma_{\mathsf{fin}}$ is an identity basis for $(S, {}^*)$.*

Proof. This is a well-known (and immediate) consequence of Birkhoff's completeness theorem for equational logic (see [1, Corollary 1.4.7] or [6, Exercise 10 of Section II.14]). □

3. Two sufficient conditions for an involution monoid to be nonfinitely based

In this section, we give two sufficient conditions under which an involution semigroup is nonfinitely based. The next two theorems are the main results.

Theorem 3.1. *Let $n \geq 2$ be a fixed integer. Suppose that $(S, {}^*)$ is an involution monoid that satisfies the following conditions:*

(I) *for each $k \geq 2$, $(S, {}^*)$ satisfies some word identity $\mathbf{p}_k \approx \mathbf{q}_k$, where*

$$\mathbf{p}_k = t^{a_1} x_1 t^{a_2} x_2 \cdots t^{a_k} x_k t^{a_0} x_1^* t^{a_{k+1}} x_2^* t^{a_{k+2}} \cdots x_k^* t^{a_{2k}},$$
$$\mathbf{q}_k = t^{b_1} x_k t^{b_2} x_{k-1} \cdots t^{b_k} x_1 t^{b_0} x_k^* t^{b_{k+1}} x_{k-1}^* t^{b_{k+2}} \cdots x_1^* t^{b_{2k}}$$

for some $a_0, a_1, a_{2k}, b_0, b_1, b_{2k} \geq 0$, $1 \leq a_2, \ldots, a_{2k-1}, b_2, \ldots, b_{2k-1} < n$;

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(II) *if* $(S, ^*)$ *satisfies a word identity* $\mathbf{w} \approx \mathbf{w}'$ *and* ${}_l x \prec {}_l y \prec {}_f x^* \prec {}_f y^*$ *in* $\mathbf{w}$ *for some* $x, y, x^*, y^* \in \mathsf{con}(\mathbf{w})$*, then either* ${}_l x \prec {}_l y \prec {}_f x^* \prec {}_f y^*$ *or* ${}_l y \prec {}_l x \prec {}_f y^* \prec {}_f x^*$ *holds in* $\mathbf{w}'$;

(III) *if* $(S, ^*)$ *satisfies a word identity of the form*

$$\mathbf{w} = \cdots {}_l x\ \mathbf{a}\ {}_l y \cdots {}_f x^*\ \mathbf{b}\ {}_f y^* \cdots \approx \cdots {}_l x\ \mathbf{a}'\ {}_l y \cdots {}_f x^*\ \mathbf{b}'\ {}_f y^* \cdots = \mathbf{w}' \quad or$$
$$\mathbf{w} = \cdots {}_l x\ \mathbf{a}\ {}_l y \cdots {}_f x^*\ \mathbf{b}\ {}_f y^* \cdots \approx \cdots {}_l y\ \mathbf{b}'\ {}_l x \cdots {}_f y^*\ \mathbf{a}'\ {}_f x^* \cdots = \mathbf{w}'$$

for some words $\mathbf{a}, \mathbf{b}, \mathbf{a}', \mathbf{b}'$*, then for any variables* $t \in \mathsf{con}(\mathbf{w})$ *and* $t^* \notin \mathsf{con}(\mathbf{w})$, $\mathsf{occ}(t, \mathbf{a}) \not\equiv 0 \pmod{n}$ *if and only if* $\mathsf{occ}(t, \mathbf{a}') \not\equiv 0 \pmod{n}$*, and* $\mathsf{occ}(t, \mathbf{b}) \not\equiv 0 \pmod{n}$ *if and only if* $\mathsf{occ}(t, \mathbf{b}') \not\equiv 0 \pmod{n}$.

Then $(S, ^*)$ *is nonfinitely based.*

The proof of Theorem 3.1 is given at the end of the section. By using this theorem, we can show that the following sufficient condition for the nonfinite basis property of involution semigroups also holds.

Theorem 3.2. *Let* $n \geq 2$ *be a fixed integer. Suppose that* $(S, ^*)$ *is an involution monoid that satisfies conditions (I) and (II) in Theorem 3.1 and the following condition*:

(IV) *if* $(S, ^*)$ *satisfies a word identity of the form*

$$\mathbf{w} = \cdots {}_l x\ \mathbf{a}\ {}_f x^* \cdots \approx \cdots {}_l x\ \mathbf{a}'\ {}_f x^* \cdots = \mathbf{w}'$$

for some words $\mathbf{a}, \mathbf{a}'$*, then for any variables* $t \in \mathsf{con}(\mathbf{w})$ *and* $t^* \notin \mathsf{con}(\mathbf{w})$, $\mathsf{occ}(t, \mathbf{a}) \equiv \mathsf{occ}(t, \mathbf{a}') \pmod{n}$.

Then $(S, ^*)$ *is nonfinitely based.*

Proof. By Theorem 3.1, it suffices to show that condition (III) is implied by condition (IV). Let t be a variable such that $t \in \mathsf{con}(\mathbf{w})$ and $t^* \notin \mathsf{con}(\mathbf{w})$. There are two cases.

Case 1. $(S, ^*)$ satisfies a word identity of the form

$$\mathbf{w} = \cdots {}_l x\ \mathbf{a}\ {}_l y\ \mathbf{c}\ {}_f x^*\ \mathbf{b}\ {}_f y^* \cdots \approx \cdots {}_l x\ \mathbf{a}'\ {}_l y\ \mathbf{c}'\ {}_f x^*\ \mathbf{b}'\ {}_f y^* \cdots = \mathbf{w}'.$$

Let φ be the substitution such that $y\varphi = x$ and $x\varphi = x$ for all $x \neq y$. Then $(S, ^*)$ satisfies the word identity

$$\mathbf{w}\varphi = \cdots x\ \mathbf{a}\ {}_l x\ \mathbf{c}\ {}_f x^*\ \mathbf{b}\ x^* \cdots \approx \cdots x\ \mathbf{a}'\ {}_l x\ \mathbf{c}'\ {}_f x^*\ \mathbf{b}'\ x^* \cdots = \mathbf{w}'\varphi.$$

From condition (IV), $\mathsf{occ}(t, \mathbf{c}) \equiv \mathsf{occ}(t, \mathbf{c}') \pmod{n}$ and $\mathsf{occ}(t, \mathbf{ac}) \equiv \mathsf{occ}(t, \mathbf{a}'\mathbf{c}') \pmod{n}$. Therefore, $\mathsf{occ}(t, \mathbf{a}) \equiv \mathsf{occ}(t, \mathbf{a}') \pmod{n}$. A similar argument shows that $\mathsf{occ}(t, \mathbf{b}) \equiv \mathsf{occ}(t, \mathbf{b}') \pmod{n}$. Consequently, condition (III) holds.

Case 2. $(S, ^*)$ satisfies a word identity of the form

$$\mathbf{w} = \cdots {}_l x\ \mathbf{a}\ {}_l y\ \mathbf{c}\ {}_f x^*\ \mathbf{b}\ {}_f y^* \cdots \approx \cdots {}_l y\ \mathbf{b}'\ {}_l x\ \mathbf{c}'\ {}_f y^*\ \mathbf{a}'\ {}_f x^* \cdots = \mathbf{w}'.$$

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Let φ be the substitution such that $y\varphi = x$ and $x\varphi = x$ for all $x \neq y$. Then $(S, {}^*)$ satisfies the word identity

$$\mathbf{w}\varphi = \cdots x\, \mathbf{a}\, {}_{l}x\, \mathbf{c}\, {}_{f}x^*\, \mathbf{b}\, x^* \cdots \approx \cdots x\, \mathbf{b}'\, {}_{l}x\, \mathbf{c}'\, {}_{f}x^*\, \mathbf{a}'\, x^* \cdots = \mathbf{w}'\varphi.$$

From condition (IV), $\mathsf{occ}(t, \mathbf{c}) \equiv \mathsf{occ}(t, \mathbf{c}') \pmod{n}$ and $\mathsf{occ}(t, \mathbf{ac}) \equiv \mathsf{occ}(t, \mathbf{c}'\mathbf{a}') \pmod{n}$. Therefore, $\mathsf{occ}(t, \mathbf{a}) \equiv \mathsf{occ}(t, \mathbf{a}') \pmod{n}$. A similar argument shows that $\mathsf{occ}(t, \mathbf{b}) \equiv \mathsf{occ}(t, \mathbf{b}') \pmod{n}$. Consequently, condition (III) holds. □

Lemma 3.3. *Let $(S, {}^*)$ be an involution monoid satisfying condition (II) in Theorem 3.1. Let $\mathbf{w} \approx \mathbf{w}'$ be any word identity satisfied by $(S, {}^*)$. Then $\mathsf{con}(\mathbf{w}) = \mathsf{con}(\mathbf{w}')$.*

Proof. Seeking a contradiction, suppose that $x \in \mathsf{con}(\mathbf{w}) \setminus \mathsf{con}(\mathbf{w}')$. The involution monoid $(S, {}^*)$ satisfies the word identity

$$xy(x^*)^a y^* = xy \cdot \mathbf{w}'(x) \cdot x^*y^* \approx xy \cdot \mathbf{w}(x) \cdot x^*y^* \in xy \cdot \{x, x^*\}^\times \cdot x \cdot \{x, x^*\}^\times \cdot x^*y^*$$

for some $a \geq 1$, where ${}_{l}x \prec {}_{l}y \prec {}_{f}x^* \prec {}_{f}y^*$ in $xy\mathbf{w}'(x)x^*y^*$, while ${}_{l}y \prec {}_{l}x$ and ${}_{f}x^* \prec {}_{f}y^*$ in $xy\mathbf{w}(x)x^*y^*$. This contradicts condition (II). Therefore, the inclusion $\mathsf{con}(\mathbf{w}) \subseteq \mathsf{con}(\mathbf{w}')$ holds. The inclusion $\mathsf{con}(\mathbf{w}') \subseteq \mathsf{con}(\mathbf{w})$ holds by symmetry. □

