THE WIELANDT SUBGROUP OF A POLYCYCLIC GROUP

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The purpose of this paper is to establish some basic properties of the Wielandt subgroup of a polycyclic group. The Wielandt subgroup of a group G is defined to be the intersection of the normalisers of all the subnormal subgroups of G and is denoted by $\omega(G)$. In 1958 Wielandt **[9]** showed that any minimal normal subgroup with the minimum condition on subnormal subgroups is contained in the Wielandt subgroup: it follows that the Wielandt subgroup of an artinian group is nontrivial. In contrast, the Wielandt subgroup of a polycyclic group can be trivial; an easy example is given by the infinite dihedral group. We will show that the Wielandt subgroup of a polycyclic group is close to being central.

For the case of nilpotent groups, the basic properties of the Wielandt subgroup are known. Schenkman **[7]** has shown that the Wielandt subgroup of a nilpotent group is always contained in the second centre. Clearly the Wielandt subgroup always contains the centre: it is not difficult to find examples of finite nilpotent groups in which the Wielandt subgroup is strictly larger than the centre. However in torsion free nilpotent groups the Wielandt subgroup always coincides with the centre (see for example Robinson **[6]**, Exercise 13.3.3).

We begin by observing that these results can be generalised to residually nilpotent groups. Note that if G is a group with a subnormal subgroup N, $\omega(G)\cap N\leq\omega(N)$ and if M is normal in G, $\omega(G)M/M\leq\omega(G/M)$: we will use these facts without further comment.

LEMMA. (i) *Let G be a residually nilpotent group. Then $\omega(G)\leq\zeta_2(G)$.*

(ii) *Let G be a residually finite p-group for some prime p and suppose that $\omega(G)$ is torsion free. Then $\omega(G)=\zeta(G)$.*

Proof. (i) is clear. To prove (ii) suppose that $x\in\omega(G)\setminus\zeta(G)$ and let $y=G$ with $[x,y]\neq 1$. Since $x=\zeta_2(G)$ by (i), $\langle x,y\rangle$ is nilpotent of class 2. Then for some normal subgroup N of G of p-power index we have $[x,y]\notin N$. Suppose that the order of yN is p^n. If M is any normal subgroup of p-power index in G, we have that xM and yM generate a metacyclic group (since xM normalises $\langle yM\rangle$) of class at most 2 (since $\langle x,y\rangle$ has class 2). Since $[x,y]$ has infinite order, we can then choose a normal subgroup $M\leq N$ of G of p-power index such that the order of $[x,y]M$ is greater than p^{2n}. Since xM normalises $\langle yM\rangle$ and $\langle x,y\rangle$ has class 2, we have $[y,x]M=y^{ap^s}M$, with a prime to p, and then $[y,x,x]M=y^{a^2p^{2s}}M=M$, giving $s\geq n$. But then $[y,x]N=y^{ap^s}N=N$ and so $[x,y]\in N$, a contradiction.

If G is a polycyclic group we show that $C_G(\omega(G))$ has finite index in G. For some groups we can prove more than this: polycyclic groups are nilpotent-by-abelian-by-finite and we show that $\zeta(G)$ has finite index in $\omega(G)$ for a nilpotent-by-abelian polycyclic group G.

THEOREM 1. (i) *Let G be a polycyclic group. Then $C_G(\omega(G))$ has finite index in G.*

(ii) *Let G be a nilpotent-by-abelian polycyclic group. Then $\zeta(G)$ has finite index in $\omega(G)$.*

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Proof. (i) If $\omega(G)$ is finite the result follows immediately and so we suppose that $\omega(G)$ is infinite; $\omega(G)$ is then abelian (Lennox and Stonehewer [**3**, Theorem 6.4.13]). Further G has a torsion free normal subgroup of finite index, T say, which is residually a finite p-group for some prime p (Segal [**8**, Proposition 1.2 and Theorem 1.4]). It then follows from the Lemma that $\omega(G) \cap T \leq \omega(T) = \zeta(T)$. Put $W = \omega(G)$, $X = W \cap T$ and then set $C = C_G(W)$, $D = C_G(X)$ and $E = C_G(W/X)$. Clearly D and E have finite index in G and $C \leq D \cap E$: to complete the proof it will be enough to show that $C = D \cap E$. Suppose that $a \in D \cap E$, $w \in W$. Then $w^a = wx$ for some $x \in X$. Since W/X is finite, $w^n \in X$ for some integer n. But then $w^n = (w^n)^a = w^n x^n$ and so $x^n = 1$. Since X is torsion free, $x = 1$ and $a \in C$ as required.

