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SYMMETRIC GEODESICS ON CONFORMAL COMPACTIFICATIONS OF EUCLIDEAN JORDAN ALGEBRAS

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In this article we define symmetric geodesics on conformal compactifications of Euclidean Jordan algebras and classify symmetric geodesics for the Euclidean Jordan algebra of all $n \times n$ symmetric real matrices. Furthermore, we show that the closed geodesics for the Euclidean Jordan algebra of all 2×2 symmetric real matrices are realised as the torus knots in the Shilov boundary of a Lie ball.

1. INTRODUCTION AND PRELIMINARIES

A commutative algebra V over field $\mathbb{R}$ or $\mathbb{C}$ with product xy is said to be a *Jordan algebra* if for all elements x, y in V, $x(x^2y) = x^2(xy)$. For $x \in V$, let $L(x)$ be the linear map of V defined by $L(x)y = xy$, and let $P(x) = 2L(x)^2 - L(x^2)$. A finite dimensional real Jordan algebra V is called a *Euclidean Jordan algebra* if it admits an inner product $\langle x \mid y\rangle$ such that $\langle xy \mid z\rangle = \langle y \mid xz\rangle$. Let V be a Euclidean Jordan algebra with identity e and let Q be the set of squares, and let Ω be the interior of Q. Then Ω is a self-dual cone and the group $G(\Omega) := \{g \in GL(V) \mid g\Omega = \Omega\}$ acts on it transitively. The tube domain $T_\Omega := V + i\Omega$ associated with Ω is a symmetric tube domain which is biholomorphically isomorphic to a bounded symmetric domain via a Cayley transform. The Lie group $G(T_\Omega)$ of all biholomorphic automorphisms on the tube domain T_Ω can be described in the following way: an element in $G(\Omega)$ acts on the tube domain T_Ω by $g(z) = g(x) + ig(y)$, $z = x + iy$. For $x \in V$, the translation by x, $t_x : z \mapsto z + x$ is a holomorphic automorphism of T_Ω and the group N^+ of all real translations is an Abelian group isomorphic to the vector space V. The map $j : z \mapsto -z^{-1}$, the symmetry at ie, belongs to $G(T_\Omega)$. Let $\widetilde{t}_x = j \circ t_x \circ j$, $N^- = j \circ N^+ \circ j$. Then $G(T_\Omega)$ is generated by $N^+, G(\Omega)$ and j [4].

Let $\mathfrak{n}^\pm$ be the Lie algebra of $N^\pm$ and let $\mathfrak{h}$ be the Lie algebra of $G(\Omega)$. Then the Lie algebra $\mathfrak{g}(T_\Omega)$ of $G(T_\Omega)$ can be identified with $\mathfrak{n}^+ + \mathfrak{h} + \mathfrak{n}^-$ in the following way:

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Let g_t be a one-parameter subgroup of $G(T_\Omega)$. Then

$$\widetilde{X}f(z) = \frac{d}{dt}f\big(g_t(z)\big)|_{t=0}, \ f \in C^1(T_\Omega),$$

defines a vector field on T_Ω and the set of vector fields obtained in this way is a real Lie subalgebra for the usual Lie bracket. We can write $\widetilde{X}f(z) = Df(z)\big(X(z)\big)$, where, for each z, $X(z)$ is a vector in $V_{\mathbb{C}} = V+iV$. Then the Lie algebra of $G(T_\Omega)$ is the set of vector fields of the form $X(z) = u+Tz+P(z)v$, where $u, v \in V$ and $T \in \mathfrak{h}$ [**4**, Theorem X.5.10]. Note that the vector field $X(z) = P(z)v$ corresponds to the one-parameter subgroup $\widetilde{t}_{tv}$ in N^- [**4**]. A vector field X in $\mathfrak{g}(T_\Omega)$, $X(z) = u + Tz + P(z)v$, can be identified with $(u, T, v) \in V \times \mathfrak{h} \times V$. Then

$$\begin{aligned} \mathfrak{n}^+ &= \big\{(u,0,0) \mid u \in V\big\} \cong V, \\ \mathfrak{h} &= \big\{(0,T,0) \mid T \in \mathfrak{g}(\Omega)\big\}, \\ \mathfrak{n}^- &= \big\{(0,0,v) \mid v \in V\big\} \cong V. \end{aligned}$$

It is known [**9**] that $\mathfrak{g}(T_\Omega)$ is a symmetric algebra of Cayley type. Hence

$$(X, h, Y) \in \mathfrak{n}^+ \times G(\Omega) \times \mathfrak{n}^- \mapsto (\exp X)h(\exp Y) \in N^+G(\Omega)N^-$$

is a diffeomorphism and $N^+G(\Omega)N^-$ is a dense open subset in $G(T_\Omega)$. Set $P = G(\Omega)N^-$. Then P is a maximal parabolic subgroup of $G(T_\Omega)$ and the homogeneous space $\mathcal{M} := G(T_\Omega)/P$ is a compact real manifold containing V as an open dense subset, that is, a real *conformal compactification* of the Jordan algebra V. The embedding $V \to \mathcal{M}$ is given by $x \mapsto t_x P$. Furthermore, the set $N^+G(\Omega)N^-$ can be characterised by the elements $g \in G(T_\Omega)$ such that $g \cdot \mathbf{0} \in V$, where $\mathbf{0}$ is the base point P corresponding to the zero vector in V [**3, 7**].

For $X \in \mathfrak{g}(T_\Omega)$, we assign $f(X) \in (0, \infty]$ as follows: if the solution $x(t)$ of the system $x'(t) = X\big(x(t)\big)$ with the initial value $x(0) = \mathbf{0}$ stays in V for all $t \in \mathbb{R}^+$, then $f(X) = \infty$. Otherwise, $f(X)$ is the first positive time reaching the boundary $\partial V \subset \mathcal{M}$. In this case the solution $x(t)$ is given by $\exp tX \cdot \mathbf{0}$. One way to evaluate $f(X)$ using the fact that $g \cdot \mathbf{0} \in V$ if and only if $g \in N^+G(\Omega)N^-$ is to decompose the exponential $\exp tX$ into $N^+G(\Omega)N^-$. Some explicit formulae for the factorisation of $\exp X$ for certain $X \in \mathfrak{q} := \mathfrak{n}^+ + \mathfrak{n}^-$ were given in [**6, 7, 10**].