For each $k \geq 2$, define

$$\mathsf{L}_k = \mathsf{X}_1\mathsf{X}_2 \cdots \mathsf{X}_k\mathsf{X}_1^*\mathsf{X}_2^* \cdots \mathsf{X}_k^*, \quad \mathsf{R}_k = \dot{\mathsf{X}}_k\dot{\mathsf{X}}_{k-1} \cdots \dot{\mathsf{X}}_1\dot{\mathsf{X}}_k^*\dot{\mathsf{X}}_{k-1}^* \cdots \dot{\mathsf{X}}_1^*,$$

where

$$\mathsf{X}_i = \{x_i, \ldots, x_k\}^\times \cdot x_i, \quad \mathsf{X}_i^* = x_i^* \cdot \{x_1^*, \ldots, x_i^*\}^\times,$$
$$\dot{\mathsf{X}}_i = \{x_1, \ldots, x_i\}^\times \cdot x_i, \quad \dot{\mathsf{X}}_i^* = x_i^* \cdot \{x_i^*, \ldots, x_k^*\}^\times.$$

Let $\mathbf{w}$ be a word. Then, by the definitions of L_k and R_k, it is easy to see that $\mathbf{w} \in \mathsf{L}_k$ if and only if ${}_{l}x_1 \prec {}_{l}x_2 \prec \cdots \prec {}_{l}x_k \prec {}_{f}x_1^* \prec {}_{f}x_2^* \prec \cdots \prec {}_{f}x_k^*$ holds in $\mathbf{w}$; and $\mathbf{w} \in \mathsf{R}_k$ if and only if ${}_{l}x_k \prec {}_{l}x_{k-1} \prec \cdots \prec {}_{l}x_1 \prec {}_{f}x_k^* \prec {}_{f}x_{k-1}^* \prec \cdots \prec {}_{f}x_1^*$ holds in $\mathbf{w}$.

Lemma 3.4. *Let $(S, {}^*)$ be an involution monoid satisfying condition (II) in Theorem 3.1. Let $\mathbf{w} \approx \mathbf{w}'$ be any word identity satisfied by $(S, {}^*)$ such that $\mathbf{w} \in \mathsf{L}_k$. Then $\mathbf{w}' \in \mathsf{L}_k \cup \mathsf{R}_k$.*

Proof. It follows from Lemma 3.3 and condition (II) that the lemma holds when $k = 2$. Therefore, we may assume that $k > 2$. By assumption,

$$\mathbf{w} = \mathbf{x}_1\mathbf{x}_2 \cdots \mathbf{x}_k\mathbf{x}_1^*\mathbf{x}_2^* \cdots \mathbf{x}_k^*$$

for some $\mathbf{x}_i \in \mathsf{X}_i$ and $\mathbf{x}_i^* \in \mathsf{X}_i^*$. Observe that for each $i < j$, ${}_{l}x_i \prec {}_{l}x_j \prec {}_{f}x_i^* \prec {}_{f}x_j^*$ holds in $\mathbf{w}$; it follows from condition (II) that either ${}_{l}x_i \prec {}_{l}x_j \prec {}_{f}x_i^* \prec {}_{f}x_j^*$ or ${}_{l}x_j \prec {}_{l}x_i \prec {}_{f}x_j^* \prec {}_{f}x_i^*$ holds in $\mathbf{w}'$. In any case, ${}_{l}x_i \prec {}_{f}x_1^*, {}_{l}x_i \prec {}_{f}x_2^*, \ldots, {}_{l}x_i \prec {}_{f}x_k^*$ hold in $\mathbf{w}'$ for each i. Without loss of generality, we assume that

$$\mathbf{w}' = \mathbf{x}_{1\pi}\mathbf{x}_{2\pi} \cdots \mathbf{x}_{k\pi}\mathbf{x}_{1\tau}^*\mathbf{x}_{2\tau}^* \cdots \mathbf{x}_{k\tau}^*$$

for some $\mathbf{x}_{i\pi} \in \{x_{i\pi}, \ldots, x_{k\pi}\}^\times \cdot x_{i\pi}$, $\mathbf{x}_{i\tau}^* \in x_{i\tau}^* \cdot \{x_{1\tau}^*, \ldots, x_{i\pi}^*\}^\times$ and some permutations π and τ on $\{1, \ldots, k\}$.

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Suppose that $(1\pi, \dots, k\pi) \neq (1\tau, \dots, k\tau)$. Then

$$(1\pi, \dots, k\pi) = (\dots, i, \dots, j, \dots) \neq (\dots, j, \dots, i, \dots) = (1\tau, \dots, k\tau)$$

must hold for some $i \neq j$. But this implies that the involution monoid $(S, {}^*)$ satisfies the word identity $\mathbf{w} \approx \mathbf{w}'$ such that ${}_l x_i \prec {}_l x_j \prec {}_f x_i^* \prec {}_f x_j^*$ if $i < j$ and ${}_l x_j \prec {}_l x_i \prec {}_f x_j^* \prec {}_f x_i^*$ if $i > j$ in $\mathbf{w}$, while ${}_l x_i \prec {}_l x_j \prec {}_f x_j^* \prec {}_f x_i^*$ in $\mathbf{w}'$. This contradicts condition (II). Therefore, $(1\pi, \dots, k\pi) = (1\tau, \dots, k\tau)$.

Suppose that $(1\pi, \dots, k\pi) \neq (1, \dots, k)$ and $(1\pi, \dots, k\pi) \neq (k, \dots, 1)$. Then one of the following holds:

(†) $(1\pi, \dots, k\pi) = (\dots, j, i, h, \dots)$ for some $i < j$ and $i < h$;
(‡) $(1\pi, \dots, k\pi) = (\dots, i, h, j, \dots)$ for some $i < h$ and $j < h$.

If (†) holds, let φ be the substitution such that $x_h\varphi = x_j$ and $x\varphi = x$ for all $x \neq x_h$. The involution monoid $(S, {}^*)$ satisfies the word identity $\mathbf{w}\varphi \approx \mathbf{w}'\varphi$, where ${}_l x_i \prec {}_l x_j \prec {}_f x_i^* \prec {}_f x_j^*$ in $\mathbf{w}\varphi$, while ${}_l x_i \prec {}_l x_j \prec {}_f x_j^* \prec {}_f x_i^*$ in $\mathbf{w}'\varphi$. This contradicts condition (II). If (‡) holds, let φ be the substitution such that $x_j\varphi = x_i$ and $x\varphi = x$ for all $x \neq x_j$. The involution monoid $(S, {}^*)$ satisfies the word identity $\mathbf{w}\varphi \approx \mathbf{w}'\varphi$, where ${}_l x_i \prec {}_l x_h \prec {}_f x_i^* \prec {}_f x_h^*$ in $\mathbf{w}\varphi$, while ${}_l x_h \prec {}_l x_i \prec {}_f x_i^* \prec {}_f x_h^*$ in $\mathbf{w}'\varphi$. This contradicts condition (II). Consequently, $(1\pi, \dots, k\pi) \in \{(1, \dots, k), (k, \dots, 1)\}$ and hence $\mathbf{w}' \in \mathsf{L}_k \cup \mathsf{R}_k$. □

In the remainder of this section, we always assume that $n \geq 2$ is a fixed integer. For each $k \geq 2$, define

$$\mathsf{P}_k = \mathsf{Y}_1\mathsf{Y}_2 \cdots \mathsf{Y}_k\mathsf{T}\mathsf{Y}_1^*\mathsf{Y}_2^* \cdots \mathsf{Y}_k^*, \quad \mathsf{Q}_k = \dot{\mathsf{Y}}_k\dot{\mathsf{Y}}_{k-1} \cdots \dot{\mathsf{Y}}_1\mathsf{T}\dot{\mathsf{Y}}_k^*\dot{\mathsf{Y}}_{k-1}^* \cdots \dot{\mathsf{Y}}_1^*,$$

where $\mathsf{T} = \{t\}^\times$,

$$\mathsf{Y}_i = \{t, x_i, \dots, x_k\}^\times \cdot x_i, \quad \mathsf{Y}_i^* = x_i^* \cdot \{t, x_1^*, \dots, x_i^*\}^\times,$$
$$\dot{\mathsf{Y}}_i = \{t, x_1, \dots, x_i\}^\times \cdot x_i, \quad \dot{\mathsf{Y}}_i^* = x_i^* \cdot \{t, x_i^*, \dots, x_k^*\}^\times$$

and $\mathsf{occ}(t, \mathbf{y}) \not\equiv 0 \pmod{n}$ for each $\mathbf{y} \in \mathsf{Y}_2 \cup \cdots \cup \mathsf{Y}_k \cup \mathsf{Y}_1^* \cup \cdots \cup \mathsf{Y}_{k-1}^* \cup \dot{\mathsf{Y}}_1 \cup \cdots \cup \dot{\mathsf{Y}}_{k-1} \cup \dot{\mathsf{Y}}_2^* \cup \cdots \cup \dot{\mathsf{Y}}_k^*$. Clearly, $t \in \mathsf{con}(\mathbf{y})$ for each $\mathbf{y}$ in any case.

By the definitions of P_k and Q_k, it is easy to see that if $\mathbf{w} \in \mathsf{P}_k$, then $\mathbf{w}(x_1, x_2, \dots, x_k) \in \mathsf{L}_k$; and, if $\mathbf{w} \in \mathsf{Q}_k$, then $\mathbf{w}(x_1, x_2, \dots, x_k) \in \mathsf{R}_k$.

