A REDUCTION FORMULA

by G. N. WATSON

(Received 11th November, 1953)

1. In this paper I construct a reduction formula for the integral

$$I_n \equiv \int_0^1 x^\alpha (1-x)^\beta \frac{d^n\{x^\gamma (1-x)^\delta\}}{dx^n}\, dx\,; \qquad \dots\dots\dots\dots\dots\dots\dots\dots (1)$$

the formula connects any three consecutive members of a set $I_0, I_1, I_2, \dots, I_m$. We regard m as given, and, in order to avoid wasting time over trivialities, we postulate that the constants $\alpha, \beta, \gamma, \delta$ (which are not restricted to be real) have real parts large enough to ensure (i) that the integrals I_n under consideration and the integrals related to them which will be introduced subsequently are all absolutely convergent, and (ii) that, in all the partial integrations which will be effected, the integrated parts vanish at both limits.

The limitations on the constants can be avoided by using double circuit integrals of Pochhammer's type instead of definite integrals, but it seems desirable to avoid the complexities thereby introduced.

The integrals I_n seem to be of some intrinsic importance ; they are readily expressible in terms of the general *terminating* hypergeometric series of the type $_3F_2$ with last element unity, so that the reduction formula may be regarded as a relation connecting three contiguous functions of this type ; and the reduction formula may be regarded as a soluble linear difference equation of the second order with a non-trivial solution, so that it augments the scanty number of such equations hitherto known.

We observe that

$$I_0 = \frac{\Gamma(\alpha+\gamma+1)\,\Gamma(\beta+\delta+1)}{\Gamma(\alpha+\beta+\gamma+\delta+2)}\cdot$$

and also that

$$I_1 = \frac{\Gamma(\alpha+\gamma)\,\Gamma(\beta+\delta)}{\Gamma(\alpha+\beta+\gamma+\delta+1)}\,(\beta\gamma - \alpha\delta)\,;$$

it is also to be remarked that I_n vanishes for *odd* values of n in two special cases, namely (i) $\alpha = \gamma$ and $\beta = \delta$, and (ii) $\alpha = \beta$ and $\gamma = \delta$; (i) can be proved by effecting n partial integrations while (ii) can be proved by writing $1-x$ in place of the dummy variable x. Strangely enough, both of these special cases occurred in a problem about Legendre polynomials which I have recently been studying, and, by keeping them in mind, I have been enabled to simplify the presentation of parts of the following work. In these special cases, the reduction formula yields simple expressions for I_n (with n even) in terms of Gamma functions.

2. The first stage in the work consists in constructing a linear differential equation of the second order satisfied by the integrand in I_n. We write

$$x^\alpha (1-x)^\beta \frac{d^n\{x^\gamma (1-x)^\delta\}}{dx^n} \equiv y_n, \qquad \dots\dots\dots\dots\dots\dots\dots\dots (2)$$

and, in order to construct the equation with a minimum of calculation, we observe that, if $x^\gamma (1-x)^\delta \equiv z$, then

$$x(1-x)\frac{dz}{dx} - \{\gamma(1-x) - \delta x\}z = 0\,;$$

E

and the result of differentiating this formula $n + 1$ times is

$$x(1-x)\frac{d^{n+2}z}{dx^{n+2}}+\{n+1-\gamma-(2n+2-\gamma-\delta)x\}\frac{d^{n+1}z}{dx^{n+1}}-(n+1)(n-\gamma-\delta)\frac{d^n z}{dx^n}=0 ;$$

this shews that $d^n z/dx^n$ is a solution of the hypergeometric equation with elements $n+1$, $n-\gamma-\delta$, $n+1-\gamma$. This equation is a Riemann P-equation with singularities $0, 1, \infty$ and exponents thereat $(0)\ 0,\ \gamma-n ;\ (1)\ 0,\ \delta-n ;\ (\infty)\ n+1,\ n-\gamma-\delta$.

Hence y_n is a solution of the Riemann P-equation with singularities $0, 1, \infty$ and exponents thereat $(0)\ \alpha,\ \alpha+\gamma-n ;\ (1)\ \beta,\ \beta+\delta-n ;\ (\infty)\ n+1-\alpha-\beta,\ n-\alpha-\beta-\gamma-\delta$. An inspection of the general Riemann P-equation now enables us to see that

$$x(x-1)\frac{d^2 y_n}{dx^2}+\{a_n(x-1)+b_n x\}\frac{dy_n}{dx}+\left(E_n-\frac{\alpha A_n}{x}-\frac{\beta B_n}{1-x}\right)y_n=0, \quad \text{...............(3)}$$

where, for brevity, we have written

$$1-2\alpha-\gamma+n\equiv a_n, \qquad 1-2\beta-\delta+n\equiv b_n, \quad \text{........................(4)}$$

$$\alpha+\gamma-n\equiv A_n, \qquad \beta+\delta-n\equiv B_n, \quad \text{.........................(5)}$$

$$(n+1-\alpha-\beta)(n-\alpha-\beta-\gamma-\delta)\equiv E_n. \quad \text{.........................(6)}$$