(ii) Let F denote the Fitting subgroup of G, so that G/F is abelian. Put $W = \omega(G)$, $Z = \zeta(G)$ and let T denote a finite normal subgroup of G. We may as well suppose that W is infinite and hence abelian: thus we assume $W \leq F$. If $Y/T = \zeta(G/T)$, then Z has finite index in Y. To see this, observe that Y has a torsion free characteristic subgroup of finite index, N say. If $x \in N$ and $y \in G$ we then have $[x, y] \in N \cap T = 1$ and so $N \leq Z$. If Z has infinite index in W, it follows that $\zeta(G/T)$ has infinite index in $\omega(G/T)$. Thus we may assume that G contains no finite normal subgroup: in particular we may assume that F is torsion free.

Suppose that G/F is cyclic of p-power order for some prime p. Since F is residually a finite p-group (Segal [**8**, Theorem 1.4]) so is G and it follows from the Lemma that $W = Z$.

Suppose that G/F is infinite cyclic. If xF generates G/F, x acts as an automorphism, of order e say, on $F/F'F^p$. It then follows as in the proof of Theorem 1.4(ii) of Segal [**8**] that $F\langle x^e \rangle$ is a residually finite p-group. Since $G/F\langle x^e \rangle$ is cyclic, if $A_p/F\langle x^e \rangle$ is the Sylow p-subgroup of $G/F\langle x^e \rangle$, A_p is a residually finite p-group, normal in G with G/A_p a finite p'-group. We then have $W \cap A_p \leq \omega(A_p) = \zeta(A_p)$ by the Lemma. Fix a prime p and let $\mathcal{P}$ be the set of primes dividing G/A_p. Let A denote the intersection of all the A_q and B denote the subgroup generated by all the A_q, where $q = p$ or $q \in \mathcal{P}$. Then $W \cap A$ is contained in the centre of B and has finite index in W. The condition on the indices ensures that $B = G$ and hence that $\zeta(G)$ has finite index in W.

Finally let $G/F = (A_1/F) \times \ldots \times (A_n/F)$, where each A_i/F is infinite cyclic or cyclic of prime power order. We then have $\zeta(A_i)$ has finite index in $\omega(A_i)$ for $i = 1, \ldots, n$ and hence Z, the intersection of the $\zeta(A_i)$, has finite index in the intersection of the $\omega(A_i)$. Since W is contained in the intersection of the $\omega(A_i)$ we have Z has finite index in W, proving the result.

That part (ii) of the Theorem cannot be extended to polycyclic groups in general is shown by the following example.

Let H be the nonabelian group of order 6 and let U be a free abelian group of rank 2 on which H acts faithfully and irreducibly as given in the second of the two representations of Example 1, p. 505 of Curtis and Reiner [**1**]. Regarded as a $\mathbb{Z}H$-module, the tensor square of U has a quotient module V of rank 1 on which H acts as a group of automorphisms of order 2. Now, following the recipe of Huppert [**2**, Hilfssatz VI.7.22], we can construct a nilpotent group A of class 2 on which H acts as a group of automorphisms, with $A/\zeta(A)$ isomorphic to $U \oplus U$ and $\zeta(A)$ isomorphic to V as $\mathbb{Z}H$-modules. Let G be the semidirect product of A and H. Then $\omega(G) \leq \zeta(A)$. We show that $\omega(G) = \zeta(A)$.

To see that $\zeta(A) \le \omega(G)$, let S be a subnormal subgroup of G. If $S \le AH'$ then S is even centralised by $\zeta(A)$. Hence suppose that S is not contained in AH': we have then $SA = G$. Suppose that $S \ne G$: then $S \cap A \ne A$. Let M be maximal in A containing $S \cap A$: then M is normal in A and $|A/M| = p$ for some prime p. It follows that $A'A^pS \ne G$ and hence $A'A^pS/A'A^p$ is a proper subnormal subgroup of $G/A'A^p$. Set $A_0 = A/A'A^p$: we can regard A_0 as a $\mathbb{Z}_pH$-module.