Lemma 3.5. *Let $(S, {}^*)$ be an involution monoid satisfying conditions (II) and (III) in Theorem 3.1. Let $\mathbf{w} \approx \mathbf{w}'$ be any word identity satisfied by $(S, {}^*)$ such that $\mathbf{w} \in \mathsf{P}_k$. Then $\mathbf{w}' \in \mathsf{P}_k \cup \mathsf{Q}_k$.*

Proof. By assumption,

$$\mathbf{w} = \mathbf{y}_1\mathbf{y}_2 \cdots \mathbf{y}_k\mathbf{t}\mathbf{y}_1^*\mathbf{y}_2^* \cdots \mathbf{y}_k^*$$

for some $\mathbf{t} \in \mathsf{T}$, $\mathbf{y}_i \in \mathsf{Y}_i$, $\mathbf{y}_i^* \in \mathsf{Y}_i^*$, $\mathsf{occ}(t, \mathbf{y}_i) \not\equiv 0 \pmod{n}$ for each $i = 2, \dots, k$ and $\mathsf{occ}(t, \mathbf{y}_i^*) \not\equiv 0 \pmod{n}$ for each $i = 1, \dots, k-1$. Then

$$\mathbf{w}(x_1, x_2, \dots, x_k) = \mathbf{x}_1\mathbf{x}_2 \cdots \mathbf{x}_k\mathbf{x}_1^*\mathbf{x}_2^* \cdots \mathbf{x}_k^*,$$

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where $\mathbf{x}_i = \mathbf{y}_i(x_1, x_2, \ldots, x_k) \in \mathsf{X}_i$ and $\mathbf{x}_i^* = \mathbf{y}_i^*(x_1, x_2, \ldots, x_k) \in \mathsf{X}_i^*$. It follows from Lemma 3.4 that $\mathbf{w}'(x_1, x_2, \ldots, x_k) \in \mathsf{L}_k \cup \mathsf{R}_k$. There are two cases.

Case 1. $\mathbf{w}'(x_1, x_2, \ldots, x_k) \in \mathsf{L}_k$. Without loss of generality, we may assume that $\mathbf{w}'(x_1, x_2, \ldots, x_k) = \mathbf{x}_1'\mathbf{x}_2' \cdots \mathbf{x}_k'(\mathbf{x}_1^*)'(\mathbf{x}_2^*)' \cdots (\mathbf{x}_k^*)'$ for some $\mathbf{x}_i' \in \mathsf{X}_i$ and $(\mathbf{x}_i^*)' \in \mathsf{X}_i^*$. Then, by Lemma 3.3, $\mathbf{w}'$ can be written in the form

$$\mathbf{w}' = \mathbf{y}_1'\mathbf{y}_2' \cdots \mathbf{y}_k'\mathbf{t}'(\mathbf{y}_1^*)'(\mathbf{y}_2^*)' \cdots (\mathbf{y}_k^*)',$$

where $\mathbf{t}' \in \{t\}^\times$, $\mathbf{y}_i' \in \{t, x_i, \ldots, x_k\}^\times \cdot x_i$, $\mathbf{y}_i'(x_1, \ldots, x_k) = \mathbf{x}_i'$, $(\mathbf{y}_i^*)' \in x_i^* \cdot \{t, x_1^*, \ldots, x_i^*\}^\times$ and $(\mathbf{y}_i^*)'(x_1, \ldots, x_k) = (\mathbf{x}_i^*)'$.

For each $i = 1, \ldots, k-1$, the number of times that t occurs between ${}_lx_i$ and ${}_lx_{i+1}$ equals $\mathsf{occ}(t, \mathbf{y}_{i+1})$ in $\mathbf{w}$, and the number of times that t occurs between ${}_fx_i^*$ and ${}_fx_{i+1}^*$ equals $\mathsf{occ}(t, \mathbf{y}_i^*)$ in $\mathbf{w}$. Since $\mathsf{occ}(t, \mathbf{y}_{i+1}) \not\equiv 0 \pmod{n}$ and $\mathsf{occ}(t, \mathbf{y}_i^*) \not\equiv 0 \pmod{n}$ in $\mathbf{w}$, it follows from condition (III) that $\mathsf{occ}(t, \mathbf{y}_{i+1}') \not\equiv 0 \pmod{n}$ and $\mathsf{occ}(t, (\mathbf{y}_i^*)') \not\equiv 0 \pmod{n}$ in $\mathbf{w}'$, respectively. Consequently, $\mathbf{w}' \in \mathsf{P}_k$.

Case 2. $\mathbf{w}'(x_1, x_2, \ldots, x_k) \in \mathsf{R}_k$. Without loss of generality, we may assume that $\mathbf{w}'(x_1, x_2, \ldots, x_k) = \dot{\mathbf{x}}_k'\dot{\mathbf{x}}_{k-1}' \cdots \dot{\mathbf{x}}_1'(\dot{\mathbf{x}}_k^*)'(\dot{\mathbf{x}}_{k-1}^*)' \cdots (\dot{\mathbf{x}}_1^*)'$ for some $\dot{\mathbf{x}}_i' \in \dot{\mathsf{X}}_i$ and $(\dot{\mathbf{x}}_i^*)' \in \dot{\mathsf{X}}_i^*$. Then $\mathbf{w}'$ can be written in the form

$$\mathbf{w}' = \dot{\mathbf{y}}_k'\dot{\mathbf{y}}_{k-1}' \cdots \dot{\mathbf{y}}_1'\mathbf{t}'(\dot{\mathbf{y}}_k^*)'(\dot{\mathbf{y}}_{k-1}^*)' \cdots (\dot{\mathbf{y}}_1^*)',$$

where $\mathbf{t}' \in \{t\}^\times$, $\dot{\mathbf{y}}_i' \in \{t, x_1, \ldots, x_i\}^\times \cdot x_i$, $\dot{\mathbf{y}}_i'(x_1, \ldots, x_k) = \dot{\mathbf{x}}_i'$, $(\dot{\mathbf{y}}_i^*)' \in x_i^* \cdot \{t, x_i^*, \ldots, x_k^*\}^\times$ and $(\dot{\mathbf{y}}_i^*)'(x_1, \ldots, x_k) = (\dot{\mathbf{x}}_i^*)'$.

For each $i = 1, \ldots, k-1$, the number of times that t occurs between ${}_lx_i$ and ${}_lx_{i+1}$ equals $\mathsf{occ}(t, \mathbf{y}_{i+1})$ in $\mathbf{w}$, and the number of times that t occurs between ${}_fx_i^*$ and ${}_fx_{i+1}^*$ equals $\mathsf{occ}(t, \mathbf{y}_i^*)$ in $\mathbf{w}$. Since $\mathsf{occ}(t, \mathbf{y}_{i+1}) \not\equiv 0 \pmod{n}$ and $\mathsf{occ}(t, \mathbf{y}_i^*) \not\equiv 0 \pmod{n}$ in $\mathbf{w}$, it follows from condition (III) that $\mathsf{occ}(t, (\dot{\mathbf{y}}_{i+1}^*)') \not\equiv 0 \pmod{n}$ and $\mathsf{occ}(t, \dot{\mathbf{y}}_i') \not\equiv 0 \pmod{n}$ in $\mathbf{w}'$, respectively. Consequently, $\mathbf{w}' \in \mathsf{Q}_k$. □

Lemma 3.6. *Suppose that* $(S, {}^*)$ *is an involution monoid satisfying condition (II) in Theorem 3.1. Let* $x^a y^b \approx \mathbf{w}$ *be a word identity satisfied by* $(S, {}^*)$*, where* $a, b \geq 1$*. Then* $\mathbf{w} = x^c y^d$ *for some* $c, d \geq 1$.

Proof. Let $x^a y^b \approx \mathbf{w}$ be a word identity satisfied by the involution monoid $(S, {}^*)$. Then it follows from Lemma 3.3 that $\mathsf{con}(\mathbf{w}) = \{x, y\}$. Suppose that some occurrence of y precedes some occurrence of x in $\mathbf{w}$. Then yx must be a factor of $\mathbf{w}$, which implies that $(S, {}^*)$ satisfies the word identity

$$y^* x^a y^b x^* \approx y^* \mathbf{w} x^* = y^* \cdots yx \cdots x^*,$$

where ${}_ly^* \prec {}_lx \prec {}_fy \prec {}_fx^*$ in $y^* x^a y^b x^*$, while ${}_ly^* \prec {}_fy \prec {}_lx \prec {}_fx^*$ in $y^* \mathbf{w} x^*$. This contradicts condition (II). Therefore, each occurrence of x precedes every occurrence of y in $\mathbf{w}$ and so the form of $\mathbf{w}$ must be $x^c y^d$ for some $c, d \geq 1$. □

A word identity $\mathbf{w} \approx \mathbf{w}'$ is said to be *k-limited* if $\mathbf{w}, \mathbf{w}' \in (\mathcal{A} \cup \mathcal{A}^*)^+$ and $|\mathcal{A}| \leq k$. For any involution semigroup $(S, {}^*)$, let $\mathsf{id}_k(S, {}^*)$ denote the set of all k-limited word identities satisfied by $(S, {}^*)$.

LEMMA 3.7. *Suppose that* $(S, {}^*)$ *is an involution monoid satisfying conditions (II) and (III) in Theorem 3.1. Let* $\mathbf{s} \approx \mathbf{s}'$ *be an identity which is directly deducible from some identity in* $\mathsf{id}_k(S, {}^*)$ *with* $\lfloor \mathbf{s} \rfloor \in \mathsf{P}_{k+1}$. *Then* $\lfloor \mathbf{s}' \rfloor \in \mathsf{P}_{k+1}$.