Since we are about to integrate the differential equation (3), we naturally regard the two integrals

$$\int_0^1 \frac{y_n\,dx}{x}\equiv U_n, \qquad \int_0^1 \frac{y_n\,dx}{1-x}\equiv V_n \quad \text{.................................(7)}$$

as additional standard forms. We shall also need an integral in which the integrand is the product of y_n and a linear function of x; by taking this function to be $1-2x$, so that we write

$$\int_0^1 (1-2x)y_n\,dx\equiv J_n, \quad \text{..(8)}$$

we introduce a combination of symmetry and skew-symmetry which more than compensates for formulae of greater length that would have been obtained if we had taken x or $1-x$.

3. The result of integrating (3) as it stands is evidently

$$\int_0^1 x(x-1)\frac{d^2 y_n}{dx^2}\,dx+\int_0^1 \{(a_n+b_n)x-a_n\}\frac{dy_n}{dx}\,dx+E_n I_n-\alpha A_n U_n-\beta B_n V_n=0.$$

We eradicate the differential coefficients of y_n from this result by performing partial integrations, two of the first term and one of the second. Hence we have

$$2I_n-(a_n+b_n)I_n+E_n I_n-\alpha A_n U_n-\beta B_n V_n=0,$$

and this reduces to

$$\alpha A_n U_n+\beta B_n V_n=(E_n-a_n-b_n+2)I_n. \quad \text{...............................(9)}$$

If we had multiplied (3) by x before integrating, a similar piece of reduction would have yielded the result

$$2\beta B_n V_n=(E_n-2\alpha A_n+2\beta B_n-2b_n+2)I_n-(E_n-2a_n-2b_n+6)J_n ; \quad \text{...........(10)}$$

while the effect of multiplying (3) by $1-x$ would have been

$$2\alpha A_n U_n=(E_n+2\alpha A_n-2\beta B_n-2a_n+2)I_n+(E_n-2a_n-2b_n+6)J_n. \quad \text{...........(11)}$$

Of course (9) is an immediate consequence of (10) and (11).

We next consider I_{n+1}. Evidently

$$y_{n+1} = x^\alpha(1-x)^\beta \frac{d\{y_n x^{-\alpha}(1-x)^{-\beta}\}}{dx} = \frac{dy_n}{dx} - \left(\frac{\alpha}{x} - \frac{\beta}{1-x}\right) y_n,$$

and consequently

$$I_{n+1} = \int_0^1 \frac{dy_n}{dx}\, dx - \int_0^1 \left(\frac{\alpha}{x} - \frac{\beta}{1-x}\right) y_n\, dx,$$

so that

$$I_{n+1} = \beta V_n - \alpha U_n. \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots(12)$$

In like manner we find that

$$J_{n+1} = 2(\alpha + \beta + 1) I_n - \alpha U_n - \beta V_n. \dots\dots\dots\dots\dots\dots(13)$$

4. The required reduction formula is now obtained by combining the results of the preceding section. We first obtain a relation connecting I_{n+1}, I_n, J_n by eliminating U_n and V_n from (10), (11) and (12) ; we transform this into a relation connecting I_{n+2}, I_{n+1}, J_{n+1} by writing $n+1$ in place of n. Next we express J_{n+1} in terms of I_{n+1} and I_n by eliminating U_n and V_n from (9), (12) and (13) ; we substitute this value of J_{n+1} in the previous (transformed) relation, and the rest is a matter of simplification. We can substantially shorten some of the calculations by writing

$$\alpha + \beta + \gamma + \delta \equiv 2\sigma, \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots(14)$$

and (at a subsequent stage of the work)

$$\left.\begin{array}{l} \alpha + \beta - \gamma - \delta \equiv 2\theta_1, \\ \alpha - \beta + \gamma - \delta \equiv 2\theta_2, \\ \alpha - \beta - \gamma + \delta \equiv 2\theta_3. \end{array}\right\} \dots\dots\dots\dots\dots\dots\dots\dots(15)$$

