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Percolation probability and critical exponents for site percolation on the UIPT

Laurent Ménard

Abstract. We derive three critical exponents for Bernoulli site percolation on the uniform infinite planar triangulation (UIPT). First, we compute explicitly the probability that the root cluster is infinite. As a consequence, we show that the off-critical exponent for site percolation on the UIPT is $\beta = 1/2$. Then we establish an integral formula for the generating function of the number of vertices in the root cluster. We use this formula to prove that, at criticality, the probability that the root cluster has at least n vertices decays like $n^{-1/7}$. Finally, we also derive an expression for the law of the perimeter of the root cluster and use it to establish that, at criticality, the probability that the perimeter of the root cluster is equal to n decays like $n^{-4/3}$. Among these three exponents, only the last one was previously known. Our main tools are the so-called gasket decomposition of percolation clusters, generic properties of random Boltzmann maps, and analytic combinatorics.

1 Introduction

Percolation on random planar maps has been studied intensively since the pioneering work of Angel [2]. The main feature of random planar maps making this study so fruitful is the spatial Markov property. It can be used with two different approaches. The first approach is to perform an exploration process of percolation interfaces with the so-called peeling process. This is the approach developed by Angel [2] to prove that the threshold for Bernoulli site percolation on the uniform infinite planar triangulation (UIPT; the limit in law of large uniform random triangulations for the local topology [4]) is 1/2. This approach has been later used by several authors to study other models of percolation on maps (see, for example, [3, 15, 21, 24, 27] and the references therein). The second approach is more global and consists in decomposing the map into the cluster of the root vertex and pieces filling the faces of this cluster. Such a decomposition is often called the Gasket decomposition (see, for instance, the works of Borot, Bouttier, Duplantier, and Guitter [8–10]). This second approach has been used very recently to study percolation on random finite triangulations by Bernardi, Curien, and Miermont [6], following the previous work by Curien and

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Kortchemski [16]; and on other natural models of random finite planar maps by Curien and Richier [17].

This work builds on the article by Bernardi, Curien, and Miermont [6] to study site percolation on the UIPT (of type 1, with loops and multiple edges allowed). Our first main result is an explicit formula for the probability that percolation from the root occurs.

Theorem 1.1 *Let $\mathbb{P}^p_\infty$ denote the law of the type 1 UIPT, with vertices colored black with probability p and white with probability $1-p$, and conditioned on the event where the root edge has both end vertices colored black. Let $\mathfrak{C}$ denote the site percolation cluster of the root vertex under $\mathbb{P}^p_\infty$. Then, for every $p \in [0,1]$, we have*

$$\mathbb{P}^p_\infty(|\mathfrak{C}| = \infty)$$
$$= 2\,\frac{\sqrt{2p-1}\left(\sqrt{3}-\cos^3\left(\frac{2}{3}\arccos(\sqrt{p})\right)\right)\left(\left(\cos\left(\frac{2}{3}\arccos(\sqrt{p})\right)\right)^{3/2}+p(2p-1)\right)}{p\left(2\sqrt{3}-3(1-p)\right)}\mathbf{1}_{p\geq 1/2}.$$

In particular, the critical exponent is $\beta = 1/2$: as $p \to 1/2^+$, one has

$$\mathbb{P}^p_\infty(|\mathfrak{C}| = \infty) = 3^{1/4}\,\frac{15}{26}\left(1+\frac{4\sqrt{3}}{3}\right)\sqrt{p-\frac{1}{2}} + \mathcal{O}\left(p-\frac{1}{2}\right).$$

To the best of our knowledge, this is the first formula of this type and the first calculation of the critical exponent β for percolation on the UIPT. A similar formula was obtained for percolation on the uniform infinite half-plane triangulation (UIHPT) by Angel and Curien [3] using the peeling process. They also calculate explicitly the probability that percolation from the root occurs and then obtain $\beta = 1$ in the UIHPT setting. The relation between these two exponents remains quite mysterious, and we do not know of any strategy to obtain the exponent β without first computing explicitly the probability of percolation (Figure 1).

Our second main result gathers estimates for the volume and perimeter of critical percolation clusters.

Theorem 1.2 *With the notation of Theorem 1.1, we have, for an explicit constant $\kappa > 0$,*

$$\mathbb{P}^{1/2}_\infty\left(|V(\mathfrak{C})| \geq n\right) \underset{n\to\infty}{\sim} \kappa\, n^{-1/7}.$$

Furthermore, let $\partial\mathfrak{C}$ denote the root face of $\mathfrak{C}$, that is, the face of $\mathfrak{C}$ containing the root face of the UIPT. Then there is an explicit constant $\kappa' > 0$ such that

$$\mathbb{P}^{1/2}_\infty\left(|V(\partial\mathfrak{C})| = n\right) \underset{n\to\infty}{\sim} \kappa'\, n^{-4/3}.$$

The perimeter exponent 4/3 was established by Curien and Kortchemski [16] using the gasket decomposition but with a different approach than the present work. The exponent 1/7 was conjectured in [21] using heuristics for the peeling process, and the present work is the first time it is established rigorously. Previous works that computed volume exponents for critical percolation models on infinite random planar maps did so for cluster hulls (part of the maps separated from infinity by a percolation interface). The reason being that all these works use variations around the peeling process, which is particularly well suited to study percolation interfaces—and therefore hulls—but not

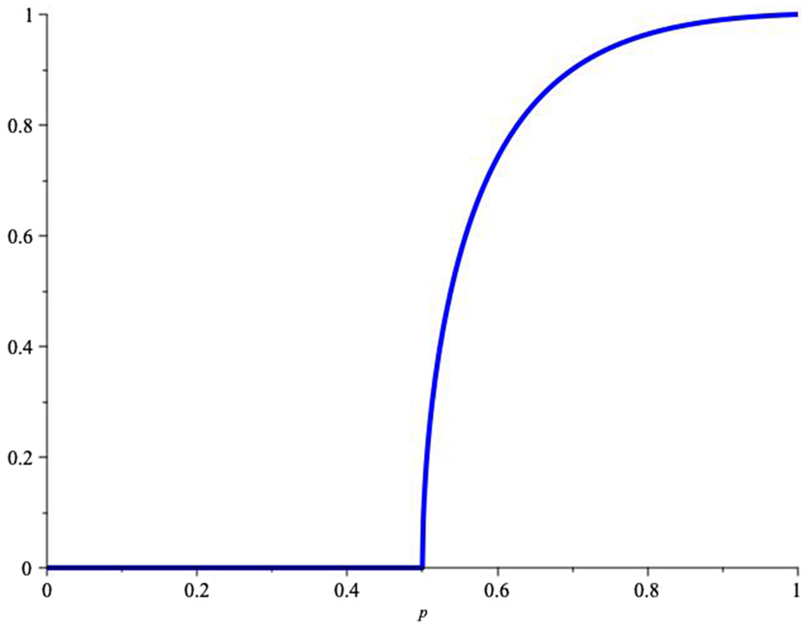

Figure 1: Plot of the probability $\mathbb{P}^{p}_{\infty}(|\mathfrak{C}| = \infty)$ for $p \in [0,1]$.

so useful to study the geometry of the clusters themselves. For instance, let $\mathfrak{H}$ denote the hull of the root cluster $\mathfrak{C}$ in the UIPT; i.e., $\mathfrak{H}$ is the complement of the only infinite connected component of $\mathbf{T}_\infty \backslash \mathfrak{C}$, where $\mathbf{T}_\infty$ is the UIPT with a critical site percolation configuration. Gorny, Maurel-Segala, and Singh [21] proved that

$$\mathbb{P}^{1/2}_{\infty}(|V(\mathfrak{H})| \geq n) \asymp n^{-1/8} \quad \text{and} \quad \mathbb{P}^{1/2}_{\infty}(|V(\partial\mathfrak{H})| \geq n) \asymp n^{-1/6},$$

where $u_n \asymp v_n$ means that u_n/v_n is bounded.

1.1 Main ingredients and organization of the paper

For every site percolated finite rooted planar triangulation $\mathfrak{t}$, we denote by $\mathrm{v}_\circ(\mathfrak{t})$ and $\mathrm{v}_\bullet(\mathfrak{t})$ its number of white (resp. black) vertices, and by $\mathrm{e}(\mathfrak{t})$ its number of edges. For $p \in (0,1)$ and for $t > 0$ small enough, we can consider a random finite percolated triangulation $\mathbf{t}$ whose law gives a probability proportional to $t^{\mathrm{e}(\mathfrak{t})}\, p^{\mathrm{v}_\bullet(\mathfrak{t})}(1-p)^{\mathrm{v}_\circ(\mathfrak{t})}$ to every finite triangulation $\mathfrak{t}$. Let us denote by $\mathcal{Z}$ the partition function of this model and by $\mathbb{P}^p$ the corresponding probability:

$$\mathcal{Z}(p,t) := \sum_{\mathfrak{t}} t^{\mathrm{e}(\mathfrak{t})}\, p^{\mathrm{v}_\bullet(\mathfrak{t})}(1-p)^{\mathrm{v}_\circ(\mathfrak{t})} < +\infty, \text{ and } \mathbb{P}^p(\mathbf{t} = \mathfrak{t}) = \frac{t^{\mathrm{e}(\mathfrak{t})}\, p^{\mathrm{v}_\bullet(\mathfrak{t})}(1-p)^{\mathrm{v}_\circ(\mathfrak{t})}}{\mathcal{Z}(p,t)}.$$

The partition function $\mathcal{Z}$ and its generalizations to triangulations with a boundary (with additional parameters counting, respectively, the number of boundary vertices and boundary edges) are studied in Section 2. In particular, we establish a rational

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parameterization for the generalized partition function. This parameterization significantly simplifies the study of its analytic properties started in [6].

The gasket decomposition resides in the following statement. There exists an explicit sequence of positive numbers $(q_k(p,t))_{k\geq 1}$ such that, for every finite nonatomic map $\mathfrak{m}$, the probability that the root cluster $\mathfrak{C}(\mathbf{t})$ of the random triangulation $\mathbf{t}$ is equal to $\mathfrak{m}$ is given by

$$\mathbb{P}^p\left(\mathfrak{C}(\mathbf{t}) = \mathfrak{m}\right) = \frac{\prod_{f\in \mathrm{Faces}(\mathfrak{m})} q_{\deg(f)}(p,t)}{\mathcal{Z}(p,t)}.$$

In this sense, the root cluster is a Boltzmann random planar map associated with the weight sequence $(q_k(p,t))_{k\geq 1}$. The properties of such random maps depend on the asymptotic behavior in k of the weight sequence, which can be inferred from the generating function of the weights. In our case, the weight generating function is closely related to the generalized partition function of percolated triangulations counted by edges, boundary edges, and boundary vertices. A crucial consequence of this relation is that it allows to calculate explicitly the so-called pointed disk generating function of the root cluster in terms of the singularities of the generalized partition function of percolated triangulations in equations (3.9) and (3.10). See Section 3 for details.

To study the origin cluster of the UIPT, we can condition the random triangulation $\mathbf{t}$ to have $3n$ edges. By continuity for the local topology, this gives

$$\text{(1.1)}\quad \mathbb{P}^p\left(\mathfrak{C}(\mathbf{t}) = \mathfrak{m} | \mathrm{e}(\mathbf{t}) = 3n\right) = \frac{[t^{3n}]\prod_{f\in \mathrm{Faces}(\mathfrak{m})} q_{\deg(f)}(p,t)}{[t^{3n}]\mathcal{Z}(p,t)} \underset{n\to\infty}{\to} \mathbb{P}^p_\infty(\mathfrak{C} = \mathfrak{m}).$$

With a careful study of the dependency in t of the weight sequence $(q_k(p,t))_{k\geq 1}$ performed in Section 6.1, we are able to compute the above limit, giving the law of the root cluster in the UIPT on the event where it is finite.

We are then able to establish an integral formula for the sum of the limiting probability over every finite map, which is the probability that the root cluster of the UIPT is finite. See Proposition 4.2. In particular, this calculation uses the explicit universal form of the pointed disk generating functions and cylinder generating functions of Boltzmann maps. With the help of our rational parameterizations, we can then calculate explicitly our integral formula for the probability that the root cluster is finite and establish Theorem 1.1. This is performed in Section 4.

The formula we obtain for the limit (1.1) is quite easy to sum over maps with the same perimeter. As a consequence, we are also able to calculate explicitly the law of the perimeter of the root cluster and obtain the second statement of Theorem 1.2 on the perimeter of the root cluster. This is done at the beginning of Section 5.2.

Finally, to compute the tail probability of the number of vertices in the root cluster at criticality ($p = 1/2$), we establish an integral formula for the generating function $\mathbb{E}^p_\infty\left[g^{|V(\mathfrak{C})|}\right]$ in Section 5.2. See in particular identity (5.7), which also originates from our explicit formula for the limit (1.1). The expression we get involves two quantities. The first is the generating series derived from the asymptotics of the coefficients in t of the weights $(q_k(p,t))_{k\geq 1}$ studied in Section 6.1 for which we have an explicit parametric expression. The second quantity involves the pointed disk generating functions of Boltzmann maps with modified weight sequence $(g^{(k-2)/2}q_k(p,t))_{k\geq 1}$. We can analyze the behavior as $g \to 1^-$ of these modified pointed disk generating functions using the bivariate generating functions associated with the Bouttier–Di Francesco–Guitter bijection [13] presented in Section 5.1 and their singularities obtained in Section 6.2. We put all that together in Section 5.2 to study the singular behavior as $g \to 1^-$ of $\mathbb{E}_\infty^{1/2}\left[g^{|V(\mathfrak{C})|}\right]$ and prove Theorem 1.2.

1.2 On the robustness of our approach

We believe that it should be possible to adapt the strategy of this paper to other models of random planar maps and other statistical models. Indeed, what is needed first is the gasket decomposition, which exists, for example, for percolation on other models of maps [17], or for the $\mathcal{O}(n)$ model on maps [8, 9]. We then need information on the generalized generating series of maps with a boundary and their singularities. In the present work, we have explicit parametric expressions to simplify calculations, but we really only need to identify the nature of the singularities for the critical exponents. Any model for which such information is available should fall into the scope of our method.

In another article [1], we derive several critical exponents for the sign clusters in finite and infinite planar triangulations coupled with an Ising model. In particular, we establish in [1] counterparts for the main results of the present work. Theorem 1.1 and its counterpart for Ising model are unrelated, but Theorem 1.2 is a particular case of its Ising version at infinite temperature.

1.3 Links with other critical exponents for percolation on the UIPT

The volume exponent 1/7 of Theorem 1.2 suggests that the scaling limit of a large critical percolation cluster in the UIPT should be of quantum dimension 7/8. There is also very strong evidence (see, for instance, the works of Bernardi, Holden, and Sun [7] and Holden and Sun [22]) that in some sense, this scaling limit is a CLE_6 on an independent pure Liouville Quantum Gravity surface. The quantum dimension 7/8 of the scaling limit of the root cluster and the dimension 91/48 of the gasket of a CLE_6 agree with the Knizhnik, Polyakov, and Zamolodchikov (KPZ) [23] relation:

$$1 - \frac{1}{2}\frac{91}{48} = \frac{2}{3}\left(1-\frac{7}{8}\right)^2 + \frac{1}{3}\left(1-\frac{7}{8}\right).$$

In a similar fashion, the perimeter exponent 4/3 of Theorem 1.2 agrees with the KPZ relation and the dimension $\frac{7}{4}$ of an SLE_6 curve [5].

