

A MAYER-VIETORIS SEQUENCE IN GROUP HOMOLOGY AND THE DECOMPOSITION OF RELATION MODULES

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Introduction. W. A. Bogley and M. A. Gutierrez [2] have recently obtained an eight-term exact homology sequence that relates the integral homology of a quotient group Γ/MN , where M and N are normal subgroups of the group Γ , to the integral homology of the free product $\Gamma/M * \Gamma/N$ in dimensions ≤ 3 by means of connecting terms constructed from commutator subgroups of Γ , M , N and $M \cap N$. In this paper we use the methods of [4] to recover this exact sequence under weaker hypotheses and for coefficients in $\mathbb{Z}/q\mathbb{Z}$ for any non-negative integer q . Further, for $q = 0$ we extend the sequence by three terms in order to capture the relation between the fourth homology groups.

Our principal application of the Mayer-Vietoris sequence is to relation modules. If $1 \rightarrow R \rightarrow F \rightarrow G \rightarrow 1$ is a free presentation of a group G , then the *relation module* is R^{ab} endowed with the G -action obtained from the conjugation action of the free group F on the normal subgroup R . There exists an extensive theory of relation modules for finite groups G (see [10]) and interesting results for classes of infinite groups, such as for polycyclic groups in [17]. A central theme of the theory has been the study of indecomposable relation modules. In the present paper we address the complementary issue: we give conditions that guarantee the decomposition of a relation module as a direct sum of submodules. In particular, as a simple consequence of the Mayer-Vietoris sequence, we see that the relation module of a presentation expressed as the union of two subpresentations is a direct sum of submodules induced from the relation modules of the subpresentations, provided that the relation subgroups of the two subpresentations are *independent* in the ambient free group.

In Section 2 we discuss independence of normal subgroups of free groups in more detail, and prove the equivalence between a relation module decomposition as a direct sum of submodules induced from the relation modules of subpresentations and independence of the relation subgroups. Whilst our methods here are essentially elementary, the main theorem applies to interesting classes of presentations, and some applications are explored in Section 3. In favourable cases the connecting terms can be precisely described, and are often trivial, indicating that independence of relation subgroups is reflected in higher-order independence conditions.

For a survey of matters closely related to the theme of this paper, the reader is referred to [3]. The authors are grateful to W. A. Bogley and to J. Howie for conversations and helpful comments.

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1. A Mayer-Vietoris sequence in group homology. Let q be a fixed non-negative integer. For any group Γ we denote by $H_n(\Gamma, \mathbb{Z}_q)$ the homology of Γ with coefficients in the trivial Γ -module $\mathbb{Z}_q = \mathbb{Z}/q\mathbb{Z}$. If N is a normal subgroup of Γ then $\Gamma \#_q N$ denotes the subgroup of Γ generated by the commutators $[g, n]$ and the elements n^q , where $g \in \Gamma$ and $n \in N$. Note that $\Gamma \#_q N$ is normal in Γ , and that when $q = 0$ we have $\Gamma \#_q N = [\Gamma, N]$.

THEOREM 1.1. *Let Γ be a group and let M and N be normal subgroups of Γ . Then there is a natural exact sequence of groups:*

$$\begin{aligned} H_4(\Gamma/M, \mathbb{Z}_q) \oplus H_4(\Gamma/N, \mathbb{Z}_q) &\rightarrow H_4(\Gamma/MN, \mathbb{Z}_q) \rightarrow V_3 \rightarrow H_3(\Gamma/M, \mathbb{Z}_q) \oplus H_3(\Gamma/N, \mathbb{Z}_q) \\ &\rightarrow H_3(\Gamma/MN, \mathbb{Z}_q) \rightarrow V_2 \rightarrow H_2(\Gamma/M, \mathbb{Z}_q) \oplus H_2(\Gamma/N, \mathbb{Z}_q) \rightarrow H_2(\Gamma/MN, \mathbb{Z}_q) \\ &\rightarrow V_1 \rightarrow H_1(\Gamma/M, \mathbb{Z}_q) \oplus H_1(\Gamma/N, \mathbb{Z}_q) \rightarrow H_1(\Gamma/MN, \mathbb{Z}_q) \rightarrow 0 \end{aligned}$$