PROOF. Let $\mathbf{w} \approx \mathbf{w}'$ be a word identity in $\mathsf{id}_k(S, {}^*)$ from which the identity $\mathbf{s} \approx \mathbf{s}'$ is directly deducible. Then there is a substitution $\varphi : \mathcal{A} \to \mathsf{T}(\mathcal{A})$ such that $\mathbf{w}\varphi$ is a subterm of $\mathbf{s}$, and replacing this particular subterm $\mathbf{w}\varphi$ of $\mathbf{s}$ with $\mathbf{w}'\varphi$ results in $\mathbf{s}'$. Then, by Remark 2.1, either $\lfloor \mathbf{w}\varphi \rfloor$ or $\lfloor (\mathbf{w}\varphi)^* \rfloor$ is a factor of $\lfloor \mathbf{s} \rfloor$. It suffices to consider the former case, since the latter is similar. Hence, there exist words $\mathbf{e}, \mathbf{f} \in (\mathcal{A} \cup \mathcal{A}^*)^\times$ such that $\lfloor \mathbf{s} \rfloor = \mathbf{e}\lfloor \mathbf{w}\varphi \rfloor \mathbf{f}$. Since $\mathbf{s}'$ is obtained by replacing $\mathbf{w}\varphi$ in $\mathbf{s}$ with $\mathbf{w}'\varphi$, it follows that $\lfloor \mathbf{s}' \rfloor = \mathbf{e}\lfloor \mathbf{w}'\varphi \rfloor \mathbf{f}$. By the assumption of this lemma,

$$\lfloor \mathbf{s} \rfloor = \mathbf{y}_1 \mathbf{y}_2 \cdots \mathbf{y}_{k+1} \mathbf{t} \mathbf{y}_1^* \mathbf{y}_2^* \cdots \mathbf{y}_{k+1}^*$$

for some $\mathbf{t} \in \mathsf{T}$, $\mathbf{y}_i \in \mathsf{Y}_i$, $\mathbf{y}_i^* \in \mathsf{Y}_i^*$, $\mathsf{occ}(t, \mathbf{y}_i) \not\equiv 0 \pmod{n}$ for $i = 2, \ldots, k+1$ and $\mathsf{occ}(t, \mathbf{y}_i^*) \not\equiv 0 \pmod{n}$ for $i = 1, \ldots, k$. Since $\mathbf{e}\lfloor \mathbf{w}\varphi \rfloor \mathbf{f} \in \mathsf{P}_{k+1}$ and the word identity $\mathbf{e}\lfloor \mathbf{w}\varphi \rfloor \mathbf{f} \approx \mathbf{e}\lfloor \mathbf{w}'\varphi \rfloor \mathbf{f}$ is satisfied by the involution monoid $(S, {}^*)$, it follows from Lemma 3.5 that $\mathbf{e}\lfloor \mathbf{w}'\varphi \rfloor \mathbf{f} \in \mathsf{P}_{k+1} \cup \mathsf{Q}_{k+1}$. Aiming for a contradiction, suppose that $\mathbf{e}\lfloor \mathbf{w}'\varphi \rfloor \mathbf{f} \in \mathsf{Q}_{k+1}$. In $\mathbf{e}\lfloor \mathbf{w}'\varphi \rfloor \mathbf{f}$, ${}_l x_{k+1} \prec {}_l x_k \prec \cdots \prec {}_l x_1 \prec {}_f x_{k+1}^* \prec {}_f x_k^* \prec \cdots \prec_f x_1^*$, while in $\mathbf{e}\lfloor \mathbf{w}\varphi \rfloor \mathbf{f}$, the reverse ${}_l x_1 \prec {}_l x_2 \prec \cdots \prec {}_l x_{k+1} \prec {}_f x_1^* \prec {}_f x_2^* \prec \cdots \prec {}_f x_{k+1}^*$ is true. Hence, the word $\lfloor \mathbf{w}\varphi \rfloor$ contains all the variables $x_1, x_2, \ldots, x_{k+1}$, $x_1^*, x_2^*, \ldots, x_{k+1}^*$, whence it contains a factor of the form

$$x_1 \mathbf{y}_2 \cdots \mathbf{y}_{k+1} \mathbf{t} \mathbf{y}_1^* \mathbf{y}_2^* \cdots \mathbf{y}_k^* x_{k+1}^*.$$

Suppose that there exists a variable $s \in \mathsf{con}(\mathbf{w})$ such that $\lfloor s\varphi \rfloor$ contains the factor $x_{k+1} \mathbf{t} x_1^*$ of $\lfloor \mathbf{w}\varphi \rfloor$. Then $x_{k+1} \mathbf{t} x_1^*$ is a factor of $\lfloor \mathbf{w}'\varphi \rfloor$, since $s \in \mathsf{con}(\mathbf{w}')$ by Lemma 3.3. But it is easy to see that none of the words in Q_{k+1} contains $x_{k+1} \mathbf{t} x_1^*$ as a factor, which contradicts $\mathbf{e}\lfloor \mathbf{w}'\varphi \rfloor \mathbf{f} \in \mathsf{Q}_{k+1}$. Therefore, there is no variable s in $\mathbf{w}$ such that $\lfloor s\varphi \rfloor$ contains the factor $x_{k+1} \mathbf{t} x_1^*$ and so we may assume that $\mathbf{w} = \mathbf{w}_1 \mathbf{w}_2$, where

(i) $\lfloor \mathbf{w}_1 \varphi \rfloor = \cdots x_1 \mathbf{y}_2 \cdots \mathbf{y}_{k+1} t^{\delta_1}$, $\lfloor \mathbf{w}_2 \varphi \rfloor = t^{\delta_2} \mathbf{y}_1^* \mathbf{y}_2^* \cdots \mathbf{y}_k^* x_{k+1}^* \cdots$ and $t^{\delta_1} t^{\delta_2} = \mathbf{t}$ for some $\delta_1, \delta_2 \geq 0$.

Since $\lfloor \mathbf{w}_1 \varphi \rfloor$ contains all of ${}_l x_1, \ldots, {}_l x_{k+1}$ in $\mathbf{w}\varphi$, the number of the last occurrence of variables in $\lfloor \mathbf{w}_1 \varphi \rfloor$ is $k+1$. But $\mathbf{w}_1$ contains at most k variables of $\mathcal{A}$, whence there must exist a variable s in $\mathbf{w}_1$ such that $\lfloor s\varphi \rfloor$ contains at least two of $\{{}_l x_1, \ldots, {}_l x_{k+1}\}$. Without loss of generality, we may assume that s_1 in $\mathbf{w}_1$ is the last variable such that $\lfloor s_1 \varphi \rfloor$ contains at least two of $\{{}_l x_1, \ldots, {}_l x_{k+1}\}$, that is, $\mathbf{w}_1 = \mathbf{p}_1 s_1 \mathbf{q}_1$, where

(ii) $\lfloor s_1 \varphi \rfloor = \cdots x_{g-1} \mathbf{y}_g \mathbf{y}$ for some $g \geq 2$ and $\mathbf{y}$ is a possibly empty prefix of $\mathbf{y}_{g+1}$ with $\mathbf{y} \neq \mathbf{y}_{g+1}$ and

(iii) $\mathsf{con}(\lfloor \mathbf{q}_1 \varphi \rfloor) = \{t, x_{g+1}, \ldots, x_{k+1}\}$ and each variable in $\lfloor \mathbf{q}_1 \varphi \rfloor$ contains at most one of $\{{}_l x_{g+1}, \ldots, {}_l x_{k+1}\}$.

Similarly, there must exist a variable s in $\mathbf{w}_2$ such that $\lfloor s\varphi \rfloor$ contains at least two of $\{{}_f x_1^*, \ldots, {}_f x_{k+1}^*\}$. Without loss of generality, we may assume that s_2 in $\mathbf{w}_2$ is the first variable such that $\lfloor s_2 \varphi \rfloor$ contains at least two of $\{{}_f x_1^*, \ldots, {}_f x_{k+1}^*\}$, that is, $\mathbf{w}_2 = \mathbf{q}_2 s_2 \mathbf{p}_2$,

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where

(iv) $\lfloor s_2\varphi\rfloor = \mathbf{y}^*\mathbf{y}_h^* x_{h+1}^* \cdots$ for some $h \le k$ and $\mathbf{y}^*$ is a possibly empty suffix of $\mathbf{y}_{h-1}^*$ with $\mathbf{y}^* \ne \mathbf{y}_{h-1}^*$ and

(v) $\mathsf{con}(\lfloor \mathbf{q}_2\varphi\rfloor) = \{t, x_1^*, \ldots, x_{h-1}^*\}$ and each variable in $\lfloor \mathbf{q}_2\varphi\rfloor$ contains at most one of $\{{}_f x_1^*, \ldots, {}_f x_{h-1}^*\}$.

It follows from (ii) and (iii) that $s_1 \notin \mathsf{con}(\mathbf{q}_1)$. Since $\lfloor s_1\varphi\rfloor = \cdots x_{g-1}\mathbf{y}_g\mathbf{y}$ is not a factor of $\lfloor \mathbf{w}_2\varphi\rfloor$ by (i), $s_1 \notin \mathsf{con}(\mathbf{w}_2)$. Since $t \in \mathsf{con}(\mathbf{y}_g) \subsetneq \mathsf{con}(\lfloor s_1\varphi\rfloor)$ by (ii) and $t^* \notin \mathsf{con}(\lfloor \mathbf{w}\varphi\rfloor)$, $\lfloor s_1^*\varphi\rfloor$ is not a factor of $\lfloor \mathbf{w}\varphi\rfloor$ and so $s_1^* \notin \mathsf{con}(\mathbf{w})$. Hence

(vi) $s_1 \notin \mathsf{con}(\mathbf{q}_1\mathbf{w}_2)$ and $s_1^* \notin \mathsf{con}(\mathbf{w})$.

Similarly,

(vii) $s_2 \notin \mathsf{con}(\mathbf{w}_1\mathbf{q}_2)$ and $s_2^* \notin \mathsf{con}(\mathbf{w})$.

Therefore, $\mathbf{w}(s_1, s_2) = s_1^a s_2^b$ for some $a, b \ge 1$. From Lemma 3.6, $\mathbf{w}'(s_1, s_2) = s_1^c s_2^d$ for some $c, d \ge 1$. Thus, $\mathbf{w}'$ can be written in the form $\mathbf{w}' = \mathbf{p}_1' s_1 \mathbf{q}' s_2 \mathbf{p}_2'$, where

(viii) $s_1 \notin \mathsf{con}(\mathbf{q}'\mathbf{p}_2')$, $s_2 \notin \mathsf{con}(\mathbf{p}_1'\mathbf{q}')$ and $s_1^*, s_2^* \notin \mathsf{con}(\mathbf{w}')$.