The value of the critical exponent β agrees heuristically with known quantities of the UIPT and Kesten's scaling relations. Indeed, Bernardi, Holden, and Sun [7] and Holden and Sun [22] established that the number of percolation pivotal points in a random triangulation of size n is of order $n^{1/4}$. In this sense, the quantum dimension of the set of pivotal points of critical percolation on the UIPT should be 1 (the map itself having dimension 4). This dimension can also be predicted with the KPZ relation and the dimension 3/4 of critical percolation pivotal points in Euclidean geometry [20]. On the other hand, the one-arm exponent α_1 of critical percolation on the UIPT

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should be 1/2 from the quantum dimension 7/8 of the clusters. Kesten's scaling relation then states that the dimension of pivotal points should be α_1/β, giving 1 with the exponent of Theorem 1.1.

2 Generating series

Let T be the generating series of rooted triangulations with a (not necessarily simple) boundary counted by edges (variable t), boundary length (parameter y), and boundary vertices (parameter p). That is, we define

$$T(p,t,y) = \sum_{k\geq 1}\sum_{\mathfrak{t}\in\mathcal{T}_k} t^{e(\mathfrak{t})} p^{\mathrm{v}_{\mathrm{out}}(\mathfrak{t})} y^k = \sum_{k\geq 1} T_k(p,t)\, y^k,$$

where $\mathcal{T}_k$ is the set of all rooted triangulations with a boundary face of degree k, and where $e(\mathfrak{t})$ and $\mathrm{v}_{\mathrm{out}}(\mathfrak{t})$ denote, respectively, the total number of edges and number of boundary vertices of the triangulation with a boundary $\mathfrak{t}$.

From [6, Lemma 3.1], we have the following equation for $T(p,t,y)$:

$$\begin{aligned} T(p,t,y) &= p + y^2 t T^2(p,t,y) + (p-1)t\frac{(T(p,t,y)-p)^2}{pyT(p,t,y)} \\ &\quad + \frac{t}{py}\left(T(p,t,y) - p - y\,T_1(p,t)\right). \end{aligned} \tag{2.1}$$

Using the quadratic method, the authors of [6] establish the following algebraic equation for $T_1 \equiv T_1(p,t)$ that will be our starting point:

$$64\,T_1^3\,t^5 - 27\,p^3\,t^5 - 96\,T_1^2\,p\,t^4 + 30\,T_1\,p^2\,t^3 + p^3\,t^2 + T_1^2\,p\,t - T_1\,p^2 = 0.$$

Up to a multiplicative constant p, the series T_1 is simply the generating series of triangulations of the 1-gon counted by edges and admits a proper rational parameterization.

Lemma 2.1 *Let U be the unique power series in t^3 having constant term 0 and satisfying*

$$t^3 = \hat{w}(U) := \frac{1}{2}U(1-U)(1-2U).$$

The series $t\,T_1(p,t)$ seen as a series in t^3 admits the following proper rational parameterization in U:

$$t\,T_1(p,t) = \hat{T}_1(p,U) := p\,U\,\frac{1-3U}{1-2U}.$$

Furthermore, the series $U(t^3)$ has a unique dominant singularity at $t^3 = (t_c)^3 := \frac{\sqrt{3}}{36}$ with the following singular behavior:

$$\begin{aligned} U(t^3) &= \frac{3-\sqrt{3}}{6} - \frac{\sqrt{2}}{6}\left(1-\left(\frac{t}{t_c}\right)^3\right)^{1/2} + \frac{\sqrt{3}}{54}\left(1-\left(\frac{t}{t_c}\right)^3\right) \\ &\quad - \frac{5\sqrt{2}}{648}\left(1-\left(\frac{t}{t_c}\right)^3\right)^{3/2} + \mathcal{O}\left(1-\left(\frac{t}{t_c}\right)^3\right)^2. \end{aligned} \tag{2.2}$$

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Proof This result could be qualified as folklore since the series T_1 is just the generating series of triangulations of the 1-gon multiplied by p. The rational parameterization given is also the classical one. The fact that $U(t^3)$ is unique comes from the Lagrangian form of the equation that defines it. We can also see that the algebraic equation satisfied by $t\,T_1(t)$ has a unique solution that is a power series with constant term 0. By composition and unicity, we can see that indeed $tT_1(t) = \hat{T}_1(p, U(t^3))$ as power series in t since both verify the algebraic equation and have constant term 0. The singular behavior of U is also very classical and without difficulties. See the Maple companion file [26] for details. ∎

Injecting the parameterization of t^3 and $t\,T_1$ by U in the equation for T then allows to establish a proper rational parameterization for T.

Lemma 2.2 *Recall the definition of the power series $U \equiv U(t^3)$ from Lemma 2.1. Let $V \equiv V(p, U, y)$ be the unique power series in $\mathbb{Q}[p, U][[y]] \subset \mathbb{Q}[p][[t^3, y]]$ with constant term in y equal to 0 satisfying*

$$y = \hat{y}(p, U, V) := \frac{2\,V\,(2 - 4\,U - V)}{4\,p\,U\,(1 - U)\,(1 - 2U) + 2\,U\,(1 - 3U)\,V + 2\,(1 - 3U)\,V^2 - V^3}.$$

The series $T(p, t, ty)$ seen as a series in t^3 and y is algebraic and admits the following proper rational parameterization in U and V:

$$\begin{aligned} T(p, t, ty) &= \hat{T}(p, U, V) \\ &:= \frac{4\,p\,U\,(1 - U)\,(1 - 2U) + 2\,U\,(1 - 3U)\,V + 2\,(1 - 3U)\,V^2 - V^3}{4\,U\,(1 - U)\,(1 - 2U)}. \end{aligned}$$

In addition, for any $p \in (0, 1]$ and any fixed $t \in (0, t_c]$, the series $V(p, U(t^3), y)$ and $T(p, t, ty)$ seen as series in y both have radius of convergence $y_+(p, t) > 1$ where it is singular. Furthermore, both series can be analytically continued in the domain $\mathbb{C}\backslash((-\infty, y_-(p, t)] \cup [y_+(p, t), +\infty))$, where $y_-(p, t)$ is on the negative real line ($-\infty$ included) and is such that $y_-(p, t) < -y_+(p, t)$.

Proof All computations are available in the Maple companion file [26].

The fact that V is uniquely defined as a power series comes from the Lagrangian form of the equation $V = y \times R(p, U, V)$ with R a rational fraction such that $R(p, U, 0) \neq 0$. This form also implies by inductive calculation of the coefficients in y of V that they are all rational in p and U. Similarly, equation (2.1) verified by $T(p, t, ty)$ takes the form

$$\begin{aligned} &p(T(p, t, ty) - p)(T(p, t, ty) - (1 - p)) \\ &\quad = y\,T(p, t, ty)\,\left(tT_1(p, t) - p(T(p, t, ty) - p) - y^2\,p\,t^3\,T(p, t, ty)^2\right). \end{aligned}$$

Here again, by inductive calculation of the coefficients, we can see that this last equation has a unique solution that is a power series in t^3 and y with constant term in y equal to p. Note that since the series $tT_1(p, t)$ is algebraic, this equation also ensures that $T(p, t, ty)$ is algebraic. By composition, $\hat{T}\left(p, U(t^3), V(p, U(t^3), y)\right)$ is a power series in t^3 and y with constant term in y equal to p. We can verify that it satisfies the same algebraic equation as $T(p, t, ty)$, and therefore the two power series are identical.

Now, fix $t^3 \in (0, t_c]$. The function $V \mapsto \hat{y}(p, U(t^3), V)$ has poles and stationary points that we can locate. Let us start with the poles. The denominator of $\hat{y}(p, U(t^3), V)$ is a polynomial of degree 3 in V. It is positive with positive derivative at $V = 0$, and changes signs between $1 - 2U$ and $2(1 - 2U)$. Since the coefficient of V^3 is -1, this leaves two possibilities for the poles of $\hat{y}(p, U(t^3), V)$: there is always a pole between $1 - 2U$ and $2(1 - 2U)$, and either there is no additional real pole or there are two negative poles.

The stationary points of $\hat{y}(p, U(t^3), V)$ are the roots of the polynomial $-2V^4 + (-16U + 8)V^3 - 4(3U - 1)(3U - 2)V^2 - 16Up(2U - 1)(U - 1)V - 16Up(U - 1)(2U - 1)^2$, where U stands for $U(t^3)$. By computing the values of the polynomial at 0, $1 - 2U$, and $2(1 - 2U)$, we can see that it has four real roots $V_-(p, U) < 0 < V_+(p, U) \le 1 - 2U \le V_l(p, U) < 2(1 - 2U) < V_r(p, U)$ (the case of a double root at $1 - 2U$ only happens when $U = U_c$ and is treated separately in the Maple file). It is also easy to see that $\partial_V \hat{y}(p, U(t^3), 0) > 0$, and therefore $y_+(p, t) := \hat{y}(p, U(t^3), V_+(p, U(t^3))) > 0$. We can define $y_-(p, t) := \hat{y}(p, U(t^3), V_-(p, U(t^3))) < 0$ when $\hat{y}(p, U(t^3), V)$ has no pole between $V_-(p, U(t^3))$ and 0, and $y_-(p, t) := -\infty$ when it has such a pole. By singular inversion, the inverse function $V(p, U(t_3), y)$ of $\hat{y}(p, U(t^3), V)$ is analytic in the domain $\mathbb{C}\setminus((-\infty, y_-(p, t)] \cup [y_+(p, t), +\infty))$ and singular at both points $y_\pm(p, t)$ when they are finite.

By composition, $y \mapsto T(p, t, ty)$ is analytic in the same domain. Since it has nonnegative coefficients, we know that it is singular at its radius of convergence. Therefore, $y_+(p, t)$ is its radius of convergence and $y_-(p, t) \le -y_+(p, t)$. Checking that $y_-(p, t) < -y_+(p, t)$ just with our current material is cumbersome but could be done. However, we do not need to go through this since this inequality is a direct consequence of equations (3.10) established with no computations at the end of Section 3. Finally, to see that $y_+(p, t) > 1$, we first note that for p fixed, $y_+(p, t)$ is nonincreasing with respect to $|t|$ and then check $y_+(p, t_c) > 1$ by direct computation. ■

3 The root cluster as a Boltzmann map

In this whole section, $p \in (0, 1)$ and $t \in (0, t_c]$ are fixed.

3.1 The weight sequence from [6] with the edge parameter

In [6, Section 2.2], it was established that for a random site-percolated triangulation (conditioned on the event where both ends of the root edge are colored in black), the cluster of the root is a random Boltzmann map with weight sequence $\mathbf{q}(p, t) = (q_k(p, t))_{k\ge1}$ given by

$$q_k(p, t) = \frac{1}{p}\left((pt)^{3/2}\delta_{\{k=3\}} + (pt^3)^{k/2}\sum_{l\ge0}\binom{k+l-1}{k-1}[y^l]T(1-p, t, ty)\right), \tag{3.1}$$

for $k \ge 1$ (see [6, equation (9)]). We briefly recall here what this statement means and how to obtain it.

For every percolated rooted planar triangulation $\mathfrak{t}$, we denote, respectively, by $\mathrm{v}_\circ(\mathfrak{t})$ and $\mathrm{v}_\bullet(\mathfrak{t})$ the number of white (resp. black) vertices. Let $\mathcal{T}^{\mathrm{perc}}$ be the set of all percolated rooted triangulations with both end vertices of the root edge black. For

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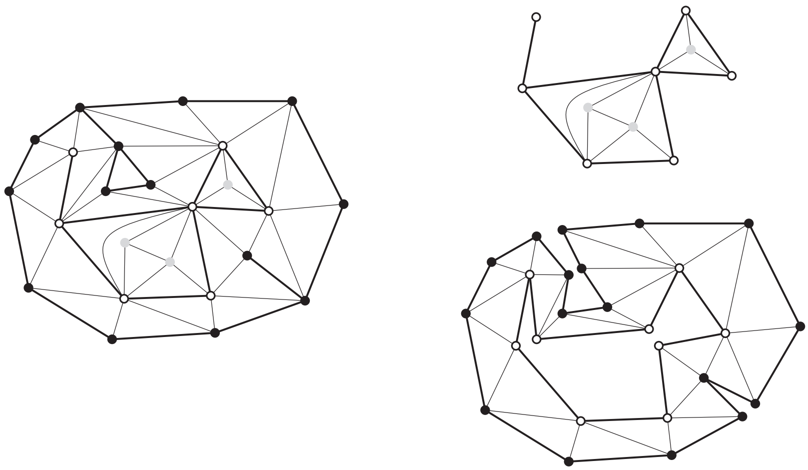

Figure 2: Gasket decomposition: A face of the root cluster (black vertices in the figure) is filled with a triangulation with arbitrary white boundary (upper-right figure, gray vertices can be either black or white) and a necklace (lower-right figure) triangulating the region between the two.

every $\mathfrak{t} \in \mathcal{T}^{\mathrm{perc}}$, we denote by $\mathfrak{C}(\mathfrak{t})$ the cluster of its root edge viewed as a planar rooted map (whose root is the root edge of the triangulation). The fact that the root cluster is a random Boltzmann map comes from the following identity (see equation (7) of [6]): for every nonatomic map $\mathfrak{m}$, one has

$$\sum_{\mathfrak{t}\in\mathcal{T}^{\mathrm{perc}}:\,\mathfrak{C}(\mathfrak{t})=\mathfrak{m}} t^{e(\mathfrak{t})} p^{\mathrm{v}_\bullet(\mathfrak{t})}(1-p)^{\mathrm{v}_\circ(\mathfrak{t})} = p^2 \cdot \prod_{f\in F(\mathfrak{m})} q_{\deg(f)}(p,t). \tag{3.2}$$

This identity stems from the classical gasket decomposition [9] sometimes called island decomposition [6], and we briefly explain how it is obtained in the following lines (see Figure 2 for an illustration of these arguments). Fix $\mathfrak{m}$ a nonatomic map and $\mathfrak{t}$ a site-percolated triangulation with root cluster $\mathfrak{m}$ colored black. For such a triangulation $\mathfrak{t}$, each face of its root cluster is filled with a triangulation with an arbitrary boundary of white vertices, and a *necklace* of triangles with no additional vertices between this triangulation with a boundary and the cluster. For each cluster face of degree $k \geq 1$ filled with a triangulation with perimeter $l \geq 1$, there are $\binom{k+l-1}{k-1}$ different possible necklaces, and each of these necklaces requires an additional $k+l$ edges. Taking into account the case where the cluster face has degree 3 and can be part of the triangulation, this writes

$$\begin{aligned}\sum_{\mathfrak{t}\in\mathcal{T}^{\mathrm{perc}}:\,\mathfrak{C}(\mathfrak{t})=\mathfrak{m}} & t^{e(\mathfrak{t})} p^{\mathrm{v}_\bullet(\mathfrak{t})}(1-p)^{\mathrm{v}_\circ(\mathfrak{t})} \\ &= p^{|V(\mathfrak{m})|}\, t^{|e(\mathfrak{m})|} \\ &\qquad \times \prod_{f\in F(\mathfrak{m})}\left(\mathbf{1}_{\deg(f)=3} + \sum_{l\geq 0}\binom{\deg(f)+l-1}{\deg(f)-1} t^{\deg(f)+l}[y^l]T(1-p,t,y)\right),\end{aligned}$$

$$= p^2 \cdot \prod_{f\in F(\mathfrak{m})} \frac{1}{p}(pt)^{\deg(f)/2}$$
$$\times \left(\mathbf{1}_{\deg(f)=3} + \sum_{l\geq 0} \binom{\deg(f)+l-1}{\deg(f)-1} t^{\deg(f)+l}[y^l]T(1-p,t,y)\right),$$

where we used Euler's relation $|V(\mathfrak{m})| - 2 = \sum_{f\in F(\mathfrak{m})}(\deg(f)/2 - 1)$ in the last display. The expression (3.2) follows easily, and we refer the reader to [6, Section 2.2] for additional details. Note for future reference that for any $k \geq 1$, the weight $q_k(p,t)$ is $1/p$ times the generating series of all triangulations of the k-gon with a weight t per inner edge, a weight $\sqrt{pt}$ per boundary edge, and a weight $1-p$ per inner vertex adjacent to the boundary.