Let $\mathrm{Con}_{G(\Omega)}(\mathfrak{q})$ be the set of closed, generating convex cones in $\mathfrak{q}$ which are invariant under the adjoint action of $G(\Omega)$. Then (see [**8, 5**]) $\mathrm{Con}_{G(\Omega)}(\mathfrak{q}) = \{C_{\mathfrak{p}}, -C_{\mathfrak{p}}, C_{\mathfrak{k}}, -C_{\mathfrak{k}}\}$, where $C_{\mathfrak{p}} = \big\{(u, 0, -v) \in \mathfrak{g}(T_\Omega) \mid u, v \in \overline{\Omega}\big\}, C_{\mathfrak{k}} = \big\{(u, 0, v) \in \mathfrak{g}(T_\Omega) \mid u, v \in \overline{\Omega}\big\}$. It turns out [**8, 9**] that the wedge $\mathfrak{h} + C_{\mathfrak{p}}$ (respectively, $\mathfrak{h} - C_{\mathfrak{p}}$) is a Lie wedge of the compression semigroup of Ω (respectively, $-\Omega$) with a triple factorisation in $N^+G(\Omega)N^-$. Hence for $X \in (\mathfrak{h} \pm C_{\mathfrak{p}})$, $f(X) = \infty$.

To complete a semisimple Jordan algebra V of classical type to a symmetric space, Makarevič [**11**] used the term of geodesics in V that originate at the zero point. In a Euclidean Jordan algebra V, these geodesics are eventually of the form $\alpha(t,a) := \exp tX_a \cdot \mathbf{0}$, where $X_a = (a,0,a) \in \mathfrak{q}$. Our main interest is in these geodesics and, in particular, in closed geodesics invariant for the symmetry j, which are said to be *symmetric* geodesics.

In this paper we shall confine ourselves to the Jordan algebra $V = \mathrm{Sym}(n,\mathbb{R})$ of $n \times n$ real symmetric matrices. An explicit expression of the exit time function f on $\pm C_t$ is given in Section 3. In Section 4, we classify symmetric geodesics and characterise some properties of these geodesics. In Section 5, we show that the closed geodesics for the Euclidean Jordan algebra $V = \mathrm{Sym}(2,\mathbb{R})$ of all 2×2 symmetric real matrices are realised as the torus knots in the Shilov boundary Σ_2 of a Lie ball.

2. Symplectic Groups

For a natural number n, $M_n(\mathbb{R})$ denotes the space of all n by n real matrices, and $A \in M_n(\mathbb{R})$ is symmetric (skew-symmetric) means that $A^t = A$ ($A^t = -A$). Let $\mathrm{Sym}(n,\mathbb{R})$ (respectively $\mathrm{Skew}(n,\mathbb{R})$) be the space of all symmetric (respectively, skew-symmetric) n by n matrices. For a positive semi-definite symmetric matrix A, by $A^{1/2}$ we shall mean the square root of A. Let $A \in \mathrm{Sym}(n,\mathbb{R})$ have the spectral decomposition $A = \sum_{i=1}^{n} \lambda_i C_i$, where $\{C_i\}$ is a complete system of orthogonal projections. Then the spectral norm $|A|$ of A is defined by $|A| = \max\{|\lambda_1|, \cdots, |\lambda_n|\}$.

Let $(\cdot \mid \cdot)$ be the skew-symmetric form on $\mathbb{R}^{2n}$ defined by $(u \mid v) = \langle Ju \mid v\rangle$ for $u, v \in \mathbb{R}^{2n}$, where $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$. Here, I stands for the $n \times n$ identity matrix. The *symplectic group* $\mathrm{Sp}(2n,\mathbb{R})$ on $\mathbb{R}^{2n}$ is the Lie group of all invertible transformations $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ satisfying one of the following equivalent conditions:

(1) g preserves $(\cdot \mid \cdot)$.

(2) $g^t J g = J$.

(3) A^tC, B^tD are symmetric and $A^tD - C^tB = I$.

The Lie algebra of $\mathrm{Sp}(2n,\mathbb{R})$ is given by

$$\mathfrak{sp}(2n,\mathbb{R}) = \left\{ \begin{pmatrix} X & Y \\ Z & -X^t \end{pmatrix} \,\middle|\, X \in M_n(\mathbb{R}),\ Y, Z \in \mathrm{Sym}(n,\mathbb{R}) \right\}.$$

It has a Cartan decomposition $\mathfrak{sp}(2n,\mathbb{R}) = \mathfrak{p} \oplus \mathfrak{k}$, where

$$\mathfrak{p} = \left\{ \begin{pmatrix} X & Y \\ Y & -X \end{pmatrix} \,\middle|\, X,\ Y \in \mathrm{Sym}(n,\mathbb{R}) \right\},$$

$$\mathfrak{k} = \left\{ \begin{pmatrix} X & Y \\ -Y & X \end{pmatrix} \,\middle|\, X \in \mathrm{Skew}(n,\mathbb{R}), Y \in \mathrm{Sym}(n,\mathbb{R}) \right\}.$$

Let $\tau = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix} \in GL(2n, \mathbb{R})$ and let $\tau(g) = \tau \cdot g \cdot \tau$ for $g \in \mathrm{Sp}(2n, \mathbb{R})$. Then τ is an involution on $\mathrm{Sp}(2n, \mathbb{R})$. The differential $d\tau$ of τ at the identity is given by

$$d\tau \begin{pmatrix} X & Y \\ Z & -X^t \end{pmatrix} = \begin{pmatrix} X & -Y \\ -Z & -X^t \end{pmatrix}.$$