Consequently,

$$\lfloor \mathbf{w}'\varphi\rfloor = \lfloor \mathbf{p}_1'\varphi\rfloor\lfloor s_1\varphi\rfloor\lfloor \mathbf{q}'\varphi\rfloor\lfloor s_2\varphi\rfloor\lfloor \mathbf{p}_2'\varphi\rfloor.$$

Since $\lfloor \mathbf{w}'\varphi\rfloor \in \mathsf{Q}_{k+1}$ by assumption, ${}_l x_{k+1} \prec {}_l x_k \prec \cdots \prec {}_l x_1 \prec {}_f x_{k+1}^* \prec {}_f x_k^* \prec \cdots \prec {}_f x_1^*$ in $\lfloor \mathbf{w}'\varphi\rfloor$. Then it follows from (ii) and (iv) that $\dot{\mathbf{y}}_{g-1} \cdots \dot{\mathbf{y}}_1 \mathbf{t} \dot{\mathbf{y}}_{k+1}^* \cdots \dot{\mathbf{y}}_{h+1}^*$ must be a factor of $\lfloor \mathbf{q}'\varphi\rfloor$ for some $\mathbf{t} \in \mathsf{T}$, $\dot{\mathbf{y}}_i \in \dot{\mathsf{Y}}_i$ and $\dot{\mathbf{y}}_i^* \in \dot{\mathsf{Y}}_i^*$. Hence, $x_1 \in \mathsf{con}(\lfloor \mathbf{q}'\varphi\rfloor)$ by $g \ge 2$. But, in the following, we will show that $x_1 \notin \mathsf{con}(\lfloor \mathbf{q}'\varphi\rfloor)$, which is a contradiction. Therefore, $\lfloor \mathbf{w}'\varphi\rfloor \notin \mathsf{Q}_{k+1}$ and so $\lfloor \mathbf{w}'\varphi\rfloor \in \mathsf{P}_{k+1}$.

Now we prove that $x_1 \notin \mathsf{con}(\lfloor \mathbf{q}'\varphi\rfloor)$. Without loss of generality, we may assume that $x_1 \in \mathsf{con}(\lfloor z\varphi\rfloor)$ for some $z \in \mathsf{con}(\mathbf{w})$. Then it suffices to show that $z \notin \mathsf{con}(\mathbf{q}')$. Seeking a contradiction, suppose that $z \in \mathsf{con}(\mathbf{q}')$. Then $z \ne s_1$ by (viii). It follows from (iii) that $z \notin \mathsf{con}(\mathbf{q}_1)$ and from (i) that $z \notin \mathsf{con}(\mathbf{w}_2)$ and $z^* \notin \mathsf{con}(\mathbf{w}_1)$. Hence

(ix) $z \in \mathsf{con}(\mathbf{p}_1) \setminus \mathsf{con}(s_1\mathbf{q}_1\mathbf{w}_2)$ and $z^* \notin \mathsf{con}(\mathbf{w}_1)$.

Now, by (vi) and (ix),

$$z\mathbf{w}(z, s_1)z^* s_1^* \in z \cdot \underbrace{\{z, s_1\}^+}_{\mathbf{p}_1(z,s_1)} \cdot s_1 \cdot \underbrace{\{z^*\}^\times}_{(\mathbf{q}_1\mathbf{w}_2)(z,s_1)} \cdot z^* s_1^*$$

and by (viii) and $\lfloor \mathbf{w}'\varphi\rfloor \in \mathsf{Q}_{k+1}$,

$$z\mathbf{w}'(z, s_1)z^* s_1^* \in z \cdot \underbrace{\{z, s_1\}^\times}_{\mathbf{p}_1'(z,s_1)} \cdot s_1 \cdot \underbrace{z \cdot \{z, z^*\}^\times}_{\mathbf{q}'(z,s_1)} \cdot \underbrace{\{z^*\}^\times}_{(s_2\mathbf{p}_2')(z,s_1)} \cdot z^* s_1^*.$$

But this implies that $(S, {}^*)$ satisfies the word identity $z\mathbf{w}(z, s_1)z^* s_1^* \approx z\mathbf{w}'(z, s_1)z^* s_1^*$, where ${}_l z \prec {}_l s_1 \prec {}_f z^* \prec {}_f s_1^*$ in $z\mathbf{w}(z, s_1)z^* s_1^*$, while ${}_l s_1 \prec {}_l z \prec {}_f z^* \prec {}_f s_1^*$ in $z\mathbf{w}'(z, s_1)z^* s_1^*$. This contradicts condition (II). Hence, $z \notin \mathsf{con}(\mathbf{q}')$, as required. □

Proof of Theorem 3.2. Let $(S, {}^*)$ be any involution monoid satisfying (I)–(III) in Theorem 3.2. Then there exists some set Σ of word identities such that (1.1) $\cup\, \Sigma$ is an

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identity basis for $(S, {}^*)$. Working toward a contradiction, suppose that the involution monoid $(S, {}^*)$ is finitely based. Then, by Lemma 2.3, there exists a finite subset Σ_{fin} of Σ such that all identities of $(S, {}^*)$ are deducible from (1.1) $\cup\, \Sigma_{\mathsf{fin}}$. Hence, there exists some fixed integer k such that $\Sigma_{\mathsf{fin}} \subseteq \Sigma \cap \mathsf{id}_k(S, {}^*)$. Since the involution monoid $(S, {}^*)$ satisfies some word identity $\mathbf{p}_{k+1} \approx \mathbf{q}_{k+1}$ by (I), there exists some sequence

$$\mathbf{p}_{k+1} = \mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_m = \mathbf{q}_{k+1}$$

of terms such that each identity $\mathbf{s}_i \approx \mathbf{s}_{i+1}$ is directly deducible from some identity $\mathbf{w}_i \approx \mathbf{w}_i'$ in (1.1) $\cup\, \Sigma_{\mathsf{fin}}$. The equality $\lfloor \mathbf{s}_1 \rfloor = \mathbf{p}_{k+1} \in \mathsf{P}_{k+1}$ holds by Remark 2.2. If $\lfloor \mathbf{s}_i \rfloor \in \mathsf{P}_{k+1}$ for some $i \geq 1$, then there are two cases depending on whether the identity $\mathbf{w}_i \approx \mathbf{w}_i'$ is from (1.1) or Σ_{fin}. If $\mathbf{w}_i \approx \mathbf{w}_i'$ is from (1.1), then $\lfloor \mathbf{s}_i \rfloor = \lfloor \mathbf{s}_{i+1} \rfloor$ by Remark 2.2, whence $\lfloor \mathbf{s}_{i+1} \rfloor \in \mathsf{P}_{k+1}$. If $\mathbf{w}_i \approx \mathbf{w}_i'$ is from Σ_{fin}, then $\lfloor \mathbf{s}_{i+1} \rfloor \in \mathsf{P}_{k+1}$ by Lemma 3.7. In any case, $\lfloor \mathbf{s}_{i+1} \rfloor \in \mathsf{P}_{k+1}$ and, by induction, $\lfloor \mathbf{s}_i \rfloor \in \mathsf{P}_{k+1}$ for all i. This gives the contradiction $\mathbf{q}_{k+1} = \lfloor \mathbf{s}_m \rfloor \in \mathsf{P}_{k+1}$, so the involution monoid $(S, {}^*)$ is nonfinitely based. □

4. Nonfinitely based involution semigroups of triangular matrices

In this section, as applications of Theorem 3.2, we explore the finite basis problem for the involution monoids $(UT_2(\mathbb{F}), {}^*)$, $(UT_2^{\pm 1}(\mathbb{F}), {}^*)$ and $(T_2(\mathbb{F}), {}^*)$. Throughout this section, $\mathsf{char}(\mathbb{F})$ means the characteristic of a field $\mathbb{F}$ and p is always a prime.

Recall that for each matrix $A = \begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix} \in T_2(\mathbb{F})$, the skew transposition of A is $A^* = \begin{pmatrix} a_{22} & a_{12} \\ 0 & a_{11} \end{pmatrix}$. Clearly, $(A)_{11} = (A^*)_{22}$, $(A)_{22} = (A^*)_{11}$ and $(A)_{12} = (A^*)_{12}$.

Remark 4.1. Let $\mathbf{w}, \mathbf{w}' \in (\mathcal{A} \cup \mathcal{A}^*)^+$ be such that $\mathsf{occ}(x, \mathbf{w}) = \mathsf{occ}(x, \mathbf{w}')$ for any $x \in \mathsf{con}(\mathbf{w}\mathbf{w}')$. Then, for any substitution $\varphi : \mathcal{A} \to T_2(\mathbb{F})$, $(\mathbf{w}\varphi)_{11} = (\mathbf{w}'\varphi)_{11}$ and $(\mathbf{w}\varphi)_{22} = (\mathbf{w}'\varphi)_{22}$. Therefore, $\mathbf{w}\varphi = \mathbf{w}'\varphi$ holds in $T_2(\mathbb{F})$ if and only if $(\mathbf{w}\varphi)_{12} = (\mathbf{w}'\varphi)_{12}$.

Lemma 4.2. *For each $k \geq 2$ and any field $\mathbb{F}$, the involution semigroup $(UT_2(\mathbb{F}), {}^*)$ satisfies the word identity $\mathbf{u}_k \approx \mathbf{u}_k'$, where*

$$\mathbf{u}_k = tx_1tx_2 \cdots tx_ktx_1^*tx_2^* \cdots tx_k^*t, \quad \mathbf{u}_k' = tx_ktx_{k-1} \cdots tx_1tx_k^*tx_{k-1}^* \cdots tx_1^*t.$$

Proof. Let φ be any substitution into $UT_2(\mathbb{F})$. From Remark 4.1, $(\mathbf{u}_k\varphi)_{11} = (\mathbf{u}_k'\varphi)_{11}$ and $(\mathbf{u}_k\varphi)_{22} = (\mathbf{u}_k'\varphi)_{22}$. Therefore, it suffices to show that $(\mathbf{u}_k\varphi)_{12} = (\mathbf{u}_k'\varphi)_{12}$. There are three cases.

Case 1. $(t\varphi)_{22} = 0$. Then

$$\begin{aligned}
(\mathbf{u}_k\varphi)_{12} &= ((tx_1tx_2 \cdots tx_ktx_1^*tx_2^* \cdots tx_k^*t)\varphi)_{12} \\
&= ((tx_1tx_2 \cdots tx_ktx_1^*tx_2^* \cdots tx_k^*)\varphi)_{11}(t\varphi)_{12} \\
&= ((tx_ktx_{k-1} \cdots tx_1tx_k^*tx_{k-1}^* \cdots tx_1^*)\varphi)_{11}(t\varphi)_{12} \quad \text{(by Remark 4.1)} \\
&= ((tx_ktx_{k-1} \cdots tx_1tx_k^*tx_{k-1}^* \cdots tx_1^*t)\varphi)_{12} = (\mathbf{u}_k'\varphi)_{12},
\end{aligned}$$

as required.