We can define the partition function of our percolated triangulations by

$$\mathcal{Z}(p,t) = \frac{1}{p^2} \cdot \sum_{\mathfrak{t}\in\mathcal{T}^{\mathrm{perc}}} t^{e(\mathfrak{t})} p^{\mathrm{v}_\bullet(\mathfrak{t})}(1-p)^{\mathrm{v}_\circ(\mathfrak{t})}. \tag{3.3}$$

From identity (3.2), denoting by $\mathcal{M}$ the set of all nonatomic rooted planar maps, we see that

$$\mathcal{Z}(p,t) = \sum_{\mathfrak{m}\in\mathcal{M}} \prod_{f\in F(\mathfrak{m})} q_{\deg(f)}(p,t). \tag{3.4}$$

Notice that this sum is finite when $p \in (0,1)$ and $t \in (0,t_c]$, meaning that the weight sequence $\mathbf{q}(p,t)$ is admissible in the sense of [25]. We will need the asymptotic behavior of the coefficients in t^3 of the series $\mathcal{Z}(p,t)$.

Proposition 3.1 *Fix $p \in (0,1)$, we have*

$$[t^{3n}]\mathcal{Z}(p,t) \underset{n\to\infty}{\sim} \frac{\sqrt{2}\left(3p-3+2\sqrt{3}\right)}{2p\sqrt{\pi}} t_c^{-3n} n^{-5/2}.$$

Proof By opening the root edge of the triangulations appearing in the sum (3.3), we can see that, for $p \in (0,1)$ and $|t| \leq t_c$, the partition function $\mathcal{Z}(p,t)$ is given by

$$\mathcal{Z}(p,t) = \frac{1}{p^2\, t}T_2(p,t) = \frac{1}{p^2\, t^3}\, t^2 T_2(p,t) = \frac{1}{t^3} tT_1(p,t)\left(1 + \frac{1-p}{p^3} tT_1(p,t)\right). \tag{3.5}$$

From Lemma 2.1, the function $tT_1(t)$ seen as a series in t^3 has a unique dominant singularity at t_c^3 and we can obtain its asymptotic expansion at t_c^3 from the expansion of U. As a consequence, the function $\mathcal{Z}(p,t)$ also has a unique dominant singularity at t_c^3 and we can easily obtain the following asymptotic expansion:

$$\mathcal{Z}(p,t) = \frac{3\sqrt{3}\left(7-4\sqrt{3}\right)+p}{4p} - \frac{\sqrt{3}\left(3p-27+16\sqrt{3}\right)}{4p}\left(1-t^3/t_c^3\right)$$
$$+ \frac{2\sqrt{2}\left(3p-3+2\sqrt{3}\right)}{3p}\left(1-(t/t_c)^3)^{3/2}\right) + \mathcal{O}\left(\left(1-t^3/t_c^3\right)^2\right).$$

The asymptotic behavior of $[t^{3n}]\mathcal{Z}(p,t)$ then follows from the classical transfer theorem [19, Theorem VI.4]. ■

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3.2 Generating series of the weights and of the associated Boltzmann maps

The generating series of the weight sequence $\mathbf{q}(p,t)$ is straightforward to compute. As shown by the following lines, the expression we obtain is valid for every $p \in (0,1)$, t in the greater domain of analyticity of $U(t^3)$, and z close enough to 0. More precisely, we need $\sqrt{pt^3}z \in \mathbb{C}\backslash[1,+\infty)$ and $\left(1-\sqrt{pt^3}z\right)^{-1}$ must belong to the domain of analyticity in y of $T(t,1-p,ty)$, which was studied in the previous section. Under these conditions for p, t, and z, the computation of the weight sequence generating function is as follows:

$$
\begin{aligned}
F_{\mathbf{q}(p,t)}(z) :&= \sum_{k\geq 1} q_k(p,t)z^k,\\
&= \frac{1}{p}(pt)^{3/2}z^3 + \frac{1}{p}\sum_{l\geq 0}[y^l]T(1-p,t,ty)\sum_{k\geq 1}\binom{k+l-1}{k-1}(pt^3)^{k/2}z^k,\\
&= \frac{1}{p}(pt)^{3/2}z^3 + \frac{1}{p}\sqrt{pt^3}z\sum_{l\geq 0}[y^l]T(1-p,t,ty)\left(1-\sqrt{pt^3}z\right)^{-l-1},\\
&= \frac{1}{p}(pt)^{3/2}z^3 + \frac{1}{p}\frac{\sqrt{pt^3}z}{1-\sqrt{pt^3}z}T\left(1-p,t,\frac{t}{1-\sqrt{pt^3}z}\right).
\end{aligned}
\tag{3.6}
$$

We will need expressions for the pointed and unpointed disk generating functions associated with the weight sequence $\mathbf{q}(p,t)$. For every $l \geq 0$, let $\mathcal{M}^l$ denote the set of all rooted planar maps with root face of degree l (for $l = 0$, this set contains only the atomic map). The unpointed disk generating function is defined as follows for $|z|$ large enough:

$$
W_{\mathbf{q}(p,t)}(z) := \sum_{l\geq 0}\left(\sum_{\mathfrak{m}\in\mathcal{M}^l}\prod_{f\in F(\mathfrak{m})\backslash\{f_r\}} q_{\deg(f)}(p,t)\right)z^{-l-1},
$$

where we denote the root face of a planar map by f_r. From our discussion establishing (3.2), we can compute the coefficients of these series. Indeed, from equation (3.2) and the fact that q_l is $1/p$ times the generating series of triangulations with a boundary of perimeter l counted with a weight t per inner edge, a weight $\sqrt{pt}$ per boundary edge, and a weight $1-p$ per inner vertex adjacent to the boundary, we have

$$
\sum_{\mathfrak{m}\in\mathcal{M}^l}\ \sum_{\mathfrak{t}\in\mathcal{T}^{\mathrm{perc}}:\mathfrak{C}(\mathfrak{t})=\mathfrak{m}} t^{e(\mathfrak{t})}p^{\mathrm{v}_\bullet(\mathfrak{t})}(1-p)^{\mathrm{v}_\circ(\mathfrak{t})} = p\sqrt{pt}^{-l}\, q_l(p,t)\sum_{\mathfrak{t}\in\mathcal{T}_l} t^{e(\mathfrak{t})}p^{\mathrm{v}_{\mathrm{out}}(\mathfrak{t})}.
$$

Comparing with the right-hand side of (3.2), this gives, for every $l \geq 1$,

$$
\sum_{\mathfrak{m}\in\mathcal{M}^l}\prod_{f\in F(\mathfrak{m})\backslash\{f_r\}} q_{\deg(f)}(p,t) = \frac{1}{p\sqrt{pt}^l}[y^l]T(p,t,y),
\tag{3.7}
$$

and thus

$$
W_{\mathbf{q}(p,t)}(z) = \frac{1}{pz}T\left(p,t,\frac{t}{\sqrt{pt^3}z}\right).
\tag{3.8}
$$

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The pointed disk generating function is defined similarly:

$$W^{\bullet}_{\mathbf{q}(p,t)}(z) = \sum_{l\geq 0}\left(\sum_{\mathfrak{m}\in\mathcal{M}^l}|V(\mathfrak{m})|\prod_{f\in F(\mathfrak{m})\setminus\{f_r\}} q_{\deg(f)}(p,t)\right) z^{-l-1}.$$

It has the following universal form: for $p\in(0,1)$ and $t\in(0,t_c]$ fixed, there exists real numbers $c_+(p,t)>2$ and $c_-(p,t)\in(-c_+(p,t),c_+(p,t))$ (the lower bound is excluded since our maps are not bipartite) such that for $z\in\mathbb{C}\setminus[c_-(p,t),c_+(p,t)]$, one has

$$W^{\bullet}_{\mathbf{q}(p,t)}(z) = \frac{1}{\sqrt{(z-c_+(p,t))(z-c_-(p,t))}}. \tag{3.9}$$

This expression, sometimes called the one-cut lemma, appears in several references. See, for example, [14, Proposition 12] or [11, Section 6.1]. It is also established in these articles that the two disk generating functions W and $W^{\bullet}$ have the same domain of analyticity. Comparing our two expressions (3.8) and (3.9), we see that as a consequence

$$c_{\pm}(p,t) = \frac{1}{\sqrt{p\,t^3}\,y_{\pm}(p,t)}, \tag{3.10}$$

where $y_{\pm}(p,t)$ are the respective positive and negative singularities in y of the series $T(p,t,ty)$ defined in Lemma 2.2. Note that, as mentioned in the proof of Lemma 2.2, this directly implies that $y_-(p,t)<-y_+(p,t)$.

4 Percolation probability

Recall that for any triangulation with a site percolation configuration, $\mathfrak{C}$ denotes the percolation cluster of its root vertex. Moreover, recall that $\mathbb{P}^p_\infty$ denotes the law of the UIPT with vertices colored independently black with probability p and white with probability $1-p$, conditioned on the event where the root edge has both end vertices colored black. We want to identify the law of $\mathfrak{C}$ under $\mathbb{P}^p_\infty$ on the event where it is finite. To express that law, we will need some additional notation.

Fix $p\in(0,1)$ and recall the definition of the power series $V(p,U,y)$ of Lemma 2.2. Define $V_c\equiv V_c(p,z)$ as the following power series in z:

$$V_c(p,z) = V\left(1-p, U(t_c^3), \frac{1}{1-z}\right). \tag{4.1}$$

From Lemma 2.2, we know that this series is analytic on $\mathbb{C}\setminus\left[1-\frac{1}{y_+(1-p,t_c)},+\infty\right)$.

Now, let $\Delta(p,z)$ be the power series in z defined as

$$\begin{aligned}\Delta(p,z) &= \hat{\Delta}(p,V_c)\\ &:= 3\,\frac{V_c\,(2\sqrt{3}-3V_c)\left(9V_c^3-9(\sqrt{3}+1)V_c^2+3(3+2\sqrt{3})V_c-2(1-p)\sqrt{3}\right)}{(3(p-1)+2\sqrt{3})\left(\sqrt{3}-3V_c\right)^3\left(9V_c^3-9V_c^2\sqrt{3}+4(1-p)\sqrt{3}\right)}.\end{aligned} \tag{4.2}$$

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We will see in Lemma 6.1 that $z \mapsto \Delta(p,z)$ is also analytic on

$$\mathbb{C}\setminus\left[1-\frac{1}{y_+(1-p,t_c)},+\infty\right)=\mathbb{C}\setminus\left[\frac{1}{y_+(p,t_c)},+\infty\right), \tag{4.3}$$

and in Lemma 6.2 that it is the generating series of positive numbers $(\delta_k(p))_{k\geq 1}$ verifying

$$\frac{[t^{3n}]q_k(p,t)\sqrt{pt^3}^{-k}}{[t^{3n}]\mathcal{Z}(p,t)} \underset{n\to\infty}{\to} \delta_k(p), \tag{4.4}$$

for every $k \geq 1$.

We can now identify the law of the root cluster on the event where it is finite.

Proposition 4.1 *For every $p \in (0,1)$ and every nonatomic rooted finite map $\mathfrak{m}$, we have*

$$\mathbb{P}^p_\infty(\mathfrak{C}=\mathfrak{m})=\left(\prod_{f\in F(\mathfrak{m})} q_{\deg(f)}(p,t_c)\right)\cdot\sum_{f\in F(\mathfrak{m})}\frac{(pt_c^3)^{\deg(f)/2}\delta_{\deg(f)}(p)}{q_{\deg(f)}(p,t_c)}.$$

Proof For $p \in (0,1)$ and $n \geq 1$, we denote by $\mathbb{P}^p_n$ the law of a uniform triangulation of the sphere with $3n$ edges and vertices colored independently black with probability p and white with probability $1-p$, conditioned on the event where both end vertices of the root edge are colored black. From equation (3.4), we can write that for every finite nonatomic rooted map $\mathfrak{m}$, we have

$$\mathbb{P}^p_n(\mathfrak{C}=\mathfrak{m})=\frac{[t^{3n}]\prod_{f\in F(\mathfrak{m})} q_{\deg(f)}(p,t)}{[t^{3n}]\mathcal{Z}(p,t)}. \tag{4.5}$$

Since the event $\{\mathfrak{C}=\mathfrak{m}\}$ is continuous for the local topology, the limit as $n\to\infty$ of the previous display gives access to the annealed law of the percolation cluster of the root in the UIPT.

As can be seen from Proposition 3.1 and Lemma 6.2, the asymptotic behaviors of $[t^{3n}]\mathcal{Z}(p,t)$ and of $[t^{3n}](pt^3)^{-k/2}q_k(p,t)$ for every fixed $k\geq 1$ are all of the form $\mathrm{Cst}\cdot t_c^{-3n}n^{-5/2}$. It is then very classical to establish the limit

$$\begin{aligned}
&\lim_{n\to\infty}\frac{[t^{3n}]\prod_{f\in F(\mathfrak{m})} q_{\deg(f)}(p,t)}{[t^{3n}]\mathcal{Z}(p,t)}\\
&\quad=\lim_{n\to\infty}\frac{[t^{3n}](pt^3)^{\mathrm{e}(\mathfrak{m})}\prod_{f\in F(\mathfrak{m})}(pt^3)^{-\deg(f)/2}q_{\deg(f)}(p,t)}{[t^{3n}]\mathcal{Z}(p,t)},\\
&\quad=\sum_{f\in F(\mathfrak{m})}(pt_c^3)^{\deg(f)/2}\delta_{\deg(f)}(p)\prod_{f'\in F(\mathfrak{m}),f'\neq f} q_{\deg(f)}(p,t_c),
\end{aligned}$$

proving the proposition. ■

Our next result is an integral formula for the probability that the root cluster $\mathfrak{C}$ of the UIPT is infinite.

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Proposition 4.2 *Recall the definition of the quantities $c_+(p,t_c)$ and $c_+(p,t_c)$ given in equations (3.9) and (3.10). For every $p \in (0,1)$, one has*

$$\mathbb{P}^p_\infty(|\mathfrak{C}| < \infty) = \frac{1}{2\pi}\int_{c_-(p,t_c)}^{c_+(p,t_c)} \frac{dz}{z}\Delta\left(p, \sqrt{pt_c^3}\, z\right)\left(z + \frac{c_+(p,t_c) + c_-(p,t_c)}{2}\right) \times \sqrt{(c_+(p,t_c) - z)(z - c_-(p,t_c))}, \tag{4.6}$$

where the function $\Delta(p,z)$ is defined in equation (4.2).