The relation of the set $(\alpha, \beta, \gamma, \delta)$ to the set $(\sigma, \theta_1, \theta_2, \theta_3)$ is of a type which occurs in work on elliptic functions. It is a reciprocal relation, so that there is no difficulty in expressing a function of either set of numbers as a function of the other set. In particular we shall subsequently need the transformations

$$E_n - a_n - b_n + 2 = (\sigma + \theta_1 - n)(2\sigma - n + 1), \dots\dots\dots\dots\dots\dots(16)$$

$$E_n - 2a_n - 2b_n + 6 = (\sigma + \theta_1 - n + 1)(2\sigma - n + 2), \dots\dots\dots\dots\dots(17)$$

$$(\alpha + \beta + 1)(A_n + B_n) - (E_n - a_n - b_n + 2) = (n+1)(\sigma - \theta_1 - n), \dots\dots\dots\dots(18)$$

$$2\alpha A_n - 2\beta B_n - a_n + b_n = \theta_2(4\sigma - 2n + 3) + \theta_3(2\sigma - 2n + 1) + 2\theta_1\theta_2. \dots\dots(19)$$

5. We now take the values of V_n and U_n given by (10) and (11) and substitute them in (12). The result is

$$A_n B_n I_{n+1} + (A_n + B_n)(\alpha A_n - \beta B_n - \tfrac{1}{2}a_n + \tfrac{1}{2}b_n) I_n$$
$$- \tfrac{1}{2}(A_n - B_n)(E_n - a_n - b_n + 2) I_n + \tfrac{1}{2}(A_n + B_n)(E_n - 2a_n - 2b_n + 6) J_n = 0,$$

and, when we replace n by $n+1$, this can be written in the modified form

$$(\alpha + \gamma - n - 1)(\beta + \delta - n - 1) I_{n+2} + (A_{n+1} + B_{n+1})(\alpha A_{n+1} - \beta B_{n+1} - \tfrac{1}{2}a_{n+1} + \tfrac{1}{2}b_{n+1}) I_{n+1}$$
$$- \tfrac{1}{2}(A_{n+1} - B_{n+1})(E_{n+1} - a_{n+1} - b_{n+1} + 2) I_{n+1} + (\sigma - n - 1)(2\sigma - n + 1)(\alpha + \beta - n) J_{n+1} = 0. \quad (20)$$

Next we solve equations (12) and (13) for αU_n and βV_n, and insert the values so obtained in (9). The result is

$$\tfrac{1}{2}(A_n + B_n)J_{n+1} = (\alpha + \beta + 1)(A_n + B_n)I_n - (E_n - a_n - b_n + 2)I_n - \tfrac{1}{2}(A_n - B_n)I_{n+1};$$

this can be written in the modified form

$$(\sigma - n)J_{n+1} = (n + 1)(\gamma + \delta - n)I_n - \tfrac{1}{2}(A_n - B_n)I_{n+1}. \quad\dots\dots\dots\dots\dots(21)$$

When we multiply (20) by $\sigma - n$ and then replace J_{n+1} by the expression supplied by (21), we obtain the required reduction formula in the form

$$P_n I_{n+2} - Q_n I_{n+1} + R_n I_n = 0, \quad\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots(22)$$

with

$$P_n \equiv (\sigma - n)(\alpha + \gamma - n - 1)(\beta + \delta - n - 1), \quad\dots\dots\dots\dots\dots\dots\dots(23)$$

$$R_n \equiv (n + 1)(\sigma - n - 1)(2\sigma - n + 1)(\alpha + \beta - n)(\gamma + \delta - n). \quad\dots\dots\dots\dots\dots(24)$$

Before we calculate the value of Q_n in its simplest form, we make some observations on the kind of expression that we may expect to obtain for it. In the special cases (i) and (ii), described in § 1, it is clear that, by reason of (22), $Q_n I_{n+1}$ vanishes whenever n is an odd positive integer. Since Q_n is a polynomial in n, we may consequently anticipate that it will vanish for all values of n in these special cases. Now both θ_3 and θ_1 vanish in case (i), while both θ_3 and θ_2 vanish in case (ii). We are thus led to aim at expressing Q_n as a sum of terms, each of which contains either θ_3 or else the product $\theta_1\theta_2$ as a factor. It was with this aim in mind that we introduced θ_1, θ_2, θ_3 in (15), and we now set about expressing Q_n in terms of σ, θ_1, θ_2, θ_3 instead of in terms of α, β, γ, δ.

The procedure by which (22) was derived from (20) and (21) makes it clear that Q_n is equal to

$$\tfrac{1}{2}(A_{n+1} - B_{n+1})(\sigma - n)(E_{n+1} - a_{n+1} - b_{n+1} + 2)$$

$$- (\sigma - n)(A_{n+1} + B_{n+1})(\alpha A_{n+1} - \beta B_{n+1} - \tfrac{1}{2}a_{n+1} + \tfrac{1}{2}b_{n+1})$$

$$+ \tfrac{1}{2}(A_n - B_n)(\sigma - n - 1)(2\sigma - n + 1)(\alpha + \beta - n)$$