such that

- (i) *there is an isomorphism $V_1 \rightarrow \Gamma/(M \cap N)(\Gamma \#_q \Gamma)$,*
- (ii) *there is a surjection $V_2 \rightarrow (M \cap N \cap (\Gamma \#_q \Gamma))/[M, N](\Gamma \#_q (M \cap N))$ which is an isomorphism if $H_2(\Gamma, \mathbb{Z}_q) = 0$,*
- (iii) *for $q = 0$ there is a surjection $V_3 \rightarrow \ker(\wedge(\Gamma; M, N) \rightarrow \wedge(\Gamma; \Gamma, \Gamma))$ which is an isomorphism if $H_3(\Gamma, \mathbb{Z}) = 0$.*

Parts (i) and (ii) of Theorem 1.1 for $q = 0$ are a slight generalisation of Theorem 5.2 of [2]. Part (iii), and parts (i) and (ii) for $q \neq 0$, are new.

In the following discussion, up to and including Corollary 1.2, we fix $q = 0$, and write $H_n(\Gamma)$ for the homology of a group Γ with integral coefficients.

The group $\wedge(\Gamma; M, N)$ in part (iii) of the theorem has a presentation with generators $x \wedge y$, where $(x, y) \in (M \times N) \cup (N \times M) \cup (\Gamma \times M \cap N) \cup (M \cap N \times \Gamma)$, subject to the relations

$$\begin{aligned} xx' \wedge y &= (xx'x^{-1} \wedge xyx^{-1})(x \wedge y), \\ x \wedge yy' &= (x \wedge y)(yxy^{-1} \wedge yy'y^{-1}), \\ x \wedge y &= (y \wedge x)^{-1}, \\ (x \wedge y)(u \wedge v)(y \wedge x) &= [x, y]u[y, x] \wedge [x, y]v[y, x], \\ z \wedge z &= 1, \quad \text{if } z \in M \cap N. \end{aligned}$$

The homomorphism $\wedge(\Gamma; M, N) \rightarrow \wedge(\Gamma; \Gamma, \Gamma)$ in (iii) is the canonical one. The group $\wedge(\Gamma; \Gamma, \Gamma)$ is isomorphic to the group (Γ, Γ) introduced by C. Miller [18] to give an interpretation of $H_2(\Gamma)$, and to the group $\Gamma \wedge \Gamma$ of [5]. As in [5, 18], we have

$$H_2(\Gamma) \cong \ker(\wedge(\Gamma; \Gamma, \Gamma) \rightarrow \Gamma).$$

Hence if $H_2(\Gamma) = 0 = H_3(\Gamma)$ then $V_3 \cong \ker(\wedge(\Gamma; M, N) \rightarrow \Gamma)$.

If $M \cap N = 1$ then $\wedge(\Gamma; M, N)$ is isomorphic to the tensor product $M^{ab} \otimes N^{ab}$ of

abelian groups. If further $H_2(\Gamma) = 0 = H_3(\Gamma)$ then we obtain from Theorem 1.1 the two exact sequences:

$$\begin{aligned}
 &H_4(\Gamma/M) \oplus H_4(\Gamma/N) \rightarrow H_4(\Gamma/MN) \rightarrow M^{ab} \otimes N^{ab} \rightarrow H_3(\Gamma/M) \oplus H_3(\Gamma/N) \rightarrow H_3(\Gamma/MN) \rightarrow 0, \\
 &0 \rightarrow H_2(\Gamma/M) \oplus H_2(\Gamma/N) \rightarrow H_2(\Gamma/MN) \rightarrow H_1(\Gamma) \rightarrow H_1(\Gamma/M) \oplus H_1(\Gamma/N) \rightarrow H_1(\Gamma/MN) \rightarrow 0.
 \end{aligned}$$