The Lie algebra $\mathfrak{sp}(2n, \mathbb{R})$ can be decomposed as the $+1$-eigenspace $\mathfrak{h}$ and the -1-eigenspace $\mathfrak{q}$ of $d\tau$:

$$\mathfrak{sp}(2n, \mathbb{R}) = \mathfrak{h} \oplus \mathfrak{q} = \mathfrak{h} \oplus \mathfrak{n}^+ \oplus \mathfrak{n}^-, \quad \mathfrak{q} = \mathfrak{n}^+ \oplus \mathfrak{n}^-,$$

where

$$\mathfrak{n}^+ = \left\{ \begin{pmatrix} 0 & Y \\ 0 & 0 \end{pmatrix} \;\middle|\; Y \in \mathrm{Sym}(n, \mathbb{R}) \right\},$$
$$\mathfrak{n}^- = \left\{ \begin{pmatrix} 0 & 0 \\ Z & 0 \end{pmatrix} \;\middle|\; Z \in \mathrm{Sym}(n, \mathbb{R}) \right\},$$
$$\mathfrak{h} = \left\{ \begin{pmatrix} X & 0 \\ 0 & -X^t \end{pmatrix} \;\middle|\; X \in M_n(\mathbb{R}) \right\}.$$

Let $N^\pm$ be the Lie subgroup of $\mathrm{Sp}(2n, \mathbb{R})$ corresponding $\mathfrak{n}^\pm$ respectively. Then

$$N^+ = \left\{ \begin{pmatrix} I & A \\ 0 & I \end{pmatrix} \;\middle|\; A \in \mathrm{Sym}(n, \mathbb{R}) \right\} = \exp \mathfrak{n}^+,$$
$$N^- = \left\{ \begin{pmatrix} I & 0 \\ A & I \end{pmatrix} \;\middle|\; A \in \mathrm{Sym}(n, \mathbb{R}) \right\} = \exp \mathfrak{n}^-.$$

Let

$$H = \left\{ \begin{pmatrix} A & 0 \\ 0 & (A^{-1})^t \end{pmatrix} \;\middle|\; A \in GL(n, \mathbb{R}) \right\}.$$

Note that $H = \{ g \in \mathrm{Sp}(2n, \mathbb{R}) \mid \tau(g) = g \}$.

Lemma 2.1. *An element $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(2n, \mathbb{R})$ can be written (uniquely) as a triple product in N^+HN^- if and only if D is invertible. In this case the (unique) factorisation is given by*

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} I & BD^{-1} \\ 0 & I \end{pmatrix} \begin{pmatrix} (D^{-1})^t & A \\ 0 & D \end{pmatrix} \begin{pmatrix} I & 0 \\ D^{-1}C & I \end{pmatrix}.$$

Proof: See [9]. $\square$

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Theorem 2.2. *Let $P = HN^-$. Then P is a closed subgroup of $G := \mathrm{Sp}(2n, \mathbb{R})$ and the homogeneous space $\mathcal{M} := G/P$ is a compact real manifold with $V := \mathrm{Sym}(n, \mathbb{R})$ as an open dense subset. The embedding of V into $\mathcal{M}$ is given by*

$$X \in V \mapsto \begin{pmatrix} I & X \\ 0 & I \end{pmatrix} \cdot P \in \mathcal{M}.$$

Furthermore, for $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(2n, \mathbb{R})$ and $X \in V$ with $g \cdot X \in V$, we have

$$g \cdot X = (AX + B)(CX + D)^{-1}.$$

Proof: This follows from the general theory of conformal compactifications of Jordan algebras [**1, 2, 3**]. ∎

3. Ad(H)-Invariant Cones

Let

$$C_{\mathfrak{k}} = \left\{ \begin{pmatrix} 0 & A \\ -B & 0 \end{pmatrix} \mid A, B \geqslant 0 \right\}, \quad C_{\mathfrak{p}} = \left\{ \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} \mid A, B \geqslant 0 \right\}.$$

Then $C_{\mathfrak{k}}, -C_{\mathfrak{k}}, C_{\mathfrak{p}}, -C_{\mathfrak{p}}$ are pointed, generating $\mathrm{Ad}(H)$-invariant convex cones in $\mathfrak{q}$.

Let $A \in V = \mathrm{Sym}(n, \mathbb{R})$ have the spectral decomposition $A = \sum_{i=1}^{n} \lambda_i C_i$. If g is a real valued function on an interval containing all eigenvalues λ_i, then we shall define an element $g(A) \in V$ by the formula $g(A) = \sum_{i=1}^{n} g(\lambda_i) C_i$. For instance, $\cos A$, $\sin A$ are defined in this way.

Lemma 3.1. *Let $A \in V$ with the spectral decomposition $A = \sum_{i=1}^{n} \lambda_i C_i$. Then $\cos(tA)$ is singular if and only if*

$$t = \frac{\pi(2k+1)}{2\lambda_i}$$

for some i and for some $k \in \mathbb{Z}$.

Proof: This follows from the fact that $\cos(tA) = \sum_{i=1}^{n} (\cos t\lambda_i) C_i$ is invertible if and only if $\cos t\lambda_i \neq 0$ for all i. ∎

For $A \in M_n(\mathbb{R})$, we let

$$c(A) = \sum_{k=0}^{\infty} \frac{A^k}{(2k)!},$$
$$s(A) = \sum_{k=0}^{\infty} \frac{A^k}{(2k+1)!}.$$

Proposition 3.2. *Let $A, B \in V$. Then*

$$\exp\begin{pmatrix} 0 & tA \\ -tB & 0 \end{pmatrix} = \begin{pmatrix} c(-t^2AB) & tAs(-t^2BA) \\ -tBs(-t^2AB) & c(-t^2BA) \end{pmatrix}.$$

In particular,

$$\exp\begin{pmatrix} 0 & tA \\ -tA & 0 \end{pmatrix} = \begin{pmatrix} \cos(tA) & \sin(tA) \\ -\sin(tA) & \cos(tA) \end{pmatrix}.$$

Moreover, if A and B are positive definite matrices, then $\exp\begin{pmatrix} 0 & tA \\ -tB & 0 \end{pmatrix}$ *equals*

$$\begin{pmatrix} B^{-1/2}\left[\cos t(B^{1/2}AB^{1/2})^{1/2}\right]B^{1/2} & B^{-1/2}(B^{1/2}AB^{1/2})^{1/2}\left[\sin t(B^{1/2}AB^{1/2})^{1/2}\right]B^{-1/2} \\ -B^{1/2}(B^{1/2}AB^{1/2})^{-1/2}\left[\sin t(B^{1/2}AB^{1/2})^{1/2}\right]B^{1/2} & B^{1/2}\left[\cos t(B^{1/2}AB^{1/2})^{1/2}\right]B^{-1/2} \end{pmatrix}.$$

Proof: This follows from a direct matrix computation. ∎

Corollary 3.3. *Let $A = \sum_{i=1}^{n} \lambda_i C_i \in V$ be the spectral decomposition of A. Then*

$$\exp\begin{pmatrix} 0 & tA \\ -tA & 0 \end{pmatrix} \cdot 0 \in \partial V \Longleftrightarrow t = \frac{\pi(2k+1)}{2\lambda_i}$$

for some i and for some $k \in \mathbb{Z}$.

