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SOME REMARKS ON RELATIVE STABILITY

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1. Introduction

In 1962 Lakshmikantham ([1], [2]) extended the concept of extreme stability (e.g. [4]) of a system described by an ordinary differential equation, not necessarily with uniqueness, to relative stability of two such systems. Here we show the restrictiveness of his definition of relative stability in that it implies not only are the solutions of two systems unique for each initial condition, they are in fact identical. We then introduce and give an example of a weaker version of relative stability which is of some interest for control systems. For greater simplicity and generality we use Roxin's attainability set defined General Control Systems [3] to describe the dynamics of our systems, as they subsume both ordinary differential equations without uniqueness and ordinary differential control equations.

2. Definitions

Let X be a complete locally compact metric space with bounded metric ρ.

A *General Control System* with state space X is given in terms of an attainability set function $F(x_0, t_0, t_1)$ which is defined for all $x_0 \in X$ and $t_1 \geqq t_0 \geqq 0$. The attainability sets are assumed nonempty and compact with $F(x_0, t_0, t_0) = \{x_0\}$. Other assumptions, not required here, are also made and be found in [3].

A *motion* of a General Control Systems is defined as a time function $\phi: [t_0, \infty) \to X$, for some $t_0 \geqq 0$, satisfying $\phi(t_2) \in F(\phi(t_1), t_1, t_2)$ for all $t_2 \geqq t_1 \geqq t_0 \geqq 0$. Their existence and continuity have been established by Roxin [3] and we denote by $\Phi(x_0, t_0; F)$ the set of all motions with $\phi(t_0) = x_0$ of a General Control System F.

Let A and B be two nonempty subsets of X. We define a distance associated with them by

$$\gamma(A, B) = \sup\{\rho(a, b);\ a \in A, b \in B\}$$

Clearly $\gamma(A, B) = 0$ if and only if $A = B = \{a_0\}$ for some $a_0 \in X$.

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The following is Lakshmikantham's definition of relative stability in terms of General Control Systems.

DEFINITION 1. Two General Control Systems F_1 and F_2 on a state space X are said to be *relatively strongly equi-stable* if for each $\varepsilon > 0$ and $t_0 \geqq 0$ there exists a $\delta = \delta(t_0, \varepsilon) > 0$ such that

$$\gamma(F_1(x_0, t_0, t), F_2(y_0, t_0, t)) < \varepsilon \tag{1}$$

for all $x_0, y_0 \in X$ with $\rho(x_0, y_0) < \delta$ and all $t \geqq t_0$.

In view of the compactness of the attainability sets of General Control Systems (1) is equivalent to $\rho(\phi_1(t), \phi_2(t)) < \varepsilon$ for all motions $\phi_1 \in \Phi(x_0, t_0; F_1)$ and $\phi_2 \in \Phi(y_0, t_0; F_2)$. This is in fact how Lakshmikantham stated his definition.

When the two General Control Systems in definition 1 are identical we call it *extreme strong equi-stability.* It should however be observed that Yoshizawa's definition of extreme stability [4] is much stronger than ours, being in fact a global asymtotic version of it.

In analogy with Roxin's weak Lyapunov stability of a set [3] we define relative weak equi-stability.

DEFINITION 2. Two General Control Systems F_1 and F_2 on a state space X are said to be *relatively weakly equi-stable* if for each $t_0 \geqq 0$ and $\varepsilon > 0$ there exists a $\delta = \delta(t_0, \varepsilon) > 0$ such that for any $x_0, y_0 \in X$ with $\rho(x_0, y_0) < \delta$ there can be found motions $\phi_1 \in \Phi(x_0, t_0; F_1)$ and $\phi_2 \in \Phi(y_0, t_0; F_2)$ satisfying $\rho(\phi_1(t), \phi_2(t)) < \varepsilon$ for all $t \geqq t_0$.

Here too we call it *extreme weak equi-stability* if the two systems are identical. Moreover, in both definition 1 and 2 we include the term "uniform" if δ is independent of t_0.

3. Relative stability

Our main result is the following theorem which shows that Lakshmikantham's definition of relative strong equi-stability is very restrictive and also superfluous as it reduces to extreme strong equi-stability.

THEOREM 1. Let F_1 and F_2 be two relatively strongly equi-stable General Control Systems on a state space X. Then they are identical and consist of a unique motion for each initial $x_0 \in X$ and $t_0 \geqq 0$.