Following [11], we say that two normal subgroups M, N of a group Γ are *independent* if $M \cap N = [M, N]$. This notion is the main concern of Section 2. If M and N are independent, and if $H_2(\Gamma) = 0 = H_3(\Gamma)$, then $V_2 = 0$ and $V_1 = H_1(\Gamma)$. If further $\Gamma = MN$ then we deduce from the Mayer-Vietoris sequence that

$$\begin{aligned}
 &H_3(\Gamma/M) \oplus H_3(\Gamma/N) \cong \ker(\wedge(\Gamma; M, N) \rightarrow \Gamma), \\
 &H_2(\Gamma/M) = 0 = H_2(\Gamma/N), \\
 &H_1(\Gamma) \cong H_1(\Gamma/M) \oplus H_1(\Gamma/N).
 \end{aligned}$$

These observations pertain to relation modules. Given free presentations

$$1 \rightarrow R_i \rightarrow F \rightarrow G_i \rightarrow 1$$

for $i = 1, 2$, with R_1 and R_2 independent, we take $\Gamma = R_1 R_2$ and note that

$$H_1(R_1 R_2 / R_2) \cong H_1(R_1 / [R_1, R_2]) \cong R_1^{ab} \otimes_{R_2} \mathbb{Z} \cong R_1^{ab} \otimes_G \mathbb{Z} G \tag{1}$$

where $G = F/\Gamma$. We deduce the following result.

COROLLARY 1.2. *If R_1 and R_2 are independent normal subgroups of a free group F , with $G_i = F/R_i$ and $G = F/R_1 R_2$, then*

$$(R_1 R_2)^{ab} \cong (R_1^{ab} \otimes_G \mathbb{Z} G) \oplus (R_2^{ab} \otimes_{G_2} \mathbb{Z} G).$$

In Section 2 we shall generalise this corollary to an independent family of normal subgroups of F , and establish the converse.

Proof of Theorem 1.1. Let $B(\Gamma)$ denote the classifying space of a group Γ . The exact sequence given in the theorem is the low-dimensional part of the exact homology sequence of the cofibration sequence

$$B(\Gamma/M) \vee B(\Gamma/N) \rightarrow B(\Gamma/MN) \rightarrow D \tag{2}$$

and to prove the theorem we need to identify $H_i D$ for $i = 2, 3, 4$. We shall exhibit a space C and a cofibration sequence

$$D \rightarrow C \rightarrow \Sigma^2 B(\Gamma) \tag{3}$$

(where Σ^2 denotes the double suspension) and compute $H_i(C)$ for $i = 2, 3, 4$. The

statements of the theorem then follow from the exact homology sequence of the cofibration sequence (3). We shall first prove the theorem for $q = 0$.

Suppose that $q = 0$. Let $B(\Gamma; M)$ denote, as in [4], the mapping cone of $B(\Gamma) \rightarrow B(\Gamma/M)$. The space C is the space $B(\Gamma; M, N)$ of [4], constructed as the mapping cone of $B(\Gamma; M) \rightarrow B(\Gamma/N; MN/N)$ (or as the mapping cone of $B(\Gamma; N) \rightarrow B(\Gamma/M, MN/M)$, which gives the same result). It follows that C is 2-connected (as it is a mapping cone of 1-connected spaces), so that $H_2C = 0$, and that C fits into a commutative diagram in which every row and column is a cofibration sequence:

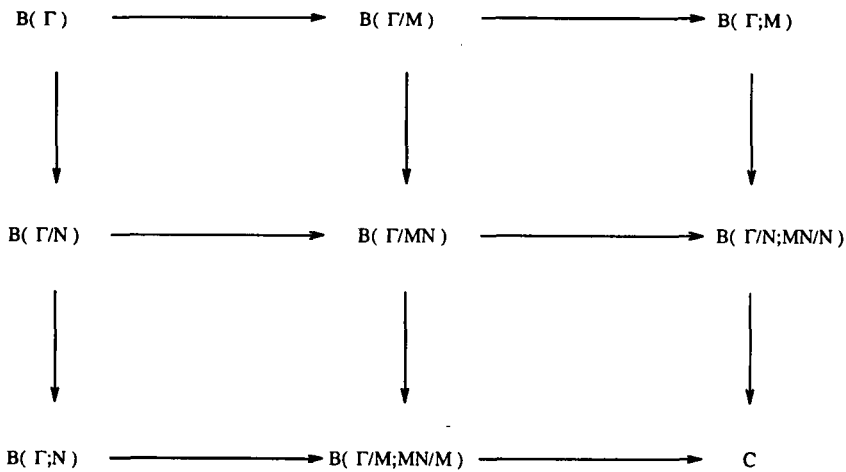


Figure 1.1

Now D in (2) is the mapping cone of $B(\Gamma/N) \rightarrow B(\Gamma/M; MN/M)$, and so there exists a map of cofibration sequences

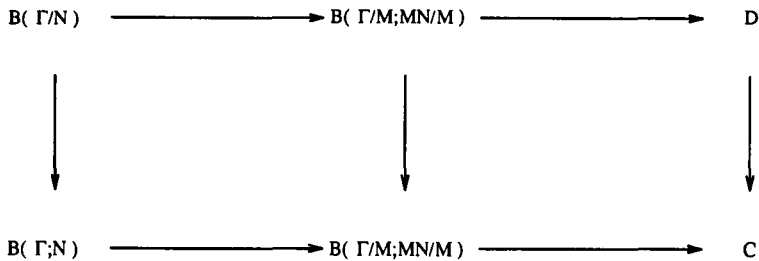


Figure 1.2

and the mapping cones of the vertical maps form the cofibration sequence

$$\Sigma B(\Gamma) \rightarrow * \rightarrow \Sigma^2 B(\Gamma),$$

from which (3) follows.

We compute $H_i C$ from $\pi_i C$. Consider the cubical diagram of spaces

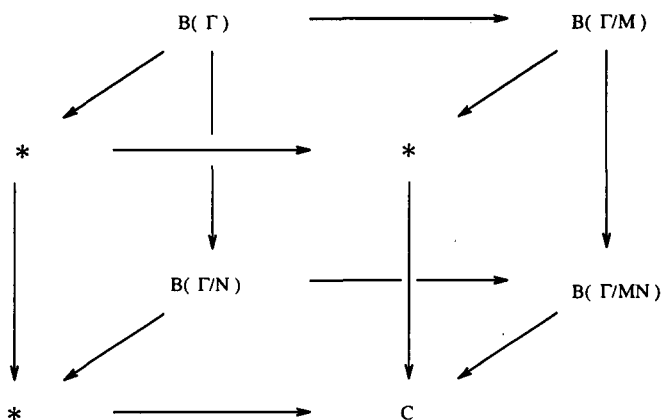


Figure 1.3

We enlarge this diagram by taking homotopy fibres:

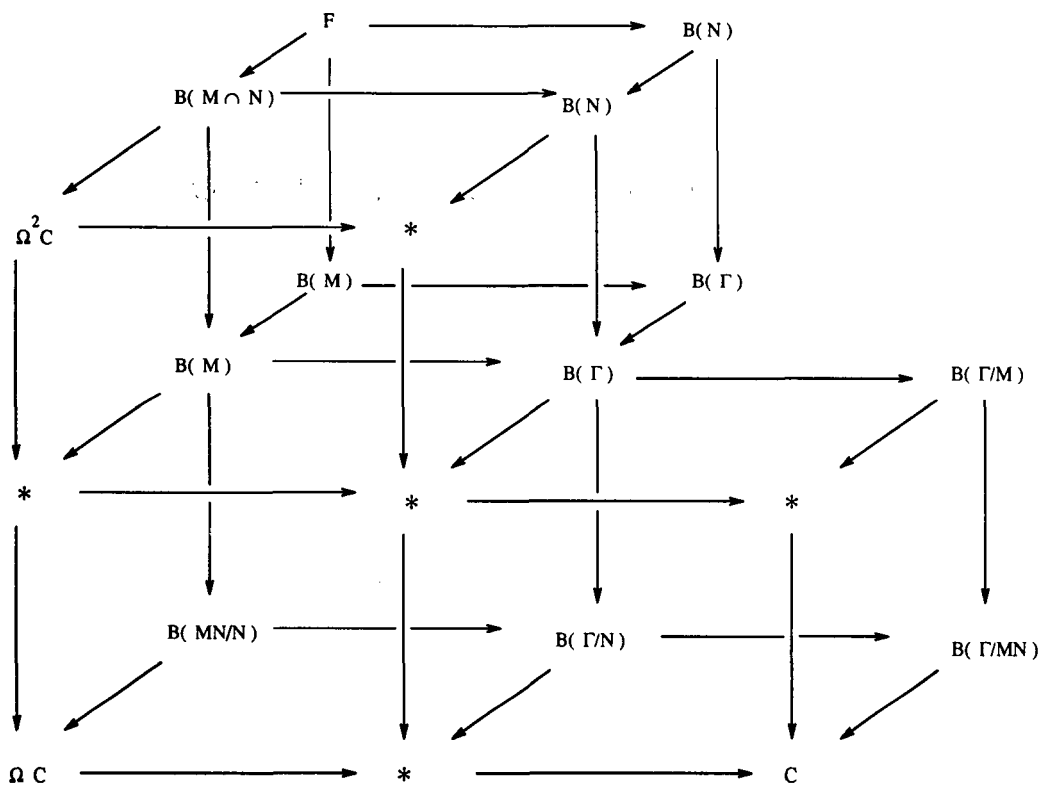


Figure 1.4

and extract the fibration sequence

$$F \rightarrow B(M \cap N) \rightarrow \Omega^2 C$$

for further perusal. We see that $\pi_{i+3}C \cong \pi_i F$ for $i \geq 2$, and the exact homotopy sequence ends

$$0 \rightarrow \pi_4 C \rightarrow \pi_1 F \rightarrow M \cap N \rightarrow \pi_3 C \rightarrow 0.$$

A presentation of $\pi_1 F$, and a description (in terms of this presentation) of the map $\pi_1 F \rightarrow M \cap N$ is given in Theorem 2.3 of [8]. From this description it is immediate that γ has image $[M, N][\Gamma, M \cap N]$, whence

$$H_3 C \cong \pi_3 C \cong M \cap N / [M, N][\Gamma, M \cap N]$$

by the Hurewicz Theorem. Furthermore, $\pi_4 C \cong \ker \gamma$. J. H. C. Whitehead's exact sequence [19] starts

$$\dots \rightarrow \pi_3 C \otimes \mathbb{Z}_2 \rightarrow \pi_4 C \rightarrow H_4 C \rightarrow 0$$

and as in Lemma 4.2 of [5] the image of $\pi_3 C \otimes \mathbb{Z}_2$ in $\pi_4 C$ can be explicitly identified in $\pi_1 F$. It follows that

$$H_4 C \cong \ker(\wedge(\Gamma; M, N) \rightarrow \Gamma).$$

Now the map $H_4 C \rightarrow H_2 G$ in the exact homotopy sequence of 2 can be identified as the restriction of the canonical homomorphism $\wedge(\Gamma; M, N) \rightarrow \wedge(\Gamma; \Gamma, \Gamma)$, since by [18] there is an isomorphism $H_2 \Gamma \cong \ker(\wedge(\Gamma; \Gamma, \Gamma) \rightarrow G)$. Now $\ker \alpha \leq \ker(\wedge(\Gamma; M, N) \rightarrow \Gamma)$, and part (iii) of the theorem follows.

Suppose now that $q \geq 1$. The short exact sequence of coefficient modules

$$0 \rightarrow \mathbb{Z} \xrightarrow{\times q} \mathbb{Z} \rightarrow \mathbb{Z}_q \rightarrow 0$$

induces a long exact sequence in the homology of the space C , part of which is

$$\dots \rightarrow H_3(C, \mathbb{Z}) \xrightarrow{\times q} H_3(C, \mathbb{Z}) \rightarrow H_3(C, \mathbb{Z}_q) \rightarrow 0.$$

It follows that $H_3(C, \mathbb{Z}_q) = M \cap N / [M, N](\Gamma \#_q (M \cap N))$. There is also an induced long exact sequence in the homology of D , part of which is

$$\dots \rightarrow H_2(D, \mathbb{Z}) \xrightarrow{\times q} H_2(D, \mathbb{Z}) \rightarrow H_2(D, \mathbb{Z}_q) \rightarrow 0.$$

It follows that $H_2(D, \mathbb{Z}_q) = \Gamma / (M \cap N)(\Gamma \#_q \Gamma)$. Parts (i) and (ii) of the theorem for $q \geq 1$ now follow from the cofibration sequence (2).