Proof: This follows from Lemma 2.1, Lemma 3.1, and Proposition 3.2. ∎

Corollary 3.4. *Let $A, B \geqslant 0$, (that is, A and B are positive semi-definite matrices). Let $\sum_{i=1}^{n} \lambda_i C_i$ be the spectral decomposition of $B^{1/2}AB^{1/2}$. Then*

$$\exp\begin{pmatrix} 0 & tA \\ -tB & 0 \end{pmatrix} \cdot 0 \in \partial V \Longleftrightarrow t = \frac{\pi(2k+1)}{2\sqrt{\lambda_i}}$$

for some i and for some $k \in \mathbb{Z}$.

Proof: First, we show that for $A, B \geqslant 0$, $\operatorname{Det} c(-BA) = \operatorname{Det} \cos(B^{1/2}AB^{1/2})^{1/2}$, where Det is the determinant function. Choose $A_n, B_n > 0$ such that $A_n \to A$ and $B_n \to B$. Then

$$c(-B_nA_n) = B_n^{1/2}\left[\cos(B_n^{1/2}A_nB_n^{1/2})^{1/2}\right]B_n^{-1/2}.$$

Thus

$$\begin{aligned} \operatorname{Det} c(-BA) &= \lim_{n\to\infty} \operatorname{Det} c(-B_nA_n) \\ &= \lim_{n\to\infty} \operatorname{Det} B_n^{1/2}\left[\cos(B_n^{1/2}A_nB_n^{1/2})^{1/2}\right]B_n^{-1/2} \\ &= \lim_{n\to\infty} \operatorname{Det} \cos(B_n^{1/2}A_nB_n^{1/2})^{1/2} \\ &= \operatorname{Det} \cos(B^{1/2}AB^{1/2})^{1/2}. \end{aligned}$$

Now by Lemma 2.1, Lemma 3.1, and Proposition 3.2,

$$\exp\begin{pmatrix} 0 & tA \\ -tB & 0 \end{pmatrix} \cdot \mathbf{0} \in \partial V$$

if and only if $c(-t^2BA)$ is singular if and only if $\cos t(B^{1/2}AB^{1/2})^{1/2}$ is singular if and only if $t = \big(\pi(2k+1)\big)/(2\sqrt{\lambda_i})$ for some i and for some $k \in \mathbb{Z}$. ∎

Theorem 3.5. *The exit function f is continuous on $C_{\mathfrak{k}}$. In particular,*

$$f^{-1}(\infty) \cap C_{\mathfrak{k}} = \left\{ \begin{pmatrix} 0 & A \\ -B & 0 \end{pmatrix} \in C_{\mathfrak{k}} \;\middle|\; B^{1/2}AB^{1/2} = 0 \right\}.$$

Proof: By Corollary 3.4, $f\begin{pmatrix} 0 & A \\ -B & 0 \end{pmatrix} = \pi/\big(2\left|B^{1/2}AB^{1/2}\right|^{1/2}\big)$. ∎

Remark 3.6. Note that for $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(2n, \mathbb{R})$, $g^{-1} = \begin{pmatrix} D^t & -B^t \\ -C^t & A^t \end{pmatrix}$. Therefore if $A, B \geqslant 0$, then

$$\begin{aligned} \exp\begin{pmatrix} 0 & -tA \\ tB & 0 \end{pmatrix} &= \exp\begin{pmatrix} 0 & tA \\ -tB & 0 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} c(-t^2BA)^t & -t\big(As(-t^2BA)\big)^t \\ t\big(Bs(-t^2AB)\big)^t & c(-t^2AB)^t \end{pmatrix}. \end{aligned}$$

Since $c(-t^2AB)^t = c(-t^2BA)$, the map f is given on $-C_{\mathfrak{k}}$ by

$$f\begin{pmatrix} 0 & -A \\ B & 0 \end{pmatrix} = \frac{\pi}{2\left|A^{1/2}BA^{1/2}\right|^{1/2}}.$$

Hence f is continuous on $-C_{\mathfrak{k}}$.

4. Geodesics on $\mathcal{M}$

Let $A \in V = \mathrm{Sym}(n, \mathbb{R})$. Set $X_A := \begin{pmatrix} 0 & A \\ -A & 0 \end{pmatrix} \in \mathfrak{k}$. Then it is known **[11]** that a geodesic in V originating at the origin $\mathbf{0}$ with direction A is of the form

$$\alpha(t, A) := \exp tX_A \cdot \mathbf{0} = \begin{pmatrix} \cos tA & \sin tA \\ -\sin tA & \cos tA \end{pmatrix} \cdot \mathbf{0}.$$

Proposition 4.1. *Let $A \in V$.*

(1) *If $\sin tA = 0$ (respectively, $\cos tA = 0$) then $\cos tA$ (respectively, $\sin tA$) is invertible.*

(2) *The geodesic $\alpha(t, A)$ touches the boundary of V at time t_0 if and only if $\cos t_0 A$ is singular. In particular, if $\cos tA$ is invertible, then $\alpha(t, A) = \tan tA$.*

(3) *The geodesic $\alpha(t, A)$ is closed if and only if $\alpha(t_0, A) = \mathbf{0}$ for some $t_0 \neq 0$ if and only if $\sin t_0 A = 0$ for some $t_0 \neq 0$.*

(4) *For the geodesic $\alpha(t, A)$, $t = \pi/(2|A|)$ is the first positive time it touches the boundary of V.*

(5) *$\alpha(t, t'A) = \alpha(tt', A)$ for any $t, t' \in \mathbb{R}$.*

PROOF: (1) and (5) are obvious. (2) follows from Lemma 2.1 and Proposition 3.2. By the definition of closed geodesic, $\alpha(t, A)$ is closed if and only if $\alpha(t_0, A) = \mathbf{0}$ for some $t_0 \neq 0$. By Lemma 2.1 and Proposition 3.2, this is equivalent to the condition $\sin t_0 A = 0$. (4) follows from Corollary 3.3. $\square$

The *period* of a non-constant closed geodesic $\alpha(t, A)$ is the smallest positive real number t_0 satisfying $\alpha(t_0, A) = \mathbf{0}$. Set

$$E_c = \{r(p_1, \cdots, p_n) \in \mathbb{R}^n \mid r \geqslant 0,\ p_i : \text{integer}\}.$$

In this setting, we always assume that the integers p_i have no common divisors.