If $p \ne 3$, A_0 is the direct sum of 2 isomorphic faithful irreducible H-modules and so the only subnormal supplement to A_0 in $G/A'A^p$ is $G/A'A^p$. Hence we must have $p = 3$. But if $p = 3$, A_0 is the direct sum of two indecomposable $\mathbb{Z}_3H$-modules, each of which has a unique maximal submodule whose quotient is nontrivial as an H-module. Again it follows that the only subnormal supplement to A_0 is the whole of $G/A'A^p$. Thus $S = G$ and so is normalised by $\zeta(A)$, as required.

We can iterate the Wielandt subgroup of a group G by defining $\omega_1(G) = \omega(G)$ and then $\omega_i(G)/\omega_{i-1}(G) = \omega(G/\omega_{i-1}(G))$: we set $\omega_\infty(G) = \bigcup \omega_n(G)$. If $\omega_{n-1}(G) \ne G$ but $\omega_n(G) = G$ for some n we say that G has Wielandt length n. Another feature enjoyed by artinian groups is that they have finite Wielandt length. In general polycyclic groups will not be of finite Wielandt length: indeed often we will have $\omega_\infty(G) \ne G$. We can give a precise answer to when a polycyclic group has finite Wielandt length.

THEOREM 2. *Let G be a polycyclic group. Then $\omega_\infty(G) = G$ if and only if G is finite-by-nilpotent; and if G is finite-by-nilpotent, G has finite Wielandt length.*

Proof. Since G is noetherian, $\omega_\infty(G) = G$ if and only if G has finite Wielandt length. Thus we need only show that G has finite Wielandt length if and only if G is finite-by-nilpotent.

If G is finite-by-nilpotent then G has either a finite normal subgroup or a nontrivial centre and hence a nontrivial Wielandt subgroup. It follows immediately from the facts that quotients of finite-by-nilpotent groups are finite-by-nilpotent and that G is noetherian that G has finite Weilandt length.

In the other direction, observe that if G has finite Wielandt length so does every quotient of G. Thus it will be enough to show that, if G is not finite-by-nilpotent but every proper quotient of G is, then $\omega(G) = 1$. Hence suppose that G is not finite-by-nilpotent but every quotient of G is, and that $\omega(G) \ne 1$. We have then that G contains no nontrivial finite normal subgroup and so the Fitting subgroup F of G is torsion free and $\omega(G) \le F$. Further G has a nilpotent-by-abelian normal subgroup H of finite index with F properly contained in H. If H were finite-by-nilpotent we would have either H nilpotent or H has a finite normal subgroup and so H is not finite-by-nilpotent. But then Theorem 1 tells us that $\zeta(H)$ has finite index in $\omega(H)$. Since $\omega(H) \ne 1$ by assumption, we have $\zeta(H) \ne 1$ and so $H/\zeta(H)$ is finite-by-nilpotent. It then follows immediately from the fact that a group is finite-by-nilpotent if and only if some finite term of the upper central series has finite index (see for example the comment after Theorem 4.25 of Robinson [**5**]) that H is finite-by-nilpotent, a contradiction. Thus we must have $\omega(G) = 1$.

(I am grateful to Carlo Casolo for pointing out to me that Theorem 2 can also be obtained as an immediate corollary of theorems of Robinson (Theorem 6.5.4 in [**3**]) and McCaughan ([**4**, Theorem 4.1]).)

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REFERENCES

1. C. W. Curtis and I. Reiner, *Representation theory of finite groups and associative algebras* (Interscience Publishers, 1962).

2. B. Huppert, *Endliche Gruppen I* (Springer-Verlag, 1967).

3. J. C. Lennox and S. E. Stonehewer, *Subnormal subgroups of groups*, Oxford Mathematical Monographs (Oxford, 1987).

4. D. J. McCaughan, Subnormality in soluble minimax groups, *J. Austral. Math. Soc.* **27** (1974), 113–128.

5. D. J. S. Robinson, *Finiteness conditions and generalized soluble groups*, Part 1 (Springer-Verlag, 1972).

6. D. J. S. Robinson, *A course in the theory of groups* (Springer-Verlag, 1982).

7. E. Schenkman, On the norm of a group, *Illinois J. Math.* **4** (1960), 150–152.

8. D. Segal, *Polycyclic groups* (Cambridge University Press, 1983).

9. H. Wielandt, Über den Normalisator der Subnormalen Untergruppen, *Math. Z.* **69** (1958), 463–465.

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