Proof We start with the formula established in Proposition 4.1. Summing over every finite nonatomic map gives, for every $p \in (0,1)$,

$$\mathbb{P}^p_\infty(|\mathfrak{C}| < \infty) = \sum_{\mathfrak{m}\in\mathcal{M}}\sum_{f\in F(\mathfrak{m})}(pt_c^3)^{\deg(f)/2}\delta_{\deg(f)}(p)\prod_{f'\in F(\mathfrak{m})\setminus\{f\}} q_{\deg(f')}(p,t_c).$$

By opening the root edges of the maps $\mathfrak{m}$ involved in the previous display, the sum transforms into a sum over maps in $\mathcal{M}^2$, the set of all rooted planar maps with root face of degree 2:

$$\mathbb{P}^p_\infty(|\mathfrak{C}| < \infty) = \sum_{\mathfrak{m}\in\mathcal{M}^2}\sum_{f\in F(\mathfrak{m})\setminus\{f_r\}}(pt_c^3)^{\deg(f)/2}\delta_{\deg(f)}(p)\prod_{f'\in F(\mathfrak{m})\setminus\{f,f_r\}} q_{\deg(f')}(p,t_c),$$

where f_r denotes the root face of a map. By rooting the nonroot face f involved in the last sum (so that f lies on the right-hand side of this additional root), we can transform the sum into a sum over maps of the cylinder. For l_1, $l_2 \geq 1$, let $\mathcal{M}^{(l_1,l_2)}$ denote the set of all planar maps with two distinct marked rooted faces f_1, f_2 of respective degree l_1, l_2 (rooted so that the corresponding marked faces lie on the right-hand side of their respective root). Taking into account the $\deg(f)$ possible roots for the face f in the sum, we have

$$\mathbb{P}^p_\infty(|\mathfrak{C}| < \infty) = \sum_{k\geq 1}\frac{1}{k}\sum_{\mathfrak{m}\in\mathcal{M}^{(2,k)}}(pt_c^3)^{k/2}\delta_k(p)\prod_{f'\in F(\mathfrak{m})\setminus\{f_1,f_2\}} q_{\deg(f')}(p,t_c). \tag{4.7}$$

Setting for every $k \geq 0$

$$\varphi_k(p) = \frac{1}{k}\sum_{\mathfrak{m}\in\mathcal{M}^{(2,k)}}\prod_{f'\in F(\mathfrak{m})\setminus\{f_1,f_2\}} q_{\deg(f')}(p,t_c),$$

and denoting the generating series of these numbers by

$$\Phi(p,z) = \sum_{k\geq 1}\varphi_k(p) z^k,$$

the sum in (4.7) takes the form of the Hadamard product of $\Delta(p, \sqrt{pt_c^3} z)$ and $\Phi(p,z)$ evaluated at $z = 1$:

$$\mathbb{P}^p_\infty(|\mathfrak{C}| < \infty) = \sum_{k\geq 1}(pt_c^3)^{k/2}\delta_k(p)\cdot\varphi_k(p) = \Delta\left(p, \sqrt{pt_c^3}\, z\right)\odot\Phi(p,z)\big|_{z=1}.$$

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We will compute this Hadamard product with the help of its contour integral representation:

$$\mathbb{P}_{\infty}^{p}\left(|\mathfrak{C}|<\infty\right)=\frac{1}{2i\pi}\oint_{\gamma}\frac{dz}{z}\Delta\left(p,\sqrt{pt_c^3}\,z\right)\,\Phi(p,1/z),$$

where the contour γ must lie in a region enclosing 0 where both functions $z\mapsto\Delta\left(p,\sqrt{pt_c^3}\,z\right)$ and $z\mapsto\Phi(p,1/z)$ are analytic (see, for example, [19, Section VI.10.2]).

To compute this integral, we first have to compute Φ. To this end, we introduce the cylinder generating functions of Boltzmann planar maps. It is defined for $|z_1|$ and $|z_2|$ large enough by

$$W^{\mathrm{cyl}}_{\mathbf{q}(p,t)}(z_1,z_2)=\sum_{l_1,l_2\geq 0}\frac{\sum_{\mathfrak{m}\in\mathcal{M}^{(l_1,l_2)}}\prod_{f'\in F(\mathfrak{m})\setminus\{f_1,f_2\}}q_{\deg(f')}(p,t)}{z_1^{l_1+1}\,z_2^{l_2+1}}.$$

In a similar fashion than the pointed disk generating function $W^{\bullet}$ of Section 2, this series has a universal form involving the two constants $c_{\pm}(p,t)$:

$$\begin{aligned}W^{\mathrm{cyl}}_{\mathbf{q}(p,t)}&(z_1,z_2)\\ &=\frac{1}{2(z_1-z_2)^2}\Bigg(W^{\bullet}_{\mathbf{q}(p,t)}(z_1)\,W^{\bullet}_{\mathbf{q}(p,t)}(z_2)\Big(z_1z_2-\frac{c_+(p,t)+c_-(p,t)}{2}(z_1+z_2)\\ &\qquad\qquad +c_+(p,t)c_-(p,t)\Big)-1\Bigg).\end{aligned}$$

This formula appears in [12, 18] and a multitude of other references. We refer to [1] for a proof and a review of the literature on this matter.

Of particular interest to us will be the generating function where the first root face has degree 2:

$$\begin{aligned}[z_1^{-3}]\,&W^{\mathrm{cyl}}_{\mathbf{q}(p,t)}(z_1,z)\\ &=\sum_{k\geq 0}z^{-k-1}\sum_{\mathfrak{m}\in\mathcal{M}^{(2,k)}}\prod_{f'\in F(\mathfrak{m})\setminus\{f_1,f_2\}}q_{\deg(f')}(p,t),\\ &=-z+W^{\bullet}_{\mathbf{q}(p,t)}(z)\Bigg(z^2-\frac{c_+(p,t)+c_-(p,t)}{2}z+\frac{1}{4}c_+(p,t)c_-(p,t)\\ &\qquad -\frac{1}{8}\left(c_+^2(p,t)+c_-^2(p,t)\right)\Bigg).\end{aligned}$$

The antiderivative of this function has a simple expression giving the identity:

$$\begin{aligned}\Phi(p,1/z)&=\sum_{k\geq 0}\frac{1}{k}z^{-k}\sum_{\mathfrak{m}\in\mathcal{M}^{(2,k)}}\prod_{f'\in F(\mathfrak{m})\setminus\{f_1,f_2\}}q_{\deg(f')}(p,t_c),\\ &=\frac{z^2}{2}-\frac{z+\frac{c_+(p,t_c)+c_-(p,t_c)}{2}}{2}\sqrt{(z-c_+(p,t_c))(z-c_-(p,t_c))}.\end{aligned}$$

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From this expression, we see that the function $z \mapsto \Phi(p,z)$ is analytic on $\mathbb{C}\backslash[c_-(p,t_c), c_+(p,t_c)]$. From Lemma 6.1, we know that $z \mapsto \Delta(p, \sqrt{pt_c^3}z)$ is analytic on $\mathbb{C}\backslash[c_+(p,t_c), +\infty)$. We cannot directly pick an appropriate contour γ to compute our Hadamard product of series evaluated at 1; however, if we take $w \in (0,1)$, we have

$$\Delta\left(p, \sqrt{pt_c^3}\,z\right) \odot \Phi(p,z)|_{z=w} = \frac{1}{2i\pi}\oint_{\gamma(w)} \frac{dz}{z}\Delta\left(p, w\sqrt{pt_c^3}\,z\right)\Phi(p,1/z), \tag{4.8}$$

where the contour $\gamma(w)$ encloses the interval $[c_-(p,t_c), c_+(p,t_c)]$ and crosses the positive real line at some point inside the interval $(c_+(p,t_c), c_+(p,t_c)+\varepsilon)$ for some $\varepsilon > 0$. Note that in the last display, the function $z \mapsto \Delta(p, w\sqrt{pt_c^3}z)/z$ is well defined and continuous at $z = 0$. Using the fact that $\Delta(p, w\sqrt{pt_c^3}\,z)$ is analytic inside $\gamma(w)$ and deforming the contour gives, for every $w \in (0,1)$,

$$\begin{aligned}
&\Delta\left(p, \sqrt{pt_c^3}\,z\right) \odot \Phi(p,z)|_{z=w} \\
&= \frac{1}{2i\pi}\oint_{\gamma(w)} \frac{dz}{z}\Delta\left(p, w\sqrt{pt_c^3}\,z\right) \\
&\qquad \times \left(\frac{z^2}{2} - \frac{z + \frac{c_+(p,t_c)+c_-(p,t_c)}{2}}{2}\sqrt{(z-c_+(p,t_c))(z-c_-(p,t_c))}\right) \\
&= -\frac{1}{4i\pi}\oint_{\gamma(w)} \frac{dz}{z}\Delta\left(p, w\sqrt{pt_c^3}\,z\right)\left(z + \frac{c_+(p,t_c)+c_-(p,t_c)}{2}\right) \\
&\qquad \times \sqrt{(z-c_+(p,t_c))(z-c_-(p,t_c))} \\
&= \frac{1}{2\pi}\int_{c_-(p,t_c)}^{c_+(p,t_c)} \frac{dz}{z}\Delta\left(p, w\sqrt{pt_c^3}\,z\right)\left(z + \frac{c_+(p,t_c)+c_-(p,t_c)}{2}\right) \\
&\qquad \times \sqrt{(c_+(p,t_c)-z)(z-c_-(p,t_c))}.
\end{aligned}$$

Taking the limit as $w \to 1$ finally gives the proposition. ∎

We are now ready to prove our first main result.

Proof of Theorem 1.1 The proof basically consists on computing the integral (4.6), which still requires some work. The change of variables $z = (1-1/\bar{y})(pt_c^3)^{-1/2}$ gives

$$\begin{aligned}
&\mathbb{P}_\infty^\nu(|\mathfrak{C}| < \infty) \\
&= \frac{1}{2\pi\, p\, t_c^3}\int_{\frac{y_-(p,t_c)}{y_-(p,t_c)-1}}^{\frac{y_+(p,t_c)}{y_+(p,t_c)-1}} \frac{d\bar{y}}{\bar{y}(\bar{y}-1)}\hat{\Delta}\left(p, V(1-p, U(t_c^3), \bar{y})\right) \\
&\qquad \times \left(\frac{\bar{y}-1}{\bar{y}} + \frac{1}{2y_-(p,t_c)} + \frac{1}{2y_+(p,t_c)}\right) \\
&\qquad \times \sqrt{\left(\frac{1}{y_+(p,t_c)} - \frac{\bar{y}-1}{\bar{y}}\right)\left(\frac{\bar{y}-1}{\bar{y}} - \frac{1}{y_-(p,t_c)}\right)},
\end{aligned}$$

where $\hat{\Delta}$ is defined in equation (4.2) and $V(1-p, U, \bar{y})$ is defined in Lemma 2.2. To calculate this new integral, we want to do the change of variables $\bar{y} = \hat{y}(1-p, U(t_c^3), 2\sqrt{3}/3 - V)$, where $\hat{y}$ is defined in Lemma 2.2 (as we will see shortly, this

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change of variables instead of simply $\bar{y} = \hat{y}(1-p, U(t_c^3), V)$ simplifies calculations a bit more).

Recall that, from Lemma 2.2 and its proof, the function $V \mapsto \hat{y}(1-p, U(t_c^3), V)$ is an increasing bijection from $[V_-(1-p, U(t_c^3)), V_+(1-p, U(t_c^3))]$ onto $[y_-(1-p, t_c), y_+(1-p, t_c)]$, and is analytic on $(V_-(1-p, U(t_c^3)), V_+(1-p, U(t_c^3)))$. In view of this, for our change of variables, we want $2\sqrt{3}/3 - V$ to be in a subinterval of $[V_-(1-p, U(t_c^3)), V_+(1-p, U(t_c^3))]$. This will also enable us to use the rational parameterization of the series Δ of Lemma 6.2. The details of the calculations that follow are available in the Maple file [26].

We have to solve for V the two equations

$$\hat{y}\left(1-p, U(t_c^3), \frac{2\sqrt{3}}{3} - V\right) = \frac{y_\pm(p, t_c)}{y_\pm(p, t_c) - 1}.$$

There is an interesting symmetry to exploit: for every $p \in (0,1)$ and every $V \in \mathbb{C}$, one has

$$\hat{y}\left(p, U(t_c^3), V\right) = \frac{\hat{y}(1-p, U(t_c^3), \frac{2\sqrt{3}}{3} - V)}{\hat{y}(1-p, U(t_c^3), \frac{2\sqrt{3}}{3} - V) - 1};$$

therefore, we want to solve for V the two equations

$$\hat{y}\left(p, U(t_c^3), V\right) = y_\pm(p, t_c). \tag{4.9}$$

Since $\hat{y}\left(p, U(t_c^3), V\right)$ is a rational fraction in V of degree 2 for its numerator and 3 for its denominator, the solutions of equation (4.9) are the roots of a polynomial of degree 3. By definition, one of the roots of this polynomial is $V_\pm(p, U(t_c^3))$, and it will even be a double root except for $V_-(p, U(t_c^3))$ when it is a negative pole of $\hat{y}$. Fortunately, we can compute explicitly these values. Indeed, the stationary points of $V \mapsto \hat{y}(p, U(t_c^3), V)$ are the roots of the polynomial

$$\left(9V^3 - 9V^2\sqrt{3} + 4p\sqrt{3}\right)\left(V - \sqrt{3}/3\right).$$

The roots of the polynomial of degree 3 are given by

$$V_m(p) = -\frac{\sqrt{3}}{3}\frac{\sqrt{p}}{\cos\left(\frac{\arccos(\sqrt{p})}{3}\right)} < 0,$$

$$V_l(p) = \frac{\sqrt{3}}{3}\frac{\sqrt{p}}{\cos\left(\frac{\arccos(\sqrt{p})}{3} - \frac{\pi}{3}\right)} \in [0, 2\sqrt{3}/3],$$

$$V_r(p) = \frac{2\sqrt{3}}{3} - V_m(p) \geq 2\sqrt{3}/3.$$

From there, we see that $V_-(p, U(t_c^3)) = V_m(p)$ when $\hat{y}$ has no negative pole, which is the case when $p > \frac{1}{2} - \frac{5\sqrt{3}}{18} \sim 0.018$. Since we are interested in $p \geq 1/2$, the case when $\hat{y}$ has negative poles will not bother us. The positive singularity is given by

$$V_+(p, U(t_c^3)) = \begin{cases} \frac{\sqrt{3}}{3}, & \text{when } p \geq 1/2, \\ V_l(p), & \text{when } p \leq 1/2. \end{cases}$$

We can compute the third root of equation (4.9) from the constant term of the polynomial, which is always $-\frac{2\sqrt{3}p}{9}$. When $p > \frac{1}{2} - \frac{5\sqrt{3}}{18}$, this root is given by

$$V_\pm^i(p) = \frac{2\sqrt{3}p}{9V_\pm(p, U(t_c^3))^2}.$$

Since $V_\pm(p, U(t_c^3)) \leq \sqrt{3}/3$, we can see that $2\sqrt{3}/3 - V_\pm(p, U(t_c^3))$ is always larger than $\sqrt{3}/3$, and therefore $V_+(1-p, U(t_c^3))$. From the explicit expression of $V_\pm(p, U(t_c^3))$, we can also check that $2\sqrt{3}/3 - V_\pm^i(p)$ is always between 0 and $V_+(1-p, U(t_c^3))$. The correct bounds for our change of variables are then $V_\pm^i(p)$ since they are the only solutions inside the interval $[V_-(1-p, U(t_c^3)), V_+(1-p, U(t_c^3))]$. Note that we have $0 < V_+^i(p) < V_-^i(p)$. This discussion also provides the following nice factorizations:

$$\begin{aligned} \frac{\hat{y}(1-p, U(t_c^3), \frac{2\sqrt{3}}{2} - V)) - 1}{\hat{y}(1-p, U(t_c^3), \frac{2\sqrt{3}}{2} - V))} - \frac{1}{y_\pm(p, t_c)} &= \frac{1}{\hat{y}(p, U(t_c^3), V))} - \frac{1}{y_\pm(p, t_c)}, \\ &= \frac{\left(V - V_\pm^i(p)\right)\left(V - V_\pm(p, U(t_c^3))\right)^2}{2\,V\left(V - \frac{2\sqrt{3}}{3}\right)}. \end{aligned}$$