THEOREM 4.2. *Let $A = \sum_{i=1}^{n} \lambda_i C_i$ be the spectral decomposition of A. Then $\alpha(t, A)$ is a closed geodesic if and only if $(\lambda_1, \cdots, \lambda_n) \in E_c$. If $A \neq 0$ and $(\lambda_1, \cdots, \lambda_n) = r(p_1, \cdots, p_n) \in E_c$, then π/r is the period of $\alpha(t, A)$.*

PROOF: Suppose that $\alpha(t, A)$ is a closed geodesic. If $A = 0$, then all λ_i are zero and hence $(\lambda_1, \cdots, \lambda_n) = 0(1, \cdots, 1) \in E_c$. Suppose that $A \neq 0$. Let λ_k be the first non-zero eigenvalue of A. By Proposition 4.1(3), there exists $t_0 \neq 0$ such that $\sin t_0 A = 0$. This implies that $\sin t_0 \lambda_i = 0$ for all $i = 1, 2, \cdots, n$. From this, we conclude that $\lambda_i/\lambda_k \in \mathbb{Q}$ (the field of rational numbers) for all $i = 1, 2, \cdots, n$. Then for $i = k+1, \cdots, n$, there exists $r_i \in \mathbb{Q}$ such that $\lambda_i = \lambda_k r_i$. By multiplying by the least common multiple of the denominators of the r_i, we can rewrite $(\lambda_1, \cdots, \lambda_k, \cdots, \lambda_n)$ as $r(0, \cdots, 0, p_k, p_{k+1}, \cdots, p_n) \in E_c$ for some $p_i \in \mathbb{Z}$, $r > 0$.

Conversely, suppose that $(\lambda_1, \cdots, \lambda_n) = r(p_1, \cdots, p_n) \in E_c$. If $r = 0$, then $\alpha(t, A) = \mathbf{0}$ is constant and hence is a closed geodesic. If $r \neq 0$, then $\sin(\pi/r)\lambda_i = \sin p_i \pi = 0$ for all i. This implies that $\sin(\pi/r)A = \mathbf{0}$ and hence $\alpha(t, A)$ is a closed geodesic.

Finally suppose that $A \neq 0$ and $\alpha(t_0, A) = \mathbf{0}$ for some $t_0 > 0$. Since $A \neq 0$, $r \neq 0$ and $p_k \neq 0$ for some k. Hence $\sin t_0 \lambda_k = \sin t_0 r p_k = 0$ implies that $t_0 = (m\pi/rp_k)$ for some integer m. If $p_k = \pm 1$, then $t_0 \geqslant \pi/r$. Suppose $p_k \neq \pm 1$. Since $\{p_i\}$ have no common divisors, there exists $j \neq k$ such that $p_j \neq 0$ and p_k and p_j are relatively prime. Since $\sin t_0 \lambda_j = \sin t_0 r p_j = 0$, $t_0 = (m'\pi/rp_j)$ for some integer m'. Therefore

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$t_0 = (m\pi/rp_k) = (m'\pi/rp_j)$. Since p_k and p_j are relatively prime, p_k must divide m and hence $t_0 \geqslant \pi/r$. $\square$

Set $j = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \in \mathrm{Sp}(2n, \mathbb{R})$. Then j is an involution on $\mathcal{M}$ and for an invertible element $A = \sum_{i=1}^{n} \lambda_i C_i$, $j \cdot A = -A^{-1}$.

Proposition 4.3. *Let $A \in V$. Then the following conditions are equivalent:*

(1) For any $t \in \mathbb{R}$, there exists t' (depends on t) such that $j \cdot \alpha(t, A) = \alpha(t', A)$.

(2) There exists a non-zero real number $t_0 \in \mathbb{R}$ such that $j \cdot \alpha(t, A) = \alpha(t+t_0, A)$ for all $t \in \mathbb{R}$.

(3) There exists a non-zero real number t_0 such that $\cos t_0 A = 0$.

In particular, if a geodesic $\alpha(t, A)$ satisfies one of these conditions, then A is invertible.

Proof: By Lemma 2.1, we have

$$(4.1) \qquad (\exp tX_A)j = \begin{pmatrix} \sin tA & -\cos tA \\ \cos tA & \sin tA \end{pmatrix} \in P \text{ if and only if } \cos tA = 0.$$

(1) implies (3) : Let t be a non-zero real number. Then there is a real number t' such that $j \cdot \alpha(t, A) = \alpha(t', A)$. That is, $j \exp tX_A \cdot \mathbf{0} = \exp t'X_A \cdot \mathbf{0}$. Since $jX_Aj^{-1} = X_A$, we have

$$\begin{aligned} \mathbf{0} &= \exp(-t'X_A) j \exp tX_A \cdot \mathbf{0} \\ &= \exp(-t'X_A) j \exp tX_A j^{-1} j \cdot \mathbf{0} \\ &= \exp(-t'X_A) \exp tX_A j \cdot \mathbf{0}. \end{aligned}$$

Hence $(\exp(t-t')X_A)j \in P$. Set $t_0 = t - t'$. Then $t_0 \neq 0$ since j is not in P. Therefore by (4.1) $\cos t_0 A = 0$.