Case 2. $(t\varphi)_{11} = 0$. This case follows from the dual result of Case 1.

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Case 3. $(t\varphi)_{11} = (t\varphi)_{22} = 1$. Then it is easy to see that $(t\varphi)^* = t\varphi$. Hence,

$$\begin{aligned}
(\mathbf{u}_k\varphi)_{12} &= ((tx_1tx_2\cdots tx_ktx_1^*tx_2^*\cdots tx_k^*t)\varphi)_{12}\\
&= ((tx_1tx_2\cdots tx_ktx_1^*tx_2^*\cdots tx_k^*t)\varphi)_{12}^*\\
&= ((tx_1tx_2\cdots tx_ktx_1^*tx_2^*\cdots tx_k^*t)^*\varphi)_{12}\\
&= ((t^*x_kt^*x_{k-1}\cdots t^*x_1t^*x_k^*t^*x_{k-1}^*\cdots t^*x_1^*t^*)\varphi)_{12}\\
&= ((t^*\varphi)(x_k\varphi)(t^*\varphi)(x_{k-1}\varphi)\cdots(t^*\varphi)(x_1\varphi)(t^*\varphi)(x_k^*\varphi)(t^*\varphi)(x_{k-1}^*\varphi)\cdots(t^*\varphi)(x_1^*\varphi)(t^*\varphi))_{12}\\
&= ((t\varphi)(x_k\varphi)(t\varphi)(x_{k-1}\varphi)\cdots(t\varphi)(x_1\varphi)(t\varphi)(x_k^*\varphi)(t\varphi)(x_{k-1}^*\varphi)\cdots(t\varphi)(x_1^*\varphi)(t\varphi))_{12}\\
&= ((tx_ktx_{k-1}\cdots tx_1tx_k^*tx_{k-1}^*\cdots tx_1^*t)\varphi)_{12} = (\mathbf{u}_k'\varphi)_{12},
\end{aligned}$$

as required. □

Lemma 4.3. *For each $k \geq 2$ and any field $\mathbb{F}$, the involution semigroup $(UT_2^{\pm 1}(\mathbb{F}), {}^*)$ satisfies the word identity $\mathbf{v}_k \approx \mathbf{v}_k'$, where*

$$\mathbf{v}_k = t^2x_1t^2x_2\cdots t^2x_kt^2x_1^*t^2x_2^*\cdots t^2x_k^*t^2, \quad \mathbf{v}_k' = t^2x_kt^2x_{k-1}\cdots t^2x_1t^2x_k^*t^2x_{k-1}^*\cdots t^2x_1^*t^2.$$

Proof. Let φ be any substitution into $UT_2^{\pm 1}(\mathbb{F})$. Then it follows from Remark 4.1 that $(\mathbf{v}_k\varphi)_{11} = (\mathbf{v}_k'\varphi)_{11}$ and $(\mathbf{v}_k\varphi)_{22} = (\mathbf{v}_k'\varphi)_{22}$. Therefore, it suffices to show that $(\mathbf{v}_k\varphi)_{12} = (\mathbf{v}_k'\varphi)_{12}$. Without loss of generality, we may assume that $t\varphi = \left(\begin{smallmatrix} t_{11} & t_{12} \\ 0 & t_{22}\end{smallmatrix}\right)$, where $t_{11}, t_{22} \in \{0, 1, -1\}$ and $t_{12} \in \mathbb{F}$. Then

$$t^2\varphi = \begin{pmatrix} t_{11}^2 & t_{11}t_{12} + t_{12}t_{22} \\ 0 & t_{22}^2 \end{pmatrix}.$$

Since $t_{11}, t_{22} \in \{0, 1, -1\}$, it follows that $t_{11}^2, t_{22}^2 \in \{0, 1\}$. There are three cases.

Case 1. $(t^2\varphi)_{22} = 0$. Then $(\mathbf{v}_k\varphi)_{12} = (\mathbf{v}_k'\varphi)_{12}$ holds by a very similar argument to Case 1 of Lemma 4.2.

Case 2. $(t^2\varphi)_{11} = 0$. This follows from the dual result of Case 1.

Case 3. $(t^2\varphi)_{11} = (t^2\varphi)_{22} = 1$. It is easy to see that $(t^2\varphi)^* = t^2\varphi$. Then $(\mathbf{v}_k\varphi)_{12} = (\mathbf{v}_k'\varphi)_{12}$ holds by a very similar argument to Case 3 of Lemma 4.2. □

Lemma 4.4. *For each $k \geq 2$ and any field $\mathbb{F}$ with* $\mathsf{char}(\mathbb{F}) = p$, *the involution monoid $(T_2(\mathbb{F}), {}^*)$ satisfies the word identity $\mathbf{p}_k \approx \mathbf{p}_k'$, where*

$$\begin{aligned}
\mathbf{p}_k &= t^{p-1}x_1t^{p-1}x_2t^{p-1}\cdots x_kt^{p-1}x_1^*t^{p-1}x_2^*t^{p-1}\cdots x_k^*t^{p-1},\\
\mathbf{p}_k' &= t^{p-1}x_kt^{p-1}x_{k-1}t^{p-1}\cdots x_1t^{p-1}x_k^*t^{p-1}x_{k-1}^*t^{p-1}\cdots x_1^*t^{p-1}.
\end{aligned}$$

Proof. Let φ be any substitution into $T_2(\mathbb{F})$. From Remark 4.1, $(\mathbf{p}_k\varphi)_{11} = (\mathbf{p}_k'\varphi)_{11}$ and $(\mathbf{p}_k\varphi)_{22} = (\mathbf{p}_k'\varphi)_{22}$. Therefore, it suffices to show that $(\mathbf{p}_k\varphi)_{12} = (\mathbf{p}_k'\varphi)_{12}$. Without loss of generality, we may assume that $t\varphi = \left(\begin{smallmatrix} t_{11} & t_{12} \\ 0 & t_{22}\end{smallmatrix}\right)$, where $t_{11}, t_{22}, t_{12} \in \mathbb{F}$. Then

$$t^{p-1}\varphi = \begin{pmatrix} t_{11}^{p-1} & s \\ 0 & t_{22}^{p-1} \end{pmatrix}$$

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for some appropriate $s \in \mathbb{F}$. It follows from Fermat's little theorem that if $(t_{11}, p) = 1$, then $t_{11}^{p-1} \equiv 1 \pmod{p}$; if $(t_{11}, p) \neq 1$, then $t_{11} = kp$ for some k and so $t_{11}^{p-1} \equiv 0 \pmod{p}$. Hence, $t_{11}^{p-1} \in \{0, 1\}$. Similarly, $t_{22}^{p-1} \in \{0, 1\}$. There are three cases.

Case 1. $(t^{p-1}\varphi)_{22} = 0$. Then $(\mathbf{p}_k\varphi)_{12} = (\mathbf{p}'_k\varphi)_{12}$ holds by a very similar argument to Case 1 of Lemma 4.2.

Case 2. $(t^{p-1}\varphi)_{11} = 0$. This follows from the dual result of Case 1.

Case 3. $(t^{p-1}\varphi)_{11} = (t^{p-1}\varphi)_{22} = 1$. Then it is easy to see that $(t^{p-1}\varphi)^* = t^{p-1}\varphi$. Then $(\mathbf{p}_k\varphi)_{12} = (\mathbf{p}'_k\varphi)_{12}$ holds by a very similar argument to Case 3 of Lemma 4.2. □

Lemma 4.5. *Suppose that* $\mathbf{w} \approx \mathbf{w}'$ *is any word identity satisfied by the involution monoid* $(UT_2(\mathbb{F}), ^*)$ *with any field* $\mathbb{F}$. *Then* $\mathsf{con}(\mathbf{w}) = \mathsf{con}(\mathbf{w}')$.

Proof. Suppose that $x \in \mathsf{con}(\mathbf{w}) \setminus \mathsf{con}(\mathbf{w}')$ for some x. Let $\varphi : \mathcal{A} \to UT_2(\mathbb{F})$ be the substitution that maps x to $\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)$ and any other variable to $\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$. Then

$$(\mathbf{w}x^*)\varphi \in \left\{\begin{pmatrix}1&0\\0&0\end{pmatrix}, \begin{pmatrix}0&0\\0&1\end{pmatrix}\right\}^\times \cdot \begin{pmatrix}1&0\\0&0\end{pmatrix} \cdot \left\{\begin{pmatrix}1&0\\0&0\end{pmatrix}, \begin{pmatrix}0&0\\0&1\end{pmatrix}\right\}^\times \cdot \begin{pmatrix}0&0\\0&1\end{pmatrix} = \begin{pmatrix}0&0\\0&0\end{pmatrix}$$

and

$$(\mathbf{w}'x^*)\varphi \in \left\{\begin{pmatrix}0&0\\0&1\end{pmatrix}\right\}^\times \cdot \begin{pmatrix}0&0\\0&1\end{pmatrix} = \begin{pmatrix}0&0\\0&1\end{pmatrix},$$

which implies the contradiction $(\mathbf{w}x^*)\varphi \neq (\mathbf{w}'x^*)\varphi$. Consequently, the inclusion $\mathsf{con}(\mathbf{w}) \subseteq \mathsf{con}(\mathbf{w}')$ holds. The inclusion $\mathsf{con}(\mathbf{w}') \subseteq \mathsf{con}(\mathbf{w})$ holds by symmetry. □

Lemma 4.6. *Suppose that the involution monoid* $(UT_2(\mathbb{F}), ^*)$ *with any field* $\mathbb{F}$ *satisfies a word identity* $\mathbf{w} \approx \mathbf{w}'$ *and* ${}_lx \prec {}_ly \prec {}_fx^* \prec {}_fy^*$ *in* $\mathbf{w}$ *for some* $x, y, x^*, y^* \in \mathsf{con}(\mathbf{w})$. *Then either* ${}_lx \prec {}_ly \prec {}_fx^* \prec {}_fy^*$ *or* ${}_ly \prec {}_lx \prec {}_fy^* \prec {}_fx^*$ *holds in* $\mathbf{w}'$.