With the explicit expression for $\hat{\Delta}$ in terms of V given in Lemma 6.2, it is easy to check

$$\begin{aligned} &\hat{\Delta}(p, V)\, \frac{\partial_V \hat{y}(1-p, U(t_c^3), V)}{\hat{y}(1-p, U(t_c^3), V)\,(\hat{y}(1-p, U(t_c^3), V) - 1)} \\ &\qquad = \frac{1}{3\,(2\sqrt{3} - 3(1-p))\left(V - \frac{\sqrt{3}}{3}\right)^2}. \end{aligned}$$

Using the change of variables $\bar{y} = \hat{y}(1-p, U(t_c^3), 2\sqrt{3}/3 - V)$, we finally get

$$\begin{aligned} &\mathbb{P}_\infty^\nu(|\mathfrak{C}| < \infty) \\ &= \frac{1}{12\,p\,t_c^3\,(2\sqrt{3} - 3(1-p))\,\pi} \\ &\qquad \int_{V_+^i(p)}^{V_-^i(p)} dV \sqrt{(V_-^i(p) - V)(V - V_+^i(p))} \\ &\qquad\qquad \times \frac{(V - V_+(p, U(t_c^3)))(V - V_-(p, U(t_c^3)))}{V\left(\frac{2\sqrt{3}}{3} - V\right)\left(V - \frac{\sqrt{3}}{3}\right)^2} \\ &\qquad\qquad \times \left(\frac{1}{\hat{y}(p, U(t_c^3), V)} + \frac{1}{2}\left(\frac{1}{y_+(p, t_c)} + \frac{1}{y_-(p, t_c)}\right)\right). \end{aligned}$$

Embarrassingly, we were not able to simplify directly this integral (it is equal to 1 when $p < 1/2$ since we know that the critical percolation threshold is 1/2 !). Fortunately, when $p > 1/2$, Maple is able to give an expression in terms of $V_\pm(p, U(t_c^3))$.

Using basic trigonometric identities, we can further simplify this expression into the one given in the theorem after injecting the expressions for $V_{\pm}(p, U(t_c^3))$. ■

5 Cluster volume and perimeter

5.1 Admissibility equations and volume-modified weight sequence

We review in this section additional background on Boltzmann planar maps that we will use in the proof of Theorem 1.2. We refer the reader to the references [6, 14, 25] for details.

For $p \in (0,1)$ and $t \in (0, t_c]$, consider the two following bivariate power series in (z_1, z_2):

$$f^{\bullet}(p,t;z_1,z_2) = \sum_{k,k'\geq 0} z_1^k z_2^{k'} \binom{2k+k'+1}{k+1,k,k'} q_{2+2k+k'}(p,t),$$
$$f^{\diamond}(p,t;z_1,z_2) = \sum_{k,k'\geq 0} z_1^k z_2^{k'} \binom{2k+k'}{k,k,k'} q_{1+2k+k'}(p,t).$$

These two functions are linked with Boltzmann maps with weight sequence $(q_k(p,t_c))_{k\geq 1}$ by the Bouttier–Di Francesco–Guitter bijection [13] and will be instrumental in the remaining of this work. In our case, we can compute alternative expressions for these functions that will be more amenable to analysis (see Lemma 6.4).

Recall the parameters $c_+(p,t)$ and $c_-(p,t)$ defined in equation (3.10). Since the weight sequence $\mathbf{q}(p,t)$ is admissible, the two functions defined above are well defined at least in the domain $|z_1| \leq z^+(p,t)$ and $|z_2| \leq z^{\diamond}(p,t)$ where $z^+(p,t)$ and $z^{\diamond}(p,t)$ are positive real numbers defined by

$$c_{\pm}(p,t) = z^{\diamond}(p,t) \pm 2\sqrt{z^+(p,t)}. \tag{5.1}$$

Furthermore, from Proposition 4.2 and Lemma 4.4 of [6], $(z^+(p,t), z^{\diamond}(p,t))$ is the minimal solution of the system of equations

$$\begin{cases} f^{\bullet}(p,t;z^+(p,t),z^{\diamond}(p,t)) = 1 - \frac{1}{z^+(p,t))}, \\ f^{\diamond}(p,t;z^+(p,t),z^{\diamond}(p,t)) = z^{\diamond}(p,t). \end{cases} \tag{5.2}$$

In addition, when $p = 1/2$ and $t = t_c$, the weight sequence is critical (see Theorem 1.1 of [6]), and $(z^+(1/2,t_c), z^{\diamond}(1/2,t_c))$ is the unique solution of the system of equations (5.2) such that

$$\left(\partial_{z_2} + \sqrt{z_1}\partial_{z_1}\right) f^{\diamond}(p,t;z^+(p,t),z^{\diamond}(p,t)) = 1. \tag{5.3}$$

For $g \in (0,1]$, let $(z^+(p,t;g), z^{\diamond}(p,t;g))$ be the unique solution in $(0, z^+(p,t)] \times (0, z^{\diamond}(p,t)]$ of the system of equations

$$\begin{cases} f^{\bullet}(p,t;z^+(p,t;g),z^{\diamond}(p,t;g)) = 1 - \frac{g}{z^+(p,t;g))}, \\ f^{\diamond}(p,t;z^+(p,t;g),z^{\diamond}(p,t;g)) = z^{\diamond}(p,t;g). \end{cases} \tag{5.4}$$

Define

$$c_\pm(p,t;g) = \frac{1}{\sqrt{g}}\left(z^\diamond(p,t;g) \pm 2\sqrt{z^+(p,t;g)}\right).$$

From, e.g., [6, equations (4.3) and (4.4)], the pointed disk generating function of Boltzmann maps with modified weight sequence $\mathbf{q}(p,t;g) := \left(g^{(k-2)/2}\, q_k(p,t)\right)_{k\geq 1}$ defined as in (3.7) is given by

(5.5)
$$W^\bullet_{\mathbf{q}(p,t;g)}(z) = \frac{1}{\sqrt{(z - c_+(p,t;g))(z - c_-(p,t;g))}}.$$

5.2 Proof of Theorem 1.2

We start with the perimeter exponent as it is the easiest of the two. From equations (3.2) and (3.7), we can write

$$\mathbb{P}^p_n(|V(\partial\mathfrak{C})| = k) = \frac{[t^{3n}]\left(q_k(p,t)\cdot[z^{-(k+1)}]W_{\mathbf{q}(p,t)}(z)\right)}{[t^{3n}]\mathcal{Z}(p,t)}$$
$$= \frac{1}{p}\,\frac{[t^{3n}]\left(\sqrt{pt^3}^{\,-k} q_k(p,t)\cdot[y^k]T(p,t,ty)\right)}{[t^{3n}]\mathcal{Z}(p,t)}.$$

Applying Lemmas 6.2 and 6.3 directly gives the limit of the previous display as $n\to\infty$. Indeed, the series $\sqrt{pt^3}^{\,-k} q_k(p,t)$ and $[y^k]T(p,t,ty)$ have the same unique dominant singularity in t^3 and it is easy to get an asymptotic expansion at t_c^3 of the product at this singularity from the two separate expansions. By continuity for the local topology, this limit is the probability that $|V(\partial\mathfrak{C})| = k$ in the UIPT:

$$\mathbb{P}^p_\infty(|V(\partial\mathfrak{C})| = k) = \frac{1}{p}\left(\delta_k(p)\, t_c^k T_k(p,t_c) + \sqrt{pt_c^3}^{\,-k} q_k(p,t_c)\,\theta_k(p)\right).$$

Using the asymptotics (6.7), (6.9), (6.11), and (6.13) derived in Section 6.3 gives that for $p = 1/2$,

$$\mathbb{P}^{1/2}_\infty(|V(\partial\mathfrak{C})| = k) \underset{k\to\infty}{\sim} \kappa'\, k^{-4/3}$$

with

(5.6)
$$\kappa' = -8\,\frac{1}{\Gamma(4/3)}\left(\frac{8\,3^{\frac{5}{6}}}{351} + \frac{2\,3^{\frac{1}{3}}}{117}\right)\frac{3^{5/6}}{2\Gamma(-2/3)} \simeq 0.454,$$

proving the second statement of Theorem 1.2.

We now turn on the first statement of the theorem on the volume of the root cluster. Although we are only interested in the case $p = 1/2$, we start the proof with a generic $p\in(0,1)$ as it will be easier to follow. From Proposition 4.1, we have, for every $g\leq 1$,

$$\mathbb{E}^p_\infty\left[g^{|V(\mathfrak{C})|}\right] = \sum_{\mathfrak{m}\in\mathcal{M}} g^{|V(\mathfrak{m})|} \sum_{f\in F(\mathfrak{m})} (pt_c^3)^{\deg(f)/2}\delta_{\deg(f)}(p) \prod_{f'\in F(\mathfrak{m})\setminus\{f\}} q_{\deg(f')}(p,t_c).$$

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Applying Euler's formula then gives

$$\mathbb{E}_\infty^p\left[g^{|V(\mathfrak{C})|}\right] = g \sum_{\mathfrak{m}\in\mathcal{M}} \sum_{f\in F(\mathfrak{m})} (g\,pt_c^3)^{\deg(f)/2}\delta_{\deg(f)}(p) \times \prod_{f'\in F(\mathfrak{m})\setminus\{f\}} g^{(\deg(f')-2)/2}\, q_{\deg(f')}(p,t_c).$$

Using the exact same line of reasoning as in the proof of Theorem 1.1, we can compute this sum as the Hadamard product evaluated at $z=1$ of the functions $z\mapsto \Delta(p,\sqrt{gpt_c^3})$ and the function $z\mapsto\Phi(p,z)$ where $c_+(p,t_c)$ and $c_-(p,t_c)$ are replaced by the two constants $c_+(p,t_c;g)$ and $c_-(p,t_c;g)$ associated with the pointed disk generating function $((z-c_+(p,t_c;g))(z-c_-(p,t_c;g)-z))^{-1/2}$ of Boltzmann maps with weight sequence $\mathbf{q}(p,t_c;g) = \left(g^{(k-2)/2}\,q_k(p,t_c)\right)_{k\geq 1}$ introduced in Section 5.1. Indeed, the first function should be obvious, and the second comes from the antiderivative of cylinder generating functions associated with the weight sequence $\mathbf{q}(p,t_c;g)$ instead of the weight sequence $\mathbf{q}(p,t_c;1)$, which has the same universal form. Of course, it remains to calculate these two constants, which are less explicit than their counterparts $c_+(p,t_c)$ and $c_-(p,t_c)$. Nevertheless, after mimicking the part of the proof of Theorem 1.1 leading to (4.6), we arrive at the formula

$$\mathbb{E}_\infty^p\left[g^{|V(\mathfrak{C})|}\right] = \frac{g}{2\pi}\int_{c_-(p,t_c;g)}^{c_+(p,t_c;g)} \frac{dz}{z}\Delta\left(p,\sqrt{g\,pt_c^3}\,z\right)\left(z+\frac{c_+(p,t_c;g)+c_-(p,t_c;g)}{2}\right) \times\sqrt{(c_+(p,t_c;g)-z)(z-c_-(p,t_c;g))}. \tag{5.7}$$

We want to study the asymptotic behavior of the integral (5.7) when $g\to 1^-$. To do so, we must first study the dependency in g of the two constants $c_\pm(p,t_c;g)$. We do not have a simple formula for the unpointed or pointed disk generating function of the Boltzmann maps as was the previously the case. However, the two constants can be studied via the solution of the system of equation (5.4). Let us denote $z^+(g) = z^+(1/2,t_c;g)$ and $z^\diamond(g) = z^\diamond(1/2,t_c;g)$ the solution of this system. We can calculate an expansion as $g\to 1^-$ of these two quantities with Lemma 6.5. Indeed, the development of $f^\diamond$ gives

$$\begin{aligned} z^\diamond - z^\diamond(g) &= \partial_{z_1} f^\diamond(z^+,z^\diamond)\ (z^+-z^+(g)) + \left(1-\sqrt{z^+}\partial_{z_1}f^\diamond(z^+,z^\diamond)\right)\ (z^\diamond - z^\diamond(g)) \\ &\quad - \kappa^\diamond\ \left((z^+-z^+(g))+\sqrt{z^+}(z^\diamond-z^\diamond(g))\right)^{7/6} \\ &\quad + o\left(\left((z^+-z^+(g))+\sqrt{z^+}(z^\diamond-z^\diamond(g))\right)^{7/6}\right). \end{aligned}$$

This yields in turn

$$\begin{aligned} z^+ - z^+(g) &= \sqrt{z^+}\ (z^\diamond - z^\diamond(g)) + \frac{\kappa^\diamond}{\partial_{z_1} f^\diamond(z^+,z^\diamond)}\ \left((z^+-z^+(g))+\sqrt{z^+}(z^\diamond-z^\diamond(g))\right)^{7/6} \\ &\quad + o\left(\left((z^+-z^+(g))+\sqrt{z^+}(z^\diamond-z^\diamond(g))\right)^{7/6}\right). \end{aligned}$$

Therefore, we have

$$z^\diamond - z^\diamond(g) = \frac{1}{\sqrt{z^+}}\,(z^+ - z^+(g)) \tag{5.8}$$
$$- \frac{\kappa^\diamond}{\sqrt{z^+}\,\partial_{z_1} f^\diamond(z^+, z^\diamond)}\,(2(z^+ - z^+(g)))^{7/6} + o\left((z^+ - z^+(g))^{7/6}\right).$$

Now, the development of $f^\bullet$ gives

$$\frac{1}{z^+} - \frac{g}{z^+(g)} = -\frac{1}{(z^+)^2}(z^+ - z^+(g))$$
$$+ \left(\frac{\kappa^\diamond}{\sqrt{z^+}} + \kappa^\bullet\right)(2(z^+ - z^+(g)))^{7/6} + o\left((2(z^+ - z^+(g)))^{7/6}\right).$$

Expanding the left-hand side of this identity then gives

$$\frac{1-g}{z^+} + \frac{1-g}{(z^+)^2}(z^+ - z^+(g))$$
$$= \left(\frac{\kappa^\diamond}{\sqrt{z^+}} + \kappa^\bullet\right)(2(z^+ - z^+(g)))^{7/6} + o\left((2(z^+ - z^+(g)))^{7/6}\right),$$

which in turns gives

$$z^+(g) = z^+ - \frac{1}{2}\left(\kappa^\diamond\sqrt{z^+} + \kappa^\bullet z^+\right)^{-6/7}(1-g)^{6/7} + o\left((1-g)^{6/7}\right).$$

Plugging this expression in equation (5.8) gives

$$z^\diamond(g) = z^\diamond - \frac{1}{2\sqrt{z^+}}\left(\kappa^\diamond\sqrt{z^+} + \kappa^\bullet z^+\right)^{-6/7}(1-g)^{6/7} + o\left((1-g)^{6/7}\right).$$