(3) implies (2) : Suppose that $\cos t_0 A = 0$ for some non-zero $t_0 \in \mathbb{R}$. Then by (4.1), $(\exp t_0 X_A) j \cdot \mathbf{0} = \mathbf{0}$ and hence for any $t \in \mathbb{R}$,

$$\begin{aligned} j(\exp tX_A) \cdot \mathbf{0} &= j(\exp(t+t_0)X_A)(\exp -t_0X_A) \cdot \mathbf{0} \\ &= j(\exp(t+t_0)X_A) j \cdot \mathbf{0} \\ &= j(\exp(t+t_0)X_A) j^{-1} \cdot \mathbf{0} \\ &= (\exp(t+t_0)X_A) \cdot \mathbf{0}. \end{aligned}$$

Therefore $j \cdot \alpha(t, A) = \alpha(t + t_0, A)$.

(2) implies (1): Trivial.

Now suppose $\cos t_0 A = 0$ for some $t_0 \neq 0$. Then $\cos t_0 \lambda_i = 0$ and hence $\lambda_i \neq 0$ for all i. Therefore A is invertible. $\square$

DEFINITION. A closed geodesic $\alpha(t, A)$ is said to be *symmetric* if it satisfies one of the conditions of Proposition 4.3.

Set $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$. Let E_s be the subset of $(\mathbb{R}^*)^n$

$$E_s = \{r(p_1, p_2, \cdots, p_n) \mid r > 0,\ p_i : \text{odd integer}\}.$$

Then E_s is a subgroup of $(\mathbb{R}^*)^n$ under the usual multiplication.

THEOREM 4.4. *Let $A = \sum_{i=1}^{n} \lambda_i C_i \in V$ be the spectral decomposition of A. Then $\alpha(t, A)$ is a symmetric geodesic if and only if $(\lambda_1, \cdots, \lambda_n) \in E_s$.*

PROOF: If $(\lambda_1, \cdots, \lambda_n) = r(p_1, \cdots, p_n) \in E_s$, then $\cos(\pi/2r)A = 0$. Hence $\alpha(t, A)$ is symmetric. Conversely, suppose that $\alpha(t, A)$ is symmetric. Then by Propositions 4.1 and 4.3, $\cos tA = 0$ and $\sin t'A = 0$ for some $t \neq 0$ and $t' \in \mathbb{R}$. By Theorem 4.2, $(\lambda_1, \cdots, \lambda_n) = r(p_1, \cdots, p_n)$ for some integers p_i which have no common divisors. Since A is invertible (by Proposition 4.3), r and the p_i's are all non-zero. Note that $\cos tA = 0$ implies that $p_i - p_j$ is even for all $i, j = 1, \cdots, n$. If p_1 is even, then all the p_i's are even. But this is impossible since they have no common divisors. Thus p_1 is odd and hence all the p_i's are odd integers. ∎

COROLLARY 4.5. *Let $\alpha(t, A)$ be a symmetric geodesic. Let $A = \sum_{i=1}^{n} rp_i C_i$ be the spectral decomposition of A. Then $j \cdot \alpha(t, A) = \alpha\big(t + (\pi/2r), A\big)$. In particular, $j \cdot \mathbf{0} = \alpha(\pi/2r,\ A)$.*

PROOF: Let $\alpha(t, A)$ be a symmetric geodesic. Note that $r > 0$ and the p_i's are all odd numbers (they have no common divisors). Set $A' = (1/r)A$. For $i = 1, \cdots, n$, $\tan(t + \pi/2)p_i = -\cot(tp_i)$ whenever it is defined. This implies that for every $t > 0$ with $j \cdot \alpha(t, A') \in V$,

$$j \cdot \alpha(t, A') = -\cot tA' = \tan\Big(t + \frac{\pi}{2}\Big)A' = \alpha\Big(+t + \frac{\pi}{2}, A'\Big).$$

Because V is dense in $\mathcal{M}$, the equality holds for all t. By Proposition 4.1, we have

$$j \cdot \alpha(t, A) = j \cdot \alpha(tr, A') = \alpha\Big(tr + \frac{\pi}{2}, A'\Big) = \alpha\Big(t + \frac{\pi}{2r}, A\Big).$$

∎

COROLLARY 4.6. *Let $\alpha(t, A)$ be a symmetric geodesic. If $\alpha(t_0, A) \in \partial V$, then $j \cdot \alpha(t_0, A) \in V$.*

PROOF: Let $\sum_{i=1}^{n} rp_i C_i$ be the spectral decomposition of A. Since $\alpha(t_0, A) \in \partial V$, $\cos t_0 A$ is singular. Hence $t_0 = (k/2rp_i)\pi$ for some i and an odd integer k. Since, by Corollary 4.5, $j \cdot \alpha(t_0, A) = \alpha(t_0 + (\pi/2r), A)$, we claim that $\cos\big(t_0 + (\pi/2r)\big)rp_j \neq 0$ for all $j = 1, \cdots, n$. It then follows from $\cos\big(t_0 + (\pi/2r)\big)rp_j = \cos\big((kp_j/2p_i) + (p_j/2)\pi\big) = -\sin\big((kp_j/2p_i)\pi\big) \neq 0$ because p_i, p_j, and k are odd integers. This ends the proof. $\square$

REMARKS 4.7. (1) By Corollary 4.6, boundary points in a symmetric geodesic curve have to move into V by the symmetry j. This property distinguishes symmetric geodesics among closed geodesics. Let $\alpha(t, A)$ be a non-constant closed geodesic. Then it touches the boundary ∂V finitely many times (Proposition 4.1 (4)). Suppose that $\alpha(t, A)$ is non-symmetric. Then an eigenvalue λ_k of A is rp_k for some even integer p_k. Here, π/r is the period of the given geodesic curve. Since $\{p_i := \lambda_i/r\}$ have no common divisors, p_j is odd for some j. This implies that $\sin\big((\pi/2r)rp_k\big) = 0$ and $\cos\big((\pi/2r)rp_j\big) = 0$. Therefore, $\alpha\big((\pi/2r), A\big) \in \partial V$ and $j \cdot \alpha\big((\pi/2r), A\big) \in \partial V$.