Proof. Clearly, $\mathbf{w}(x, y) \in \{x, y\}^\times \cdot x \cdot \{y\}^\times \cdot yx^* \cdot \{x^*\}^\times \cdot y^* \cdot \{x^*, y^*\}^\times$. It follows from Lemma 4.5 that $\mathsf{con}(\mathbf{w}'(x, y)) = \{x, y, x^*, y^*\}$. To complete the proof, it remains to show that none of xx^*, yy^*, x^*x, y^*y, x^*y and y^*x is a factor of $\mathbf{w}'(x, y)$. Working toward a contradiction, suppose that one of xx^*, yy^*, x^*x, y^*y, x^*y and y^*x is a factor of $\mathbf{w}'(x, y)$. Then, by letting φ be the substitution that maps x to $\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)$ and y to $\left(\begin{smallmatrix}1&1\\0&0\end{smallmatrix}\right)$,

$$\begin{aligned}\mathbf{w}(x, y)\varphi \in{} & \left\{\begin{pmatrix}1&0\\0&0\end{pmatrix}, \begin{pmatrix}1&1\\0&0\end{pmatrix}\right\}^\times \cdot \begin{pmatrix}1&0\\0&0\end{pmatrix} \cdot \left\{\begin{pmatrix}1&1\\0&0\end{pmatrix}\right\}^+ \cdot \left\{\begin{pmatrix}0&0\\0&1\end{pmatrix}\right\}^+ \cdot \begin{pmatrix}0&1\\0&1\end{pmatrix} \\ & \cdot \left\{\begin{pmatrix}0&0\\0&1\end{pmatrix}, \begin{pmatrix}0&1\\0&1\end{pmatrix}\right\}^\times \\ ={} & \left\{\begin{pmatrix}1&0\\0&0\end{pmatrix}, \begin{pmatrix}1&1\\0&0\end{pmatrix}\right\}^\times \cdot \begin{pmatrix}0&1\\0&0\end{pmatrix} \cdot \left\{\begin{pmatrix}0&0\\0&1\end{pmatrix}, \begin{pmatrix}0&1\\0&1\end{pmatrix}\right\}^\times = \begin{pmatrix}0&1\\0&0\end{pmatrix}\end{aligned}$$

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and

$$\mathbf{w}'(x,y)\varphi \in \begin{cases} \cdots \begin{pmatrix}1&0\\0&0\end{pmatrix}\cdot\begin{pmatrix}0&0\\0&1\end{pmatrix}\cdots = \begin{pmatrix}0&0\\0&0\end{pmatrix} & \text{if } xx^* \text{ is a factor of } \mathbf{w}'(x,y),\\ \cdots \begin{pmatrix}1&1\\0&0\end{pmatrix}\cdot\begin{pmatrix}0&1\\0&1\end{pmatrix}\cdots \in \left\{\begin{pmatrix}0&0\\0&0\end{pmatrix},\begin{pmatrix}0&2\\0&0\end{pmatrix}\right\} & \text{if } yy^* \text{ is a factor of } \mathbf{w}'(x,y),\\ \cdots \begin{pmatrix}0&0\\0&1\end{pmatrix}\cdot\begin{pmatrix}1&0\\0&0\end{pmatrix}\cdots = \begin{pmatrix}0&0\\0&0\end{pmatrix} & \text{if } x^*x \text{ is a factor of } \mathbf{w}'(x,y),\\ \cdots \begin{pmatrix}0&1\\0&1\end{pmatrix}\cdot\begin{pmatrix}1&1\\0&0\end{pmatrix}\cdots = \begin{pmatrix}0&0\\0&0\end{pmatrix} & \text{if } y^*y \text{ is a factor of } \mathbf{w}'(x,y),\\ \cdots \begin{pmatrix}0&0\\0&1\end{pmatrix}\cdot\begin{pmatrix}1&1\\0&0\end{pmatrix}\cdots = \begin{pmatrix}0&0\\0&0\end{pmatrix} & \text{if } x^*y \text{ is a factor of } \mathbf{w}'(x,y),\\ \cdots \begin{pmatrix}0&1\\0&1\end{pmatrix}\cdot\begin{pmatrix}1&0\\0&0\end{pmatrix}\cdots = \begin{pmatrix}0&0\\0&0\end{pmatrix} & \text{if } y^*x \text{ is a factor of } \mathbf{w}'(x,y),\end{cases}$$

which implies the contradiction $\mathbf{w}(x,y)\varphi \neq \mathbf{w}'(x,y)\varphi$. Therefore, none of xx^*, yy^*, x^*x, y^*y, x^*y and y^*x is a factor of $\mathbf{w}'(x,y)$, whence the only possibility of $\mathbf{w}'(x,y)$ is either $\{x,y\}^\times \cdot x \cdot \{y\}^\times \cdot yx^* \cdot \{x^*\}^\times \cdot y^* \cdot \{x^*,y^*\}^\times$ or $\{x,y\}^\times \cdot y \cdot \{x\}^\times \cdot xy^* \cdot \{y^*\}^\times \cdot x^* \cdot \{x^*,y^*\}^\times$. Hence, either ${}_lx \prec {}_ly \prec {}_fx^* \prec {}_fy^*$ or ${}_ly \prec {}_lx \prec {}_fy^* \prec {}_fx^*$ holds in $\mathbf{w}'$. □

Lemma 4.7. *Suppose that the involution monoid $(UT_2(\mathbb{F}),{}^*)$ satisfies a word identity of the form $\mathbf{w} = \cdots {}_lx\,\mathbf{a}\,{}_fx^* \cdots \approx \cdots {}_lx\,\mathbf{a}'\,{}_fx^* \cdots = \mathbf{w}'$ for some words $\mathbf{a},\mathbf{a}'$. Then, for any variable $t \in \mathsf{con}(\mathbf{w})$ and $t^* \notin \mathsf{con}(\mathbf{w})$:*

(i) $\mathsf{occ}(t,\mathbf{a}) \equiv \mathsf{occ}(t,\mathbf{a}') \pmod p$ *if* $\mathsf{char}(\mathbb{F}) = p$; *and*

(ii) $\mathsf{occ}(t,\mathbf{a}) = \mathsf{occ}(t,\mathbf{a}')$ *if* $\mathsf{char}(\mathbb{F}) = 0$.

Proof. From Lemma 4.5, $t \in \mathsf{con}(\mathbf{w}')$, $t^* \notin \mathsf{con}(\mathbf{w}')$. Let $a = \mathsf{occ}(t,\mathbf{a})$, $b = \mathsf{occ}(t,\mathbf{a}')$. Let φ be the substitution that maps t to $\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$, x to $\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)$ and $y \neq x, t$ to $\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$. Then

$$\begin{aligned}\mathbf{w}\varphi &= (\cdots {}_lx\,\mathbf{a}\,{}_fx^*\cdots)\varphi\\ &= \left\{\begin{pmatrix}1&1\\0&1\end{pmatrix},\begin{pmatrix}1&0\\0&0\end{pmatrix}\right\}^\times\cdot\left(\begin{pmatrix}1&0\\0&0\end{pmatrix}\cdot\begin{pmatrix}1&1\\0&1\end{pmatrix}^a\cdot\begin{pmatrix}0&0\\0&1\end{pmatrix}\right)\cdot\left\{\begin{pmatrix}1&1\\0&1\end{pmatrix},\begin{pmatrix}0&0\\0&1\end{pmatrix}\right\}^\times\\ &= \left\{\begin{pmatrix}1&1\\0&1\end{pmatrix},\begin{pmatrix}1&0\\0&0\end{pmatrix}\right\}^\times\cdot\begin{pmatrix}0&a\\0&0\end{pmatrix}\cdot\left\{\begin{pmatrix}1&1\\0&1\end{pmatrix},\begin{pmatrix}0&0\\0&1\end{pmatrix}\right\}^\times = \begin{pmatrix}0&a\\0&0\end{pmatrix}\end{aligned}$$

and

$$\begin{aligned}\mathbf{w}'\varphi &= (\cdots {}_lx\,\mathbf{a}'\,{}_fx^*\cdots)\varphi\\ &= \left\{\begin{pmatrix}1&1\\0&1\end{pmatrix},\begin{pmatrix}1&0\\0&0\end{pmatrix}\right\}^\times\cdot\left(\begin{pmatrix}1&0\\0&0\end{pmatrix}\cdot\begin{pmatrix}1&1\\0&1\end{pmatrix}^b\cdot\begin{pmatrix}0&0\\0&1\end{pmatrix}\right)\cdot\left\{\begin{pmatrix}1&1\\0&1\end{pmatrix},\begin{pmatrix}0&0\\0&1\end{pmatrix}\right\}^\times\\ &= \left\{\begin{pmatrix}1&1\\0&1\end{pmatrix},\begin{pmatrix}1&0\\0&0\end{pmatrix}\right\}^\times\cdot\begin{pmatrix}0&b\\0&0\end{pmatrix}\cdot\left\{\begin{pmatrix}1&1\\0&1\end{pmatrix},\begin{pmatrix}0&0\\0&1\end{pmatrix}\right\}^\times = \begin{pmatrix}0&b\\0&0\end{pmatrix}.\end{aligned}$$

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Since the involution monoid $(UT_2(\mathbb{F}), {}^*)$ satisfies the word identity $\mathbf{w} \approx \mathbf{w}'$, it follows that $\left(\begin{smallmatrix} 0 & a \\ 0 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & b \\ 0 & 0 \end{smallmatrix}\right)$. Therefore, if $\mathsf{char}(\mathbb{F}) = p$, then $a \equiv b \pmod{p}$; and, if $\mathsf{char}(\mathbb{F}) = 0$, then $a = b$. □

THEOREM 4.8. *For any field* $\mathbb{F}$*, the involution monoid* $(UT_2(\mathbb{F}), {}^*)$ *is nonfinitely based.*

PROOF. Let $\mathsf{char}(\mathbb{F}) = p$. Take $n = p$. By Lemmas 4.2, 4.6 and 4.7(i), the involution monoid $(UT_2(\mathbb{F}), {}^*)$ satisfies conditions (I), (II) and (IV) of Theorem 3.2, respectively. The result thus follows from Theorem 3.2.