2. Independent normal subgroups of free groups. Let F be a free group with basis \mathbf{x} and suppose that $\{R_1, \dots, R_n\}$ is a family of normal subgroups of F , with $n > 1$. We write R for the product $R_1 \dots R_n$ of all the R_i with $1 \leq i \leq n$, and N_i for the product $R_1 \dots R_{i-1} R_{i+1} \dots R_n$ that omits R_i . Since each R_i is normal, the order of the factors in such products is irrelevant. The quotient F/R we denote by G , and the quotient F/R_i by G_i . The family $\{R_1, \dots, R_n\}$ is *independent* if, for all i with $1 \leq i \leq n$, we have $R_i \cap N_i = [R_i, N_i]$. This notion has been studied in [1], [9], [11] and [16]. Independence may be considered as ensuring that certain intersections are as small as possible, or as ensuring that certain commutator subgroups are as large as possible: the following

remarks will indicate that the second point of view is generally the most appropriate. None of these remarks depends upon the freeness of F .

REMARKS

1. The trivial group may be added to, or deleted from, an independent family to leave an independent family.
2. If $\{R_1, \dots, R_n\}$ is an independent family with K a normal subgroup of F contained in R_1 , and if $\{K, R_2, \dots, R_n\}$ is independent, then $K \cap [R_1, N_1] = [K, N_1]$.

We now state the generalisation of Corollary 1.2.

THEOREM 2.1. *Let F be a free group and $\{R_1, \dots, R_n\}$ a family of normal subgroups of F , with $n > 1$. Then the following are equivalent:*

- (1) *the family $\{R_1, \dots, R_n\}$ is independent,*
- (2) *the quotient maps $R \rightarrow R/N_i$, ($1 \leq i \leq n$) induce an isomorphism*

$$R^{ab} \cong \bigoplus_{i=1}^n R_i^{ab} \otimes_{\mathbb{Z}G_i} \mathbb{Z}G$$

of $\mathbb{Z}G$ -modules.

Proof. (1) \Rightarrow (2). The quotient maps $R \rightarrow R/N_i$ certainly induce a surjection $\theta: R^{ab} \rightarrow \bigoplus_{i=1}^n (R/N_i)^{ab}$ and, as in (1) in Section 1 there is an isomorphism of $\mathbb{Z}G$ -modules $(R/N_i)^{ab} \cong R_i^{ab} \otimes_{G_i} \mathbb{Z}G$. So it suffices to show that θ is an isomorphism of $\mathbb{Z}G$ -modules. Since the $\mathbb{Z}G$ -action on both R^{ab} and $(R/N_i)^{ab}$ is induced by conjugation in F , it follows that θ is a $\mathbb{Z}G$ -map. The kernel of θ is isomorphic to

$$\frac{N_1 \cap \dots \cap N_n}{[R, R] \cap N_1 \cap \dots \cap N_n}$$

so that it suffices to show that $N_1 \cap \dots \cap N_n \subseteq [R, R]$. we set

$$N_{i_1, \dots, i_k} = \prod_{j \neq i_1, \dots, i_k} R_j.$$

LEMMA 2.2. *For any subset $\{i_1, \dots, i_k\} \subseteq \{1, \dots, n\}$ we have*

$$N_{i_1} \cap \dots \cap N_{i_k} = (R_{i_1} \cap N_{i_1}) \dots (R_{i_k} \cap N_{i_k}) N_{i_1, \dots, i_k}.$$