(2) Let $\alpha(t, A)$ be a symmetric geodesic with the spectral decomposition $A = \sum_{i=1}^{n} rp_iC_i$. Let $A' = (1/r)A$. If $\alpha(t_0, A') \in \partial V$, then t_0 is of the form $(k/2p_i)\pi$ for some i and some odd integer k. Since $2p_i - k$ is an odd number, $\cos(\pi - t_0)p_i = \cos\big((2p_i - k)/2)\big)\pi = 0$. Hence $\alpha(\pi - t_0, A') \in \partial V$. If $0 < t_0 < \pi/2$, then $\pi/2 < \pi - t_0 = \big(1 - (k/2p_i)\big)\pi < \pi$. Similarly if $\pi/2 < t_0 < \pi$, then $0 < \pi - t_0 < \pi/2$. This implies the number of boundary points in the geodesic curve $\alpha(t, A')$ is odd, centred at the time $\pi/2$. This holds for A since $\alpha(t, A') = \alpha\big((t/r), A\big)$. Moreover $\pi/r - \pi/(2|A|)$ is the last time of touching the boundary since $\pi/(2|A|)$ is the first one. In the case $n = 2$, one may easily show that the number of boundary points in the symmetric geodesic curve $\alpha(t, A)$ is $p_1 + p_2 - 1$.

5. CLOSED GEODESICS FOR Sym(2, $\mathbb{R}$)

Let $V = \mathrm{Sym}(2, \mathbb{R})$ and let Ω be the symmetric cone of positive definite 2×2 symmetric real matrices. Then the tube domain $T_\Omega := V + i\Omega$ can be realised as a bounded symmetric domain $\mathcal{D}$ in the complex plane $V^{\mathbb{C}} := V + iV$ as follows: we define

$$D(p) = \{Z \in V^{\mathbb{C}} \mid Z + iI \in GL(2, \mathbb{C})\},$$
$$D(c) = \{W \in V^{\mathbb{C}} \mid I - W \in GL(2, \mathbb{C})\}$$

and for all $Z \in D(p)$, $W \in D(c)$,

$$p(Z) = (Z - iI)(Z + iI)^{-1}$$
$$c(W) = i(I + W)(I - W)^{-1}.$$

Then the map p is a holomorphic bijection of $D(p)$ onto $D(c)$ and c, called the *Cayley transform*, is its inverse. The closure of T_Ω in $V^{\mathbb{C}}$ is contained in $D(p)$. The image $\mathcal{D} := p(T_\Omega)$ of p is known as a bounded symmetric domain, called a *Lie ball* (open unit ball with respect to the spectral norm) [4]. We define Σ_2 as the set of invertible elements in $V^{\mathbb{C}}$ such that $Z^{-1} = \overline{Z}$. It is known [4] that Σ_2 is the Shilov boundary of $\mathcal{D}$, which is a compact connected 3-dimensional manifold, and is exactly equal to $\overline{p(V)}$.

Let $\mathbf{c} = \{C_i\}_{i=1}^2$ be a complete system of orthogonal projections (Jordan frame) and let $V(\mathbf{c})$ be the subspace of V generated by C_i's. Then for $A = \sum\limits_{i=1}^{2} \lambda_i C_i$,

$$p(A) = \sum_{i=1}^{2} \frac{\lambda_i - i}{\lambda_i + i} C_i. \tag{5.2}$$

Since $(\lambda_i - i)/(\lambda_i + i) \in S^1$, the unit circle in $\mathbb{C}$, for $i = 1, 2$, we conclude that $\overline{p(V(\mathbf{c}))}$ is diffeomorphic to the torus $T^2 = S^1 \times S^1$ embedded in Σ_2.

For a geodesic curve $\alpha(t, A)$ in $\mathcal{M}$, we let $\widehat{\alpha}(t, A) := p(\alpha(t, A))$ be the corresponding geodesic on Σ_2. From (5.2), we have the following.

Proposition 5.1. *Let $A = \sum\limits_{i=1}^{2} \lambda_i C_i$ be the spectral decomposition of A. Then*

$$\widehat{\alpha}(t, A) = \sum_{i=1}^{2} e^{i(\pi + 2\lambda_i t)} C_i.$$

Proof: For $t > 0$ with $\alpha(t, A) \in V$, $p(\alpha(t, A)) = \sum\limits_{i=1}^{2} ((\tan \lambda_i t - i)/(\tan \lambda_i t + i)) C_i$. The result then follows from

$$\frac{\tan \lambda_i t - i}{\tan \lambda_i t + i} = \sin^2 \lambda_i t \cos^2 \lambda_i t - 2i \sin \lambda_i t \cos \lambda_i t = e^{i(\pi + 2\lambda_i t)}.$$

∎

A *torus knot* is a simple closed curve embedded on the standard torus $T^2 = S^1 \times S^1$ in $\mathbb{C}^2$. Any point of T^2 has coordinates $(e^{i\psi}, e^{i\phi})$. We have the standard meridian-longitude generator system of $H_1(T^2) \cong \mathbb{Z} \oplus \mathbb{Z}$:

$$m : e^{i\theta} \mapsto (e^{i\theta}, 1), \quad \text{meridian,}$$
$$\ell : e^{i\theta} \mapsto (1, e^{i\theta}), \quad \text{longitude.}$$

A torus knot is said to be of type (p, q), denoted by $T(p, q)$, if it is homologous to $pm + q\ell$ on T^2 and p and q are coprime integers. A torus knot $T(p, q)$ is trivial if and only if either $p = \pm 1$ or $q = \pm 1$. Two nontrivial torus knots $T(p, q)$ and $T(p', q')$ are equivalent if and only if (p', q') is equal to one of $(p, q), (q, p), (-p, q)$, or $(-p, -q)$ [**12**].