We finally get an expansion for $c_\pm(g)$:

$$c_+(g) = z^\diamond(g) + 2\sqrt{z^+(g)}$$
$$= c_+(1) - \frac{1}{\sqrt{z^+}}\left(\kappa^\diamond\sqrt{z^+} + \kappa^\bullet z^+\right)^{-6/7}(1-g)^{6/7} + o\left((1-g)^{6/7}\right),$$
$$c_-(g) = z^\diamond(g) - 2\sqrt{z^+(g)}$$
$$= c_-(1) + o\left((1-g)^{6/7}\right).$$

The change of variable $z = \phi(g,\xi) := c_-(g) + (c_+(g) - c_-(g))\,\xi$ in equation (5.7) gives

$$\mathbb{E}_\infty^p\left[g^{|V(\mathfrak{C})|}\right]$$
$$= \frac{g(c_+(g) - c_-(g))^2}{2\pi}\int_0^1 d\xi\sqrt{\xi(1-\xi)}\,\frac{\Delta\left(1/2, \sqrt{g\,t_c^3/2}\,\phi(g,\xi)\right)}{\phi(g,\xi)}$$
$$\times\left(\phi(g,\xi) + \frac{c_+(g) + c_-(g)}{2}\right),$$

$$= \frac{(c_+(1) - c_-(1))^2}{2\pi} \int_0^1 d\xi \sqrt{\xi(1-\xi)} \frac{\Delta\left(1/2, \sqrt{g\, t_c^3/2}\, \phi(g,\xi)\right)}{\sqrt{g}\, \phi(g,\xi)} \times \left(\phi(1,\xi) + \frac{c_+(1) + c_-(1)}{2}\right) + \mathcal{O}\left((1-g)^{6/7}\right),$$

where we used the developments of $c_\pm(g)$ to obtain the second equality. In a similar fashion than in the proof of Lemma 6.5, the dominant singular term of the integral comes for the singularity at $g = 1$ and $\xi = 1$ of the term $\frac{\Delta\left(1/2, \sqrt{g\, t_c^3/2}\, \phi(g,\xi)\right)}{\sqrt{g}\, \phi(g,\xi)}$. The asymptotic expansion (6.10) established in Section 6.3 gives

$$\frac{\Delta(1/2, z)}{z} \underset{z \to 1/2}{\sim} \left(\frac{16\, 3^{\frac{5}{6}}}{351} + \frac{4\, 3^{\frac{1}{3}}}{117}\right) (1 - 2z)^{-4/3}.$$

Applying the same techniques as in the proof of Lemma 6.5, we see that the main singular term of the expansion of $\mathbb{E}_\infty^p\left[g^{|V(\mathfrak{C})|}\right]$ comes from the integral

$$\frac{\tilde{\kappa}}{\pi} \int_0^1 d\xi \sqrt{\xi(1-\xi)} \left(1 - 2\sqrt{g\, t_c^3/2}\, \phi(g,\xi)\right)^{-4/3} = \frac{\tilde{\kappa}}{8\left(1 - 2\sqrt{g\, t_c^3/2}\, c_-(g)\right)^{4/3}}\, {}_2F_1\left(\frac{4}{3}, \frac{3}{2}; 3; \frac{2\sqrt{g\, t_c^3/2}\,(c_+(g) - c_-(g))}{1 - 2\sqrt{g\, t_c^3/2}\, c_-(g)}\right),$$

with

$$\tilde{\kappa} = \frac{(c_+(1) - c_-(1))^2 \left(\phi(1,1) + \frac{c_+(1) + c_-(1)}{2}\right) \sqrt{t_c^3/2}}{2} \left(\frac{4\, 3^{\frac{5}{6}}}{351} + \frac{3^{\frac{1}{3}}}{117}\right).$$

Using the singular expansion of the hypergeometric function at 1 and the developments of $c_\pm(g)$, we finally get

$$\mathbb{E}_\infty^p\left[g^{|V(\mathfrak{C})|}\right] = 1 - \frac{\tilde{\kappa}}{8\left(1 - 2\sqrt{t_c^3/2}\, c_-(1)\right)^{4/3}} \frac{36\sqrt{3}\,\Gamma\left(\frac{5}{6}\right)\Gamma\left(\frac{2}{3}\right)}{\pi^{\frac{3}{2}}} \left(\frac{2\sqrt{t_c^3/2}}{1 - 2\sqrt{t_c^3/2}\, c_-(1)} \frac{1}{\sqrt{z^+}} \left(\kappa^\diamond \sqrt{z^+} + \kappa^\bullet z^+\right)^{-6/7} (1-g)^{6/7}\right)^{1/6} + o\,(1-g)^{1/7}.$$

This expansion is unfortunately not enough to extract the asymptotic behavior of the probabilities $\mathbb{P}_\infty^{1/2}\left(|V(\mathfrak{C})| = n\right)$ as $n \to \infty$. Indeed, the generating series of these probabilities is $\mathbb{E}_\infty^p\left[g^{|V(\mathfrak{C})|}\right]$, but this function could have singularities of modulus 1 other than 1 contributing to the asymptotic. However, we do not have this problem for tail probabilities. For every n, denote $p_n = \mathbb{P}_\infty^{1/2}\left(|V(\mathfrak{C})| \geq n\right)$. A simple computation gives

$$\sum_{n \geq 0} p_n g^n = \frac{1 - g\,\mathbb{E}_\infty^p\left[g^{|V(\mathfrak{C})|}\right]}{1 - g} \underset{g \to 1^-}{\sim} \kappa_1 (1-g)^{-6/7},$$

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with

$$\kappa_1 = \frac{\tilde{\kappa}}{8\left(1-2\sqrt{t_c^3/2}c_-(1)\right)^{4/3}} \frac{36\sqrt{3}\,\Gamma\left(\frac{5}{6}\right)\Gamma\left(\frac{2}{3}\right)}{\pi^{\frac{3}{2}}}$$
$$\times\left(\frac{2\sqrt{t_c^3/2}}{1-2\sqrt{t_c^3/2}c_-(1)}\frac{1}{\sqrt{z^+}}\right)^{1/6}\left(\kappa^{\diamond}\sqrt{z^+}+\kappa^{\bullet}z^+\right)^{-8/7}.$$

From there, a classical Tauberian theorem (see, e.g., Theorem VI.13 of [19] and the following discussion) establishes the asymptotic behavior of p_n given in the theorem with

$$\kappa = \frac{\kappa_1}{\Gamma(8/7)} = \frac{63\left(3^{\frac{1}{3}}+\frac{4\,3^{\frac{5}{6}}}{3}\right)3^{\frac{17}{21}}7^{\frac{1}{7}}137^{\frac{6}{7}}\Gamma\left(\frac{2}{3}\right)^{\frac{18}{7}}2^{\frac{3}{14}}5^{\frac{13}{14}}}{56,992\pi^{\frac{12}{7}}\Gamma\left(\frac{1}{7}\right)} \simeq 0.278. \tag{5.9}$$

6 Technical lemmas

6.1 Dependency in t of the weights

Recall that V_c be the power series in z defined in (4.1) by $V_c(p,z) = V(1-p, U(t_c^3), 1/(1-z))$, where V is defined in Lemma 2.2. Furthermore, recall that $\Delta(p,z)$ is the power series in z defined in (4.2) as

$$\begin{aligned}
&\Delta(p,z)\\
&= \hat{\Delta}(p, V_c(p,z))\\
&:= 3\,\frac{V_c\,(2\sqrt{3}-3V_c)\,\left(9V_c^3-9(\sqrt{3}+1)V_c^2+3(3+2\sqrt{3})V_c-2(1-p)\sqrt{3}\right)}{(3(p-1)+2\sqrt{3})\,\left(\sqrt{3}-3V_c\right)^3\,\left(9V_c^3-9V_c^2\sqrt{3}+4(1-p)\sqrt{3}\right)}\\
&= \sum_{k\geq 0}\delta_k(p)\,z^k.
\end{aligned}$$

Lemma 6.1 *Fix $p\in(0,1)$. The series $\Delta(p,z)$ is analytic in the domain*

$$\mathbb{C}\backslash\left[1-\frac{1}{y_+(1-p,t_c)},+\infty\right) = \mathbb{C}\backslash\left[\frac{1}{y_+(p,t_c)},+\infty\right).$$

Proof By definition, the series $z\mapsto V_c(p,z)$ is analytic on $\mathbb{C}\backslash\left[1-\frac{1}{y_+(1-p,t_c)},+\infty\right)$. In addition, if z is in this domain, the denominator of $\hat{\Delta}$ does not vanish (see the discussion in the proof of Theorem 1.1 after equation (4.9)). The only statement left to prove is the equality

$$1-\frac{1}{y_+(1-p,t_c)} = \frac{1}{y_+(p,t_c)}.$$

Since we have an explicit expression for $y_+(p,t_c)$ (again, see the discussion in the proof of Theorem 1.1 after equation (4.9)), it is a straightforward verification. ∎

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Lemma 6.2 *Fix $p \in (0,1)$. For every $k \geq 1$, one has*

$$\frac{[t^{3n}]q_k(p,t)\sqrt{pt^3}^{-k}}{[t^{3n}]\mathcal{Z}(p,t)} \underset{n\to\infty}{\to} \delta_k(p).$$

Proof In the whole proof, $p \in (0,1)$ is fixed. All calculations are available in the Maple companion file [26]. We start by proving that, for every $k \geq 1$, the series

$$\tilde{q}_k(p,t) = \sqrt{pt^3}^{-k} \cdot \left(q_k(p,t) - (pt)^{3/2}\mathbf{1}_{k=3}\right)$$

seen as a series in t^3 is algebraic and has a unique dominant singularity at t_c^3. In view of (3.6), the generating series of these modified weights is given by

$$\tilde{F}(p,t,z) = \sum_{k\geq 1} \tilde{q}_k(p,t)\, z^{k-1} = \frac{1}{p}\frac{1}{1-z} T\left(1-p, t, \frac{t}{1-z}\right). \tag{6.1}$$

Injecting this into the algebraic equation verified by T, we get an algebraic equation verified by $\tilde{F}$:

$$\begin{aligned}&\left(p\,\tilde{F}(p,t,z) - T(1-p,t,t)\right) \mathrm{Pol}_1(p\,\tilde{F}(p,t,z), p, t^3, T(1-p,t,t), tT_1(p,t))\\ &\quad = z \cdot \mathrm{Pol}_2(p\,\tilde{F}(p,t,z), p, z),\end{aligned}$$

where Pol_1 and Pol_2 are explicit polynomials. The form of this equation has the following consequences. First, using Lemma 2.2, the series $\tilde{F}(p,t,z)$ is algebraic. Second, it is the unique solution of this equation with constant term in z equal to $T(1-p,t,t)/p$, and we can compute its coefficients in z inductively. These coefficients are the modified weights $\tilde{q}_k(p,t)$ and their expressions are then rational fractions in p, t^3, $tT_1(p,t)$, and $T(1-p,t,t)$, whose denominator (up to a factor p) are the kth power of

$$\begin{aligned}&\mathrm{Pol}_1(\tilde{F}(p,t,0), p, t^3, T(1-p,t,t), tT_1(p,t))\\ &\qquad = \mathrm{Pol}_1(T(1-p,t,t), p, t^3, T(1-p,t,t), tT_1(p,t)),\\ &\qquad = 3t^3(p-1)\,T^2(1-p,t,t) + p(p-1) + tT_1(t).\end{aligned}$$

A quick glance at equation (2.1) with $y = 1$ shows that the last display is the derivative of the algebraic equation verified by $T(1-p,t,t)$ with respect to T. Therefore, this quantity can only be 0 at singularities of $T(1-p,t,t)$, which leaves only t_c^3 according to Lemma 2.2. As a consequence, we just proved that the series $\tilde{q}(p,t)$ are all algebraic series in t^3 and all have a unique dominant singularity at t_c^3.

Now that we know that $\tilde{q}_k(p,t)$ has a unique dominant singularity at t_c^3, it will follow from the general form of Puiseux expansions of algebraic series near their singularities (see [19, Theorem VII.7, p. 498]) that $[t^{3n}]\tilde{q}_k(p,t)$ has the same asymptotic behavior as $[t^{3n}]\mathcal{Z}(p,t)$ if we can prove that there exist two positive constants c, c', depending on k and p such that, for n large enough,

$$c \cdot [t^{3n}]\mathcal{Z}(p,t) \leq [t^{3n}]\tilde{q}_k(p,t) \leq c' \cdot [t^{3n}]\mathcal{Z}(p,t).$$

The upper bound is easily obtained by putting a triangulation of the 1-gon with a white boundary vertex inside a cycle of k black vertices and summing over every possible necklace between the two. The lower bound is obtained similarly starting from cycle of

k black vertices by putting a single white vertex inside the cycle and an edge joining this additional vertex to every boundary vertex, and putting an arbitrary triangulation with white boundary together with a matching necklace on the outside of the cycle. To sum up, we have proved that for every k, $\tilde{q}_k(p,t)$ seen as a series in t^3 has a unique dominant singularity at t_c^3 and that its asymptotic expansion at t_c^3 is of the form

$$\tilde{q}_k(p,t) = \tilde{q}_k(p,t_c) - \tilde{a}_k(p)\left(1-\frac{t^3}{t_c^3}\right) + \tilde{b}_k(p)\left(1-\frac{t^3}{t_c^3}\right)^{3/2} + o\left(1-\frac{t^3}{t_c^3}\right)^{3/2}. \tag{6.2}$$

This finishes the proof of the first statement (4.4) of the lemma, and we now have to identify the generating series of the numbers $\delta_k(p) = \tilde{b}_k(p)/\kappa(p)$, where $\kappa(p)$ is the coefficient of the term $\left(1-\frac{t^3}{t_c^3}\right)^{3/2}$ in the asymptotic expansion of $\mathcal{Z}(p,t)$ calculated in Proposition 3.1.

Using the rational parameterization of Lemma 2.2, we can find a rational parameterization for $\tilde{F}$. Indeed, if $V = V(1-p,U,y)$ is the power series in $\mathbb{Q}[p,U][[y]] \subset \mathbb{Q}[p][[t^3,y]]$ defined in Lemma 2.2, we have

$$\tilde{F}(p,t,z) = \frac{1}{p}\hat{y}\left(1-p, U(t^3), V\left(1-p,U(t^3),1/(1-z)\right)\right) \times \hat{T}\left(1-p,U(t^3),V\left(1-p,U(t^3),1/(1-z)\right)\right),$$

where $\hat{y}$ and $\hat{T}$ are rational fractions defined in Lemma 2.2. For z fixed such that $|1/(1-z)| < y_+(1-p,t_c)$ (which includes a neighborhood of 0 for z since $y_+(1-p,t_c) > 1$), the series $\tilde{F}(p,t,z)$ seen as a series in t^3 has nonnegative coefficients and has radius of convergence t_c^3. In addition, this implies that $(t^3,z) \mapsto \tilde{F}(p,t,z)$ is analytic in the larger domain $\mathcal{D}(0,t_c^3) \times \mathcal{D}(0, y_+(1-p,t_c)/(y_+(1-p,t_c)-1))$. We will produce an asymptotic expansion of $\tilde{F}(p,t,z)$ near t_c^3 using our rational parameterization. To do so, we start by computing the asymptotic expansion of $V\left(p,U(t^3),1/(1-z)\right)$ near t_c^3, with z fixed.