PROOF. Let $\delta = \delta(t_0, \varepsilon) > 0$ be that from definition 1 and let $\varepsilon = \varepsilon(t_0, \delta)$ be its inverse for each $t_0 \geqq 0$. We can assume without loss of generality that $\varepsilon(t_0, \delta) \to 0$ as $\delta \to 0$ for each $t_0 \geqq 0$.

Now $0 = \rho(x_0, x_0) < \delta$ for any $x_0 \in X$ and $\delta > 0$, so from the relative strong equi-stability of F_1 and F_2 we have

$$0 \leqq \gamma(F_1(x_0, t_0, t), F_2(x_0, t_0, t)) < \varepsilon(t_0, \delta)$$

and hence on taking $\delta \to 0$

$$\gamma(F_1(x_0, t_0, t), F_2(x_0, t_0, t)) = 0$$

for all $x_0 \in X$ and $t \geqq t_0 \geqq 0$.

Thus $F_1(x_0, t_0, t) = F_2(x_0, t_0, t) = \{\phi(x_0, t_0; t)\} \in X$ for all $x_0 \in X$ and $t \geqq t_0 \geqq 0$. Clearly for each $x_0 \in X$ and $t_0 \geqq 0$ this $\phi(x_0, t_0; \cdot)$ is a motion.

This completes the proof.

COROLLARY. If F is an extremely strongly equi-stable General Control System on a state space X then it has a unique motion for each initial $x_0 \in X$ and $t_0 \geqq 0$.

For relative weak equi-stability we have an analogous, yet far less restrictive result to that of theorem 1. As the proof is quite similar we omit it.

THEOREM 2. If F_1 and F_2 are two relatively weakly equi-stable General Control Systems on a state space X then

$$\bigcap_{i=1}^{2} \Phi(x_0, t_0; F_i) \neq \varnothing$$

for all $x_0 \in X$ and $t_0 \geqq 0$ i.e. they have at least one motion in common for each initial condition.

As an example we define two General Control Systems F_1 and F_2 on $X = R^+$ by

$$F_1(x_0, t_0, t) = [x_0, x_0 \exp(t - t_0)]$$

and

$$F_2(x_0, t_0, t) = [x_0 \exp(t_0 - t), x_0]$$

for all $x_0 \in X$ and $t \geqq t_0 \geqq 0$. (These two systems are actually generated by the autonomous ordinary differential control equation $x' = ux$ on X, with openloop controls constrained to $0 \leqq u \leqq 1$ and $-1 \leqq u \leqq 0$ respectively.)

It is easy to see that F_1 and F_2 are uniformly relatively weakly equi-stable with $\delta(\varepsilon) = \varepsilon$. Moreover, each is also uniformly extremely weakly equi-stable.

A possible application for relative weak equi-stability arises in cooperative two-person dynamical games in which the control actions of the players are uncoupled e.g. the inflight refuelling of a fighter by a tanker aircraft.

4. Lyapunov conditions

We could derive from first principles necessary and sufficient Lyapunov conditions for these relative strong and weak equi-stabilities. Instead, we indicate how they follow as paraphrases of already known results on strong and weak Lyapunov stabilities of a set for General Control Systems.

Let F_1 and F_2 be two General Control Systems on a state space X. We define their *cartesian product* $F_1 \times F_2$ on $X \times X$ by

$$F_1 \times F_2(x_0, y_0, t_0, t) = F_1(x_0, t_0, t) \times F_2(y_0, t_0, t)$$

for all $x_0, y_0 \in X$ and $t \geqq t_0 \geqq 0$. It is easily seen, with any convenient product metric on $X \times X$ that $F_1 \times F_2$ is a General Control System on $X \times X$. Moreover for any $x_0, y_0 \in X$ and $t_0 \geqq 0$ its set of motions is

$$\Phi(x_0, y_0, t_0; F_1 \times F_2) = \Phi(x_0, t_0; F_1) \times \Phi(y_0, t_0; F_2).$$

Now let $\Delta = \cup\{(x, x);\, x \in X\}$ be the diagonal subset of $X \times X$. Then it is easily seen that definition 1 of relative strong equi-stability of F_1 and F_2 is equivalent to the strong Lyapunov stability of the set Δ with respect to their cartesian product $F_1 \times F_2$. Similarly, the relative weak equi-stability of F_1 and F_2 is equivalent to the weak Lyapunov stability of Δ with respect to $F_1 \times F_2$.

We leave to the reader the actual paraphrasing of the necessary and sufficient Lyapunov conditions in [3] for the relative stabilities considered here.

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