Theorem 5.2. *Let $A = \sum\limits_{i=1}^{2} r p_i C_i$ ($r \geqslant 0$, $p_i \in \mathbb{Z}$) be the spectral decomposition of $A \in \mathrm{Sym}(2, \mathbb{R})$ and let $\alpha(t, A)$ be the corresponding closed geodesic in $\mathcal{M}$. Then $\widehat{\alpha}(t, A)(0 \leqslant t \leqslant \pi/r)$ is the torus knot of type $(|p_1|, |p_2|)$.*

PROOF: Let $K = \{\widehat{\alpha}(t, A) \in V = \Sigma_2 \mid 0 \leqslant t \leqslant \pi/r\}$. By Proposition 5.1, K is a simple closed curve on the standard torus $T^2 = S^1 \times S^1$ embedded in $\mathbb{C}^2$. Since $\alpha(t, A)$ is a closed geodesic, p_1 and p_2 are relatively prime integers. Now

$$\begin{aligned} K &= \{e^{i\pi}(e^{i2rp_1t}, e^{i2rp_2t}) \in T^2 \mid 0 \leqslant t \leqslant \pi/r\} \\ &= \{(-e^{ip_1s}, -e^{ip_2s}) \in T^2 \mid 0 \leqslant s \leqslant 2\pi\}. \end{aligned}$$

Hence K is the torus knot of type $(|p_1|, |p_2|)$. ∎

FUTURE DIRECTIONS: In subsequent papers, we shall study the periodicity of symmetric geodesics for the Euclidean Jordan algebra $\mathrm{Sym}(2, \mathbb{R})$ of all 2×2 symmetric real matrices, realised in the Shilov boundary Σ_2 which is a non-orientable closed 3-manifold. By an investigation of some finite group actions on an orientable double covering space of the Shilov boundary Σ_2, we shall give a certain class of knots and links in S^3 corresponding to the Euclidean Jordan algebra $\mathrm{Sym}(2, \mathbb{R})$ and will discuss their properties.

6. APPENDIX

Let $\mathcal{X}(\mathbb{R})$ be the Lie algebra of all smooth vector fields on $\mathbb{R}$. For $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{R})$, we consider the linear fractional transformation $g(x) = (ax+b)(cx+d)^{-1}$. This defines a local action on $\mathbb{R}$ and a global action on S^1. The induced Lie homomorphism $\phi : \mathfrak{sl}(2, \mathbb{R}) \longrightarrow \mathcal{X}(\mathbb{R})$ is given by

$$\phi \begin{pmatrix} a & b \\ c & -a \end{pmatrix} = (b + 2ax - cx^2)\frac{\partial}{\partial x}.$$

In this case, $\mathfrak{q}, \mathfrak{h}$, and $C_{\mathfrak{p}}$ are given by

$$\begin{aligned} \mathfrak{q} &= \left\{ \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} \;\middle|\; a, b \in \mathbb{R} \right\}, \\ C_{\mathfrak{p}} &= \left\{ \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} \;\middle|\; a, b \geqslant 0 \right\}, \\ \mathfrak{h} &= \left\{ \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} \;\middle|\; a \in \mathbb{R} \right\}. \end{aligned}$$

It is easy to show that $f(\mathfrak{h} \pm C_{\mathfrak{p}}) = \infty$. Now let $X = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in \mathfrak{sl}(2, \mathbb{R})$. Note that $X(x) = b + 2ax - cx^2$. Set $\alpha = \left|a^2 + bc\right|^{1/2}$. For $b \neq 0$ and $c \neq 0$, the differential equation

$x'(t) = X\big(x(t)\big), x(0) = 0$, has the solution:

$$x(t) = \begin{cases} \frac{a}{c}(1 + \frac{1}{at-1}) & \text{if } a^2+bc=0, \\ \frac{a}{c} + \frac{\alpha}{c}\tanh\Big[\alpha t + \tanh^{-1}\Big(\frac{-a}{\alpha}\Big)\Big] & \text{if } a^2+bc>0,\ bc>0, \\ \frac{a}{c} + \frac{\alpha}{c}\coth\Big[\alpha t + \coth^{-1}\Big(\frac{-a}{\alpha}\Big)\Big] & \text{if } a^2+bc>0,\ bc<0, \\ \frac{a}{c} - \frac{\alpha}{c}\tan\Big[\alpha t + \tan^{-1}\frac{a}{\alpha}\Big] & \text{if } a^2+bc<0. \end{cases}$$

Then the function f is given by

$$f(X) = \begin{cases} \infty & \text{if } X \in \mathfrak{h} \pm C_{\mathfrak{p}}, \\ \infty & \text{if } a^2+bc=0,\ a \leqslant 0, \\ \infty & \text{if } a^2+bc>0,\ bc<0,\ a<0 \\ a^{-1} & \text{if } a^2+bc=0,\ a>0, \\ \dfrac{\tanh^{-1}\Big(\frac{\alpha}{a}\Big)}{\alpha} & \text{if } a^2+bc>0,\ bc<0,\ a>0, \\ \dfrac{\tan^{-1}\frac{\alpha}{a}}{\alpha} & \text{if } a^2+bc<0,\ a>0, \\ \dfrac{\pi}{2\alpha} & \text{if } a^2+bc<0,\ a \leqslant 0. \end{cases}$$

References

[1] W. Bertram, 'Un théorème de Liouville pour les algèbres de Jordan', *Bull. Soc. Math. France* **124** (1996), 299–327.

[2] W. Bertram, 'On some classes and conformal groups', *J. Lie Theory* **6** (1996), 215–247.

[3] W. Bertram, 'Algebraic structures of Makarevič spaces I', *Transform. Groups* **1** (1998), 3–32.

[4] J. Faraut and A. Koranyi, *Analysis on symmetric cones*, Oxford Mathematical Monographs (Oxford University Press, Oxford, 1994).

[5] J. Hilgert and G. Olafsson, *Causal symmetric spaces.Geometry and harmonic analysis*, Perspectives in Mathematics **18** (Academic Press, San Diego, 1997).

[6] W. Kaup, 'Algebraic characterization of symmetric complex Banach manifolds', *Math. Ann.* **228** (1977), 39–64.

[7] W. Kaup, 'A Riemann mapping theorem for bounded symmetric domains in complex banach spaces', *Math. Z.* **183** (1983), 503–529.

[8] K. Koufany, 'Semi-groupe de Lie associé a un cône symétriques', *Ann. Inst. Fourier (Grenoble)* **45** (1995), 1–29.

[9] J. D. Lawson and Y. Lim, 'Lie semigroups with triple decompositions', (submitted).

[10] O. Loos, *Bounded symmetric domains and Jordan pairs*, Lecture Notes (University California, Irvine, 1977).

[11] B. Makarevič, 'Ideal points of semisimple Jordan algebras', *Mat. Zametki* **15** (1974), 295–305.

[12] D. Rolfsen, *Knots and links*, Mathematics Lecture Series **7** (Publish or Perish Press, Berkley, CA, 1976).

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