First, writing

$$\frac{1}{1-z} = \hat{y}\left(1-p,U(t^3),V\left(1-p,U(t^3),1/(1-z)\right)\right) = \hat{y}\left(1-p,U(t_c^3),V\left(1-p,U(t_c^3),1/(1-z)\right)\right),$$

we get an algebraic equation between $V_c^z = V\left(1-p,U(t_c^3),1/(1-z)\right)$, $V\left(1-p,U(t^3),1/(1-z)\right)$, and $U(t^3)$. Plugging the asymptotic expansion (2.2) of $U(t^3)$ in this equation, we obtain an asymptotic expansion for $V\left(p,U(t^3),1/(1-z)\right)$ of the form:

$$V\left(1-p,U(t^3),1/(1-z)\right) = V_c^z - a_1(V_c^z)\left(1-\frac{t^3}{t_c^3}\right)^{1/2} + a_2(V_c^z)\left(1-\frac{t^3}{t_c^3}\right) + a_3(V_c^z)\left(1-\frac{t^3}{t_c^3}\right)^{3/2} + o\left(1-\frac{t^3}{t_c^3}\right)^{3/2},$$

where the a_i's are explicit rational fractions whose expressions are given in the Maple companion file [26]. Injecting in turn the asymptotic expansions in t^3 of U and V in $\hat{T}$ and $\hat{y}$, we find an asymptotic expansion for $\tilde{F}$ of the form:

$$\tilde{F}(p,t,z) = \tilde{F}(p,t_c,z) + \tilde{A}(p,V_c^z)\left(1-\frac{t^3}{t_c^3}\right) + \tilde{B}(p,V_c^z)\left(1-\frac{t^3}{t_c^3}\right) + o\left(1-\frac{t^3}{t_c^3}\right)^{3/2},$$

where $\tilde{A}$ and $\tilde{B}$ are explicit rational functions, and are analytic on the disk $\mathcal{D}(0, y_+(1-p,t_c)/(y_+(1-p,t_c)-1)$ (this is obvious from their expressions: $z \mapsto V_c^z$ is analytic in this region, and the poles of $\tilde{A}$ and $\tilde{B}$ fall outside it; see the Maple companion file [26]). Note that the error term in the previous expansion is *a priori* not uniform in z. To ensure that $\tilde{A}$ and $\tilde{B}$ are the respective generating series of the numbers $\tilde{a}_k$ and $\tilde{b}_k$, we see that, as power series in (z,t^3), we have

$$\tilde{A}(p,V_c^z) = \lim_{t\to t_c}\left(\tilde{F}(p,t,z)-\tilde{F}(p,t_c,z)\right)\cdot(1-t^3/t_c^3)^{-1},$$

and

$$\tilde{B}(p,V_c^z) = \lim_{t\to t_c}\left(\tilde{F}(p,t,z)-\tilde{F}(p,t_c,z)-\tilde{A}(p,V_c^z)\left(1-\frac{t^3}{t_c^3}\right)\right)\cdot(1-t^3/t_c^3)^{-3/2}.$$

Combined with the analyticity properties of these series, this ensures that $\tilde{A}(p,V_c^z)$ and $\tilde{B}(p,V_c^z)$ are indeed the generating series of the numbers $\tilde{a}_k$ and $\tilde{b}_k$.

Finally, the generating series of the numbers δ_k is then given by

$$\Delta(p,z) = \frac{z}{\kappa(p)\,(1-z)}\,\tilde{B}(p,V_c^z) = \frac{\hat{y}\left(1-p,U(t_c^3),V_c^z\right)-1}{\kappa(p)}\,\tilde{B}(p,V_c^z),$$

giving the expression of the lemma. See the Maple file [26] for detailed computations. ∎

Applying the same proof to the function $T(p,t,ty)$ instead of $\tilde{F}(p,t,z)$ defined in equation (6.1) allows to establish the asymptotic behavior in n of $[t^{3n}]t^k T_k(p,t)$ for every $k \geq 0$. We do not reproduce the proof as it will be almost exactly the same as the proof of Lemma 6.2, with no additional difficulties but with the function

$$\frac{1-p}{y}\,\tilde{F}\left(1-p,t,1-\frac{1}{y}\right)$$

instead of $\tilde{F}(p,t,z)$. The statement is as follows.

Lemma 6.3 *Fix $p \in (0,1)$ and $k \geq 1$. One has*

$$\frac{[t^{3n}]t^k T_k(p,t)}{[t^{3n}]\mathcal{Z}(p,t)} \underset{n\to\infty}{\to} \theta_k(p), \tag{6.3}$$

where the generating series of the numbers $\theta_k(p)$ is given by

$$\Theta(p,y) := \sum_{k\geq 0}\theta_k(p)\,y^k = \frac{1-p}{y}\,\Delta\left(1-p,1-\frac{1}{y}\right), \tag{6.4}$$

which is analytic on $\mathbb{C}\setminus[y_+(p,t_c),+\infty)$.

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6.2 BDFG functions

We start with an integral formula for the functions $f^\bullet$ and $f^\diamond$.

Lemma 6.4 *Fix $p \in (0,1)$ and $t \in (0,t_c]$. Then, for $0 < z_1 \le z^+(p,t)$ and $0 < z_2 \le z^\diamond(p,t)$, we have*

$$\begin{aligned} &f^\bullet(p,t;z_1,z_2) \\ &= \sqrt{pt^3}\, 2z_2 + \frac{t^3}{\pi} \int_{\left(1-\sqrt{pt^3}(z_2-2\sqrt{z_1})\right)^{-1}}^{\left(1-\sqrt{pt^3}(z_2+2\sqrt{z_1})\right)^{-1}} dz\, \frac{1-\sqrt{pt^3}z_2 - \frac{1}{z}}{2pt^3\, z_1} \\ &\qquad \cdot \frac{T(1-p,t,tz)}{\sqrt{\left(1-z\left(1-\sqrt{pt^3}(z_2+2\sqrt{z_1})\right)\right)\left(z\left(1-\sqrt{pt^3}(z_2-2\sqrt{z_1})\right)-1\right)}}, \\ &f^\diamond(p,t;z_1,z_2) \\ &= \sqrt{pt^3}\,(2z_2+z_1^2) + \frac{\sqrt{t^3/p}}{\pi} \int_{\left(1-\sqrt{pt^3}(z_2-2\sqrt{z_1})\right)^{-1}}^{\left(1-\sqrt{pt^3}(z_2+2\sqrt{z_1})\right)^{-1}} dz \\ &\qquad \cdot \frac{T(1-p,t,tz)}{\sqrt{\left(1-z\left(1-\sqrt{pt^3}(z_2+2\sqrt{z_1})\right)\right)\left(z\left(1-\sqrt{pt^3}(z_2-2\sqrt{z_1})\right)-1\right)}}. \end{aligned}$$

Proof Fix $p \in (0,1)$ and $t \in (0,t_c]$. Replacing the weights by their expression (3.1) in the definitions of the two functions gives

$$\begin{aligned} &f^\bullet(p,t;z_1,z_2) \\ &= \frac{1}{p} 2(pt)^{3/2} z_2 \\ &\quad + \frac{pt^3}{p} \sum_{k,k',l\ge 0} \binom{2k+k'+l+1}{k+1,k,k',l} (pt^3\, z_1)^k \left(\sqrt{pt^3}\, z_2\right)^{k'} [y^l] T(1-p,t,ty), \\ &f^\diamond(p,t;z_1,z_2) \\ &= \frac{1}{p}(pt)^{3/2}(2z_2+z_1^2) \\ &\quad + \frac{\sqrt{pt^3}}{p} \sum_{k,k',l\ge 0} \binom{2k+k'+l}{k,k,k',l} (pt^3\, z_1)^k \left(\sqrt{pt^3}\, z_2\right)^{k'} [y^l] T(1-p,t,ty). \end{aligned}$$

We can express these two functions as Hadamard products. Indeed, define the trivariate power series in $(z_1,z_2;z)$:

$$\begin{aligned} &h^\bullet(z_1,z_2;z) \\ &:= \sum_{k,k',l\ge 0} \binom{2k+k'+l+1}{k+1,k,k',l} (pt^3\, z_1)^k \left(\sqrt{pt^3}\, z_2\right)^{k'} z^l, \\ &= \frac{1}{2pt^3\, z_1} \left(\frac{1-z-\sqrt{pt^3}z_2}{\sqrt{\left(1-z-\sqrt{pt^3}(z_2+2\sqrt{z_1})\right)\left(1-z-\sqrt{pt^3}(z_2-2\sqrt{z_1})\right)}} - 1 \right), \end{aligned}$$

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and

$$h^{\diamond}(z_1,z_2;z) := \sum_{k,k',l\geq 0} \binom{2k+k'+l}{k,k,k',l} (pt^3 z_1)^k (\sqrt{pt^3}\, z_2)^{k'} z^l,$$
$$= \frac{1}{\sqrt{\left(1-z-\sqrt{pt^3}(z_2+2\sqrt{z_1})\right)\left(1-z-\sqrt{pt^3}(z_2-2\sqrt{z_1})\right)}}.$$

Then we have

$$f^{\bullet}(p,t;z_1,z_2) = \sqrt{pt^3}\, 2z_2 + t^3\, T(1-p,t,tz) \odot h^{\bullet}(z_1,z_2;z)|_{z=1},$$
$$f^{\diamond}(p,t;z_1,z_2) = \sqrt{pt^3}\,(2z_2+z_1^2) + \frac{\sqrt{pt^3}}{p}\, T(1-p,t,tz) \odot h^{\diamond}(z_1,z_2;z)|_{z=1}.$$

We can calculate these two Hadamard products as contour integrals in a similar fashion than in the proof of Theorem 1.1 where we established (4.8). For $(z_1,z_2) \in (0, z^+(p,t)] \times (0, z^{\diamond}(p,t)]$, the functions $h^{\bullet}$ and $h^{\diamond}$ are analytic in the domain $|z| < 1-\sqrt{pt^3}(z_2+2\sqrt{z_1})$ that contains the domain $|z| < 1-\sqrt{pt^3}c_+(p,t) = 1-\frac{1}{y_+(p,t)}$. This last domain is not empty since $y_+(p,t) > 1$ from Lemma 2.2. Therefore, we can represent the Hadamard product as a contour integral similar to (4.8) if $\left(1-\frac{1}{y_+(p,t)}\right)^{-1} \leq y_+(1-p,t)$. We can check that this is the case since we have explicit formulas for $y_+(p,t_c)$ and $y_+(p,t)$ is increasing in t (see the Maple file [26] for details). Therefore, for a contour γ enclosing 0 and a point in the interval $\left[\left(1-\frac{1}{y_+(p,t)}\right)^{-1}, y_+(1-p,t)\right]$, we have

$$f^{\bullet}(p,t;z_1,z_2) = \sqrt{pt^3}\, 2z_2 + \frac{t^3}{2i\pi}\oint_{\gamma} \frac{dz}{z} T(1-p,t,tz)\, h^{\bullet}(z_1,z_2,1/z),$$
$$f^{\diamond}(p;z_1,z_2) = \sqrt{pt^3}\,(2z_2+z_1^2) + \frac{\sqrt{t_c^3/p}}{2i\pi}\oint_{\gamma} \frac{dz}{z} T(1-p,t,tz)\, h^{\diamond}(z_1,z_2,1/z).$$

We obtain the expressions of the lemma after simplifications and taking contours converging to the cut $\left[\left(1-\sqrt{pt^3}(z_2-2\sqrt{z_1})\right)^{-1}, \left(1-\sqrt{pt^3}(z_2+2\sqrt{z_1})\right)^{-1}\right]$. ∎

When $p = 1/2$ and $t = t_c$, we can compute explicitly an asymptotic expansion of $f^{\bullet}$ and $f^{\diamond}$ at the point $(z^+(p,t), z^{\diamond}(p,t))$ that will be used in the proof of Theorem 1.2.

Lemma 6.5 *Write $f^{\bullet}(z_1,z_2) = f^{\bullet}(1/2,t_c;z_1,z_2)$, $f^{\diamond}(z_1,z_2) = f^{\diamond}(1/2,t_c;z_1,z_2)$, $z^+ = z^+(1/2,t_c)$, and $z^{\diamond} = z^{\diamond}(1/2,t_c)$. Then $z_+ = \frac{27\sqrt{3}}{32}$ and $z^{\diamond} = \frac{3^{1/4}\sqrt{2}}{4}$ and we have the following asymptotic expansions at $(z^+, z^{\diamond})^-$:*

$$f^{\bullet}(z_1,z_2)$$
$$= 1 - \frac{1}{z^+} + \frac{1}{z^+}\left(\frac{1}{z^+} - \sqrt{z_+}\partial_{z_1} f^{\diamond}(z^+,z^{\diamond})\right)(z_1-z^+) + \partial_{z_1} f^{\diamond}(z^+,z^{\diamond})\,(z_2-z^{\diamond})$$
$$+ \kappa^{\bullet}\left((z^+-z_1)+\sqrt{z^+}(z^{\diamond}-z_2)\right)^{7/6} + o\left(\left((z^+-z_1)+\sqrt{z^+}(z^{\diamond}-z_2)\right)^{7/6}\right),$$

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$$
\begin{aligned}
f^{\diamond}(z_1,z_2)
&= z^{\diamond} + \partial_{z_1} f^{\diamond}(z^+,z^{\diamond})\,(z_1 - z^+) + \Big(1 - \sqrt{z^+}\partial_{z_1} f^{\diamond}(z^+,z^{\diamond})\Big)\,(z_2 - z^{\diamond})\\
&\quad + \kappa^{\diamond}\Big((z^+ - z_1) + \sqrt{z^+}(z^{\diamond} - z_2)\Big)^{7/6} + o\Big(\Big((z^+ - z_1) + \sqrt{z^+}(z^{\diamond} - z_2)\Big)^{7/6}\Big),
\end{aligned}
$$

where

$$
\kappa^{\diamond} = \frac{4\,2^{\frac{1}{6}}\Gamma\left(\frac{2}{3}\right)^3 3^{\frac{2}{3}}\sqrt{5}}{63\pi^2} \quad \textit{and} \quad \kappa^{\bullet} = \frac{512\,3^{\frac{1}{4}}\sqrt{2}}{81}\kappa^{\diamond}.
$$

Proof The respective values of z^+ and $z^{\diamond}$ are computed from (3.10) and the explicit values of the singularities $y_+(1/2,t_c) = 2$ and $y_-(1/2,t_c) = -4$ of the function $y \mapsto T(1/2,t_c,t_c y)$ (see the Maple file [26] for details):

$$
\begin{aligned}
z^+ &= \frac{c_+(1/2,t_c) + c_-(1/2,t_c)}{2} = \frac{27\sqrt{3}}{32},\\
z^{\diamond} &= \left(\frac{c_+(1/2,t_c) - c_-(1/2,t_c)}{4}\right)^2 = \frac{3^{3/4}\sqrt{2}}{4}.
\end{aligned}
$$

Now, the change of variable

$$
\begin{aligned}
z &= \phi(z_1,z_2;\xi)\\
&:= \frac{1}{1-\sqrt{pt^3}(z_2 - 2\sqrt{z_1})} + \left(\frac{1}{1-\sqrt{pt^3}(z_2 + 2\sqrt{z_1})} - \frac{1}{1-\sqrt{pt^3}(z_2 - 2\sqrt{z_1})}\right)\xi
\end{aligned}
$$

in the expressions of Lemma 6.4 gives

$$
\begin{aligned}
f^{\bullet}(p,t;z_1,z_2)
&= \sqrt{pt^3}\,2z_2 + \frac{1}{2pz_1}\frac{1}{\pi}\int_0^1 d\xi\,\xi^{-1/2}(1-\xi)^{-1/2}\,\frac{\phi(z_1,z_2;\xi)(1-\sqrt{pt^3}z_2) - 1}{\phi(z_1,z_2;\xi)}\\
&\qquad \times T\left(1-p,t,t\,\phi(z_1,z_2;\xi)\right),\\
f^{\diamond}(p,t;z_1,z_2)
&= \sqrt{pt^3}\,(2z_2 + z_1^2) + \sqrt{\frac{t^3}{p}}\frac{1}{\pi}\int_0^1 d\xi\,\xi^{-1/2}(1-\xi)^{-1/2}\,T\left(1-p,t,t\,\phi(z_1,z_2;\xi)\right).
\end{aligned}
$$

We will see that, when $p = 1/2$ and $t = t_c$, the main term in the expansion of these functions stems from the singular behavior of T at $y_+ = 2$.