$$= \theta_2(\sigma - n)(2\sigma - n)(\sigma + \theta_1 - n - 1)$$

$$- (\sigma - n)(\sigma - n - 1)\{\theta_2(4\sigma - 2n + 1) + \theta_3(2\sigma - 2n - 1) + 2\theta_1\theta_2\}$$

$$+ \theta_2(\sigma - n - 1)(2\sigma - n + 1)(\sigma + \theta_1 - n).$$

Consider this expression *qua* function of θ_1, θ_2, θ_3. It is a homogeneous linear function of θ_2, θ_3, $\theta_1\theta_2$ with coefficients which are functions of n and σ. The coefficient of θ_2 is

$$(\sigma - n)(\sigma - n - 1)\{(2\sigma - n) - (4\sigma - 2n + 1) + (2\sigma - n + 1)\} = 0 ;$$

the coefficient of θ_3 is

$$- (\sigma - n)(\sigma - n - 1)(2\sigma - 2n - 1) ;$$

and, finally, the coefficient of $\theta_1\theta_2$ is

$$(\sigma - n)(2\sigma - n) - 2(\sigma - n)(\sigma - n - 1) + (\sigma - n - 1)(2\sigma - n + 1) = (\sigma + 1)(2\sigma - 2n - 1).$$

When we collect these results, we see that

$$Q_n = (2\sigma - 2n - 1)\{\theta_1\theta_2(\sigma + 1) - \theta_3(\sigma - n)(\sigma - n - 1)\}. \quad\dots\dots\dots\dots\dots(25)$$

It is a simple matter to throw this formula into the alternative equivalent forms

$$Q_n = (2\sigma - 2n - 1)\{(\theta_1\theta_2 - \sigma\theta_3)(\sigma + 1) + \theta_3(n + 1)(2\sigma - n)\}, \quad \dots\dots\dots\dots(26)$$

$$Q_n = (2\sigma - 2n - 1)\{(\beta\gamma - \alpha\delta)(\sigma + 1) + \theta_3(n + 1)(2\sigma - n)\}.\dots\dots\dots\dots(27)$$

We have now proved that I_n satisfies the reduction formula (22) with P_n, Q_n and R_n defined by (23), (27) and (24).

6. As was mentioned in § 1, I_n is easily expressible in terms of generalised hypergeometric functions, so that (22) yields a relation connecting three contiguous functions of this type. Such a relation proves to be less elegant than the relation connecting the numerators of the functions when they are expressed as quotients of polynomials in α, β, γ, δ. We calculate these numerators directly from (1).

We use the continental notation for binomial coefficients together with the notation for products which is now standard, namely

$$(x)_0 \equiv 1, \qquad (x)_r \equiv x(x + 1) \dots (x + r - 1), \qquad (r \geqslant 1).$$

From the theorem of Leibniz it follows that

$$I_n = \int_0^1 x^\alpha (1 - x)^\beta \left\{ \sum_{r=0}^n \binom{n}{r} \frac{d^{n-r}x^\gamma}{dx^{n-r}} \cdot \frac{d^r(1 - x)^\delta}{dx^r} \right\} dx$$

$$= \sum_{r=0}^n (-)^{n-r} \binom{n}{r} (-\gamma)_{n-r} (-\delta)_r \int_0^1 x^{\alpha+\gamma-n+r}(1 - x)^{\beta+\delta-r} \, dx$$

$$= \frac{\Gamma(\alpha + \gamma - n + 1)\Gamma(\beta + \delta - n + 1)}{\Gamma(2\sigma - n + 2)} H_n,$$

where

$$H_n = \sum_{r=0}^n (-)^{n-r} \binom{n}{r} (-\gamma)_{n-r}(\beta + \delta - n + 1)_{n-r}(-\delta)_r(\alpha + \gamma - n + 1)_r. \quad \dots\dots\dots(28)$$

In (22) we now replace I_n by its value in terms of H_n, and we immediately get

$$(2\sigma - n)(\sigma - n)H_{n+2} - Q_n H_{n+1} + S_n H_n = 0, \quad \dots\dots\dots\dots(29)$$

where

$$S_n \equiv (n + 1)(\sigma - n - 1)(\alpha + \beta - n)(\gamma + \delta - n)(\alpha + \gamma - n)(\beta + \delta - n). \quad \dots\dots\dots(30)$$

Formula (29) has now been proved on the hypothesis that the real parts of α, β, γ, δ are sufficiently large; but, since the formula asserts that a *polynomial* in α, β, γ, δ is zero for such values of α, β, γ, δ, the polynomial must vanish identically. We have consequently proved that (29) holds for *all* values of α, β, γ, δ without restriction when H_n is defined by (28), while Q_n and S_n are defined by (27) and (30).

46 WARWICK NEW ROAD
LEAMINGTON, WARWICKSHIRE