Let $\mathsf{char}(\mathbb{F}) = 0$. Let $n \geq 2$ be any integer. By Lemmas 4.2, 4.6 and 4.7(ii), the involution monoid $(UT_2(\mathbb{F}), {}^*)$ satisfies conditions (I), (II) and (IV) of Theorem 3.2, respectively. The result thus follows from Theorem 3.2. This result can also be derived from [24, Theorem 12]. □

THEOREM 4.9. *Let* $n = p$ *be a prime and* $(S, {}^*)$ *be any involution monoid that satisfies condition (I). Suppose that the involution monoid* $(UT_2(\mathbb{F}), {}^*)$ *with* $\mathsf{char}(\mathbb{F}) = p$ *belongs to the variety generated by* $(S, {}^*)$*. Then* $(S, {}^*)$ *is nonfinitely based.*

PROOF. From Lemmas 4.6 and 4.7(i), the involution monoid $(UT_2(\mathbb{F}), {}^*)$ satisfies conditions (II) and (IV) of Theorem 3.2, respectively. Since $(UT_2(\mathbb{F}), {}^*)$ belongs to the variety generated by the involution monoid $(S, {}^*)$, it follows that $(S, {}^*)$ also satisfies conditions (II) and (IV) of Theorem 3.2. By Theorem 3.2, $(S, {}^*)$ is nonfinitely based. □

THEOREM 4.10. *Let* $n \geq 2$ *be an integer and* $(S, {}^*)$ *be any involution monoid that satisfies condition (I). Suppose that the involution monoid* $(UT_2(\mathbb{F}), {}^*)$ *with* $\mathsf{char}(\mathbb{F}) = 0$ *belongs to the variety generated by* $(S, {}^*)$*. Then* $(S, {}^*)$ *is nonfinitely based.*

PROOF. From Lemmas 4.6 and 4.7(ii), the involution monoid $(UT_2(\mathbb{F}), {}^*)$ satisfies conditions (II) and (IV) of Theorem 3.2, respectively. Since $(UT_2(\mathbb{F}), {}^*)$ belongs to the variety generated by the involution monoid $(S, {}^*)$, it follows that $(S, {}^*)$ also satisfies conditions (II) and (IV) of Theorem 3.2. By Theorem 3.2, $(S, {}^*)$ is nonfinitely based. □

COROLLARY 4.11. *For any field* $\mathbb{F}$*, the involution monoid* $(UT_2^{\pm 1}(\mathbb{F}), {}^*)$ *is nonfinitely based.*

PROOF. Since $(UT_2(\mathbb{F}), {}^*)$ is an involution submonoid of $(UT_2^{\pm 1}(\mathbb{F}), {}^*)$, the result follows from Lemma 4.3 and Theorem 4.9 if $\mathsf{char}(\mathbb{F}) = p$, and from Lemma 4.3 and Theorem 4.10 if $\mathsf{char}(\mathbb{F}) = 0$. □

COROLLARY 4.12. *For any field* $\mathbb{F}$*, the involution monoid* $(T_2(\mathbb{F}), {}^*)$ *is nonfinitely based.*

PROOF. Note that $(UT_2(\mathbb{F}), {}^*)$ is an involution submonoid of $(T_2(\mathbb{F}), {}^*)$. If $\mathsf{char}(\mathbb{F}) = p$, then the result follows from Lemma 4.4 and Theorem 4.9; if $\mathsf{char}(\mathbb{F}) = 0$, then the result follows from [24, Corollary 14]. □

THEOREM 4.13. *For any field* $\mathbb{F}$*, the involution monoids* $(UT_2(\mathbb{F}), {}^*)$*,* $(UT_2^{\pm 1}(\mathbb{F}), {}^*)$ *and* $(T_2(\mathbb{F}), {}^*)$ *are nonfinitely based.*

PROOF. It follows from Theorem 4.8 and Corollaries 4.11 and 4.12. □

REMARK 4.14. It is clear that the proof of Theorem 4.13 works for all involution monoids $(UT_2(R), {}^*)$, $(UT_2^{\pm 1}(R), {}^*)$ and $(T_2(R), {}^*)$, where R is an associative ring with a unit element 1 such that either:

(†) for every positive integer n, $\underbrace{1 + 1 + \cdots + 1}_{n} \neq 0$; or

(‡) for some prime p, $\underbrace{1 + 1 + \cdots + 1}_{p} = 0$.

Therefore, Theorem 4.13 is still true if we substitute the field $\mathbb{F}$ by an arbitrary associative ring R with a unit element 1 satisfying either (†) or (‡). In particular, Theorem 4.13 is true for any integral domain or division ring. For example, the involution monoids $(UT_2(\mathbb{Z}), {}^*)$, $(UT_2^{\pm 1}(\mathbb{Z}), {}^*)$, $(T_2(\mathbb{Z}), {}^*)$, $(UT_2(\mathbb{Z}_p, {}^*)$, $(UT_2^{\pm 1}(\mathbb{Z}_p), {}^*)$ and $(T_2(\mathbb{Z}_p), {}^*)$, where $\mathbb{Z}$ is the ring of integers and $\mathbb{Z}_p$ is the ring of integers modulo p, are nonfinitely based.

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References

[1] J. Almeida, *Finite Semigroups and Universal Algebra* (World Scientific, Singapore, 1994).
[2] K. Auinger, I. Dolinka, T. V. Pervukhina and M. V. Volkov, 'Unary enhancements of inherently non-finitely based semigroups', *Semigroup Forum* **89**(1) (2014), 41–51.
[3] K. Auinger, I. Dolinka and M. V. Volkov, 'Matrix identities involving multiplication and transposition', *J. Eur. Math. Soc. (JEMS)* **14**(3) (2012), 937–969.
[4] K. Auinger, I. Dolinka and M. V. Volkov, 'Equational theories of semigroups with involution', *J. Algebra* **369** (2012), 203–225.
[5] Y. A. Bahturin and A. Y. Ol'shanskii, 'Identical relations in finite Lie rings', *Math. Sb. (N.S.)* **96**(138) (1975), 543–559; (in Russian); English translation, *Math. USSR Sb.* **25** (1975), 507–523.
[6] S. Burris and H. P. Sankappanavar, *A Course in Universal Algebra* (Springer, New York, 1981).
[7] Y. Z. Chen, X. Hu and Y. F. Luo, 'On the finite basis problem for a certain semigroup of upper triangular matrices over a field', *J. Algebra Appl.* **15** (2016), 1650177.
[8] S. Crvenkoic, I. Dolinka and Z. Ésik, 'The variety of Kleene algebras with conversion is not finitely based', *Theoret. Comput. Sci.* **230** (2000), 235–245.
[9] J. Ježek, 'Nonfinitely based three-element idempotent groupoids', *Algebra Universalis* **20** (1985), 292–301.
[10] R. Kruse, 'Identities satisfied in a finite ring', *J. Algebra* **26** (1973), 298–318.
[11] E. W. H. Lee, 'Finite involution semigroups with infinite irredundant bases of identities', *Forum Math.* **28**(3) (2016), 587–607.
[12] J. R. Li and Y. F. Luo, 'On the finite basis problem for the monoids of triangular boolean matrices', *Algebra Universalis* **65** (2011), 353–362.
[13] I. V. L'vov, 'Varieties of associative rings I', *Algebra Logika* **12** (1973), 269–297.
[14] R. McKenzie, 'Equational bases for lattice theories', *Math. Scand.* **27** (1970), 24–38.
[15] R. McKenzie, 'Tarski's finite basis problem is undecidable', *Internat. J. Algebra Comput.* **6** (1996), 49–104.

https://doi.org/10.1017/S0004972719001035 Published online by Cambridge University Press

[16] V. L. Murskiĭ, 'The existence in the three-valued logic of a closed class with a finite basis having no finite complete system of identities', *Sov. Math. Dokl.* **6** (1965), 1020–1024.
[17] S. Oates and M. B. Powell, 'Identical relations in finite groups', *J. Algebra* **1** (1964), 11–39.
[18] P. Perkins, 'Bases for equational theories of semigroups', *J. Algebra* **11** (1969), 298–314.
[19] P. Perkins, 'Finite axiomatizability for equational theories of computable groupoids', *J. Symbolic Logic* **54** (1989), 1018–1022.
[20] L. N. Shevrin and M. V. Volkov, 'Identities of semigroups', *Izv. Vyssh. Uchebn. Zaved. Mat.* **29**(11) (1985), 3–47; (in Russian); English translation, *Sov. Math. (Iz. VUZ)* **29**(11) (1985), 1–64.
[21] M. V. Volkov, 'The finite basis problem for finite semigroups', *Sci. Math. Jpn.* **53** (2001), 171–199.
[22] M. V. Volkov, 'A nonfinitely based semigroup of triangular matrices', in: *Semigroups, Algebra and Operator Theory* (eds. J. Meakin, A. R. Rajan and P. G. Romeo) (Springer, New Delhi–Heidelberg–New York–Dordrecht–London, 2015), 27–38.
[23] M. V. Volkov and I. A. Goldberg, 'Identities of semigroups of triangular matrices over finite fields', *Math. Notes* **73** (2003), 474–481; English translation of Mat. Zametki **73** (2003), 502–510.
[24] W. T. Zhang, Y. D. Ji and Y. F. Luo, 'The finite basis problem for infinite involution semigroups of triangular 2×2 matrices', *Semigroup Forum* **94** (2017), 426–441.
[25] W. T. Zhang, J. R. Li and Y. F. Luo, 'On the variety generated by the monoid of triangular 2×2 matrices over a two-element field', *Bull. Aust. Math. Soc.* **86**(1) (2012), 64–77.
[26] W. T. Zhang, J. R. Li and Y. F. Luo, 'Hereditarily finitely based semigroups of triangular matrices over the two-element field', *Semigroup Forum* **86**(2) (2013), 229–261.

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