Proof of Lemma. For ease of notation we shall assume that $i_j = j$ and we proceed by induction on k . The case $k = 1$ is clear. For $k \geq 1$ we assume that

$$N_1 \cap \dots \cap N_k \cap N_{k+1} = (R_1 \cap N_1) \dots (R_k \cap N_k) N_{1, \dots, k} \cap N_{k+1}.$$

Suppose that x lies in the right-hand side. Then

$$x = r_1 r_2 \dots r_k r_{k+1} \dots r_n = r'_1 r'_2 \dots r'_{k-1} r'_k r'_{k+2} \dots r'_n,$$

where $r_j, r'_j \in R_j$ and if $j \leq k$, $r_j \in R_j \cap N_j$. Then

$$r_{k+1} = (r_1 \dots r_k)^{-1} r'_1 \dots r'_k r'_{k+2} \dots r'_n (r_{k+2} \dots r_n)^{-1} \in N_{k+1}$$

and so $x \in (R_1 \cap N_1) \dots (R_k \cap N_k) (R_{k+1} \cap N_{k+1}) N_{1, \dots, k, k+1}$. This completes the inductive step and establishes one inclusion of the lemma, whilst the reverse inclusion is clear.

We apply the lemma with $\{i_1, \dots, i_k\} = \{1, \dots, n\}$, which tells us that $N_1 \cap \dots \cap N_n = \prod_{i=1}^n R_i \cap N_i$, and so assuming independence of the R_i we have

$$N_1 \cap \dots \cap N_n = \prod_{i=1}^n [R_i, N_i] \subseteq [R, R].$$

(2) \Rightarrow (1). There exist minimal and pairwise disjoint generating sets \mathbf{r}_i for R_i^{ab} as a $\mathbb{Z}G$ -module, ($i = 1, \dots, n$), with preimages $\hat{\mathbf{r}}_i \subseteq F$ such that no two elements of $\hat{\mathbf{r}}_1 \cup \dots \cup \hat{\mathbf{r}}_n$ are F -conjugate. For each i with $1 \leq i \leq n$ there is a short exact sequence

$$0 \rightarrow \pi_2 Y_i \rightarrow \bigoplus_{\mathbf{r}_i} \mathbb{Z}G_i \rightarrow R_i^{ab} \rightarrow 0 \tag{4}$$

where Y_i is the standard 2-complex model of the presentation $\langle \mathbf{x} \mid \hat{\mathbf{r}}_i \rangle$ of G_i , and a short exact sequence

$$0 \rightarrow \pi_2 Y \rightarrow \bigoplus_{\mathbf{r}} \mathbb{Z}G \rightarrow R^{ab} \rightarrow 0$$

where $\mathbf{r} = \mathbf{r}_1 \cup \dots \cup \mathbf{r}_n$ and Y is the 2-complex model of the presentation $\langle \mathbf{x} \mid \hat{\mathbf{r}} \rangle$ of G . Tensoring each sequence (4) with $\mathbb{Z}G$ and summing over i yields the commutative diagram

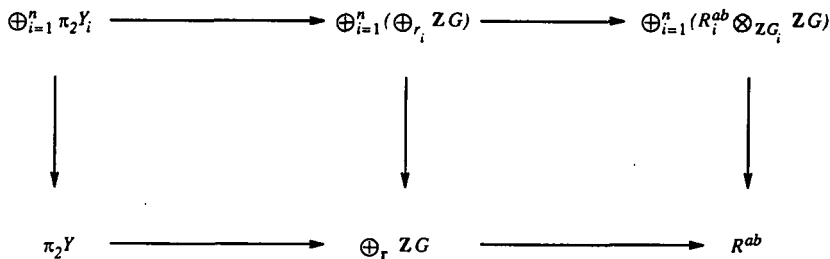


Figure 2.1

from which it follows that the map $\bigoplus_{i=1}^n \pi_2 Y_i \xrightarrow{\alpha} \pi_2 Y$ is surjective.

Let Z_i be the 2-complex model of the presentation

$$\langle \mathbf{x} \mid \hat{\mathbf{r}}_1 \cup \dots \cup \hat{\mathbf{r}}_{i-1} \cup \hat{\mathbf{r}}_{i+1} \cup \dots \cup \hat{\mathbf{r}}