From Lemma 6.6 and the discussion that follows, we know that the function $y \mapsto T(1/2,t_c,t_c y)$ is analytic for $y \in [0,2)$ and has the following expansion as $y \to 2^-$:

$$
\begin{aligned}
T&(1/2,t_c,t_c y)\\
&= \frac{\sqrt{3}}{2} - \frac{3^{5/6}}{2}\left(1-\frac{y}{2}\right)^{2/3} + \frac{\sqrt{3}}{2}\left(1-\frac{y}{2}\right) - 3^{1/6}\left(1-\frac{y}{2}\right)^{4/3} + \mathcal{O}\left(\left(1-\frac{y}{2}\right)^{5/3}\right).
\end{aligned}
$$

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The function

$$\varphi(y) := T(1/2, t_c, t_c\, y) - \left(\frac{\sqrt{3}}{2} - \frac{3^{5/6}}{2}\left(1-\frac{y}{2}\right)^{2/3} + \frac{\sqrt{3}}{2}\left(1-\frac{y}{2}\right) + 3^{1/6}\left(1-\frac{y}{2}\right)^{4/3}\right)$$

is twice differentiable on $[0,2)$. In addition, for $\xi \in (0,1)$ and if $(z_1, z_2) \in [0, z^+] \times [0, z^\diamond]$, the quantity $\phi(z_1, z_2; \xi)$ varies in a subset of $(0, (1 - 1/y_+(1/2, t_c))^{-1}) = (0,2)$. We then have

$$\int_0^1 d\xi\, \xi^{-1/2}(1-\xi)^{-1/2}\, |\varphi'\left(\phi(z_1,z_2;\xi)\right)| < +\infty$$

and

$$\int_0^1 d\xi\, \xi^{-1/2}(1-\xi)^{-1/2}\, |\varphi''\left(\phi(z_1,z_2;\xi)\right)| < +\infty.$$

As a consequence, the function

$$\Phi(z_1,z_2) = \frac{1}{\pi}\int_0^1 d\xi\, \xi^{-1/2}(1-\xi)^{-1/2}\varphi\left(\phi(z_1,z_2;\xi)\right)$$

is twice differentiable on $[0, z^+] \times [0, z^\diamond]$ and has the following asymptotic expansion when $(z_1, z_2) \to (z^+, z^\diamond)$:

$$\begin{aligned}&\Phi(z_1,z_2)\\ &\quad = \Phi(z^+,z^\diamond) + \nabla\Phi(z^+,z^\diamond)\cdot(z_1 - z^+, z_2 - z^\diamond) + \mathcal{O}\left((z_1 - z^+)^2 + (z_2 - z^\diamond)^2\right).\end{aligned}$$

The singular parts of the expansions of $f^\bullet$ and $f^\diamond$ come from the singularities of the form $(1 - y/2)^\alpha$ in the development of $T(1/2, t_c, t_c\, y)$ at $y = 2$ for $\alpha \in \{2/3, 4/3\}$ (it is straightforward to check that the linear term $(1 - y/2)$ contributes only to nonsingular parts in the expansion). Indeed, for $\alpha \in \{2/3, 4/3\}$, set

$$\begin{aligned}&I_\alpha(z_1,z_2)\\ &\quad = \frac{1}{\pi}\int_0^1 d\xi\, \xi^{-1/2}(1-\xi)^{-1/2}\left(1 - \frac{\phi(z_1,z_2;\xi)}{2}\right)^\alpha,\\ &\quad = \left(1 - \frac{1}{2(1-\sqrt{t_c^3/2}(z_2 - 2\sqrt{z_1}))}\right)^\alpha \frac{1}{\pi}\int_0^1 d\xi\, \xi^{-1/2}(1-\xi)^{-1/2}\\ &\qquad\qquad \cdot\left(1 - \xi\frac{\frac{1-\sqrt{t_c^3/2}(z_2-2\sqrt{z_1})}{1-\sqrt{t_c^3/2}(z_2+2\sqrt{z_1})} - 1}{1 - 2\sqrt{t_c^3/2}(z_2 - 2\sqrt{z_1})}\right)^\alpha,\\ &\quad = \left(1 - \frac{1}{2(1-\sqrt{t_c^3/2}(z_2 - 2\sqrt{z_1}))}\right)^\alpha {}_2F_1\left(-\alpha, \frac{1}{2}; 1; \frac{\frac{1-\sqrt{t_c^3/2}(z_2-2\sqrt{z_1})}{1-\sqrt{t_c^3/2}(z_2+2\sqrt{z_1})} - 1}{1 - 2\sqrt{t_c^3/2}(z_2 - 2\sqrt{z_1})}\right),\end{aligned}$$

where we used Euler's integral representation of the hypergeometric function ${}_2F_1$ in the last line. This last equality is valid when $0 < z_1 \le z^+$ and $0 < z_2 \le z^\diamond$ since in this case $\sqrt{t_c^3/2}(z_2 + 2\sqrt{z_1}) \le 1/2$ and the variable in the hypergeometric function is in

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$(0,1]$. Furthermore, using the values of z^+ and $z^\diamond$, a simple computation done in the Maple file [26] gives

$$\frac{\frac{1-\sqrt{t_c^3/2}(z_2-2\sqrt{z_1})}{1-\sqrt{t_c^3/2}(z_2+2\sqrt{z_1})}-1}{1-2\sqrt{t_c^3/2}(z_2-2\sqrt{z_1})} = 1-\frac{20\sqrt{3}}{81}\Big((z^+-z_1)+\sqrt{z^+}(z^\diamond-z_2)\Big)+\mathcal{O}\big((z^+-z_1)^2+(z^\diamond-z_2)^2\big).$$

Using the standard asymptotic development of hypergeometric functions at 1, we see that the first singular term in the development of $I_{2/3}$ is

$$-\kappa\Big((z^+-z_1)+\sqrt{z^+}(z^\diamond-z_2)\Big)^{7/6} \text{ with } \kappa=\frac{18\,2^{\frac{1}{3}}\,\Gamma\left(\frac{2}{3}\right)^3}{7\pi^2}\left(\frac{3}{5}\right)^{2/3}\left(\frac{20\sqrt{3}}{81}\right)^{7/6}.$$

The first singular term of $I_{4/3}(z_1,z_2)$ is similar, but with exponent $11/6$ instead of $7/6$. This means that the first singular term in the development of $f^\diamond(1/2,z_1,z_2)$ is from $-\frac{3^{5/6}}{2}\sqrt{t_c^3/p}\,I_{2/3}(z_1,z_2)$, and we have

$$\begin{aligned} f^\diamond(z_1,z_2) &= f^\diamond(z^+,z^\diamond)+\nabla f^\diamond(z^+,z^\diamond)\cdot(z_1-z^+,z_2-z^\diamond)\\ &\quad+\frac{3^{5/6}}{2}\sqrt{2t_c^3}\,\kappa\Big((z^+-z_1)+\sqrt{z^+}(z^\diamond-z_2)\Big)^{7/6}\\ &\quad+o\Big(\Big((z^+-z_1)+\sqrt{z^+}(z^\diamond-z_2)\Big)^{7/6}\Big). \end{aligned}$$

The statement for $f^\diamond$ follows using the fact that $f^\diamond(z^+,z^\diamond)=z^\diamond$ and the criticality equation (5.3).

The expansion for $f^\bullet(1/2,z_1,z_2)$ is obtained similarly by replacing φ by

$$\begin{aligned} &\frac{1-\sqrt{pt_c^3}z_2-\frac{1}{y}}{2pt_c^3\,z_1}T(1/2,t_c,t_c\,y)-\frac{1-\sqrt{t_c^3/2}z_2-\frac{1}{2}}{t_c^3\,z_1}\left(\frac{\sqrt{3}}{2}-\frac{3^{5/6}}{2}\left(1-\frac{y}{2}\right)^{2/3}\right.\\ &\qquad\left.+\frac{\sqrt{3}}{2}\left(1-\frac{1}{2t_c^3z_1}\right)\left(1-\frac{y}{2}\right)-3^{1/6}\left(1-\frac{y}{2}\right)^{4/3}\right). \end{aligned}$$

The first singular term in the development of $f^\bullet$ is then the one from $-\frac{3^{5/6}}{2}\frac{1-\sqrt{t_c^3/2}z^\diamond-\frac{1}{2}}{t_c^3\,z^+}\frac{1}{z^+}I_{2/3}$, and we have

$$\begin{aligned} f^\bullet(z_1,z_2) &= f^\bullet(z^+,z^\diamond)+\nabla f^\bullet(z^+,z^\diamond)\cdot(z_1-z^+,z_2-z^\diamond)\\ &\quad+\frac{3^{5/6}}{2}\frac{1-2\sqrt{t_c^3/2}z^\diamond}{2t_c^3\,(z^+)^2}\kappa\Big((z^+-z_1)+\sqrt{z^+}(z^\diamond-z_2)\Big)^{7/6}\\ &\quad+o\Big(\Big((z^+-z_1)+\sqrt{z^+}(z^\diamond-z_2)\Big)^{7/6}\Big). \end{aligned}$$

The statement for $f^\bullet$ the follows from $f^\bullet(z^+,z^\diamond)=1-\frac{1}{z^+}$ and from the identities $\partial_{z_2}f^\bullet=\partial_{z_1}f^\diamond$ and $z_1\partial_{z_1}f^\bullet+f^\bullet=\partial_{z_2}f^\diamond$. ■

6.3 Perimeter asymptotics at criticality

We gather in this section asymptotics of several quantities appearing in this paper when $p = 1/2$ and $t = t_c$. They are all consequences of the following lemma characterizing the singularities of the function $y \mapsto V(1/2, U(t_c^3), y)$ defined in Lemma 2.2.

Lemma 6.6 *The function* $y \mapsto V(1/2, U(t_c^3), y)$ *is analytic on* $\mathbb{C} \setminus ((-\infty, -4] \cup [2, +\infty))$. *In addition, it has the following asymptotic expansion in a slit neighborhood of* 2*:*

(6.5)
$$V(1/2, U(t_c^3), y) = \frac{\sqrt{3}}{3} - \frac{1}{3^{\frac{1}{3}}}\left(1 - \frac{y}{2}\right)^{\frac{1}{3}} + \frac{1}{3}\left(1 - \frac{y}{2}\right) - \frac{1}{3^{\frac{4}{3}}}\left(1 - \frac{y}{2}\right)^{\frac{4}{3}} + \mathcal{O}\left(\left(1 - \frac{y}{2}\right)^{\frac{5}{3}}\right).$$

Proof From Lemma 2.2, we already know that $y \mapsto V(1/2, U(t_c^3), y)$ is analytic on the domain $\mathbb{C} \setminus ((-\infty, y_-(1/2, t_c)] \cup [y_+(1/2, t_c), +\infty))$. We also know that $y_\pm(1/2, t_c)$ are the values of $\hat{y}(1/2, U(t_c^3), V)$ at $V = V_\pm(1/2, t_c)$ the two stationary points of $\hat{y}$ enclosing 0. We can easily compute the corresponding values in the Maple file [26]. The expansion is then easily obtained by singular inversion. ∎

This lemma combined with the rational expressions in terms of V that we have allows to immediately compute asymptotic expansions. Indeed, the expressions $\hat{T}(1/2, U(t_c^3), V)$ and $\hat{\Delta}(1/2, V)$ defined, respectively, in Lemmas 2.2 and 6.2 are singular only when V is singular, and we can get an asymptotic expansion at their unique dominant singularity by plugging the expansion of V in their expression. As usual, calculations are available in the Maple companion file [26].

We get the following expansion for T:

$$T(1/2, t_c, t_c\, y)$$
$$= \frac{\sqrt{3}}{2} - \frac{3^{5/6}}{2}\left(1 - \frac{y}{2}\right)^{2/3} + \frac{\sqrt{3}}{2}\left(1 - \frac{y}{2}\right) - 3^{1/6}\left(1 - \frac{y}{2}\right)^{4/3} + \mathcal{O}\left(\left(1 - \frac{y}{2}\right)^{5/3}\right), \tag{6.6}$$

and as a consequence,

$$t_c^k\, T_k(1/2, t_c) \underset{k\to\infty}{\sim} \frac{-3^{5/6}}{2\Gamma(-2/3)}\, 2^k\, k^{-5/3}. \tag{6.7}$$

Similarly, using the expression (6.1) gives the asymptotic expansion of the weights $\sqrt{t_c^3/2}^{\,-k}\, q_k(1/2, t_c)$:

$$\tilde{F}(1/2, t_c, z) = 2\sqrt{3} - 2\, 3^{5/6}\,(1 - 2z)^{2/3} + +\mathcal{O}\left((1 - 2z)^{5/3}\right), \tag{6.8}$$

and as a consequence,

$$\sqrt{t_c^3/2}^{\,-k}\, q_k(1/2, t_c) \underset{k\to\infty}{\sim} \frac{-2\, 3^{5/6}}{\Gamma(-2/3)}\, 2^{-k}\, k^{-5/3}. \tag{6.9}$$

Note that these two singular expansions and the corresponding asymptotics were established in [6] using different and more involved techniques. The rational parameterization that we have simplifies this analysis a lot.

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Using $\hat{\Delta}$, we get the following expansion:

$$\Delta(1/2, z) \underset{z\to 1/2}{\sim} \left(\frac{32\, 3^{\frac{5}{6}}}{351} + \frac{8\, 3^{\frac{1}{3}}}{117}\right) (1-2z)^{-4/3}, \tag{6.10}$$

giving

$$\delta_k(1/2) \underset{k\to\infty}{\sim} \frac{1}{\Gamma(4/3)} \left(\frac{32\, 3^{\frac{5}{6}}}{351} + \frac{8\, 3^{\frac{1}{3}}}{117}\right) 2^{-k}\, k^{1/3}. \tag{6.11}$$

Finally, using the expression (6.4), we get

$$\Theta(1/2, y) \underset{y\to 2}{\sim} \left(\frac{8\, 3^{\frac{5}{6}}}{351} + \frac{2\, 3^{\frac{1}{3}}}{117}\right) (1-y/2)^{-4/3}, \tag{6.12}$$

giving

$$\theta_k(1/2) \underset{k\to\infty}{\sim} \frac{1}{\Gamma(4/3)} \left(\frac{8\, 3^{\frac{5}{6}}}{351} + \frac{2\, 3^{\frac{1}{3}}}{117}\right) 2^{k}\, k^{1/3}. \tag{6.13}$$

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New York University Shanghai, Shanghai, China

and

Laboratoire Modal'X, Université Paris Nanterre, Nanterre, France

e-mail: laurent.menard@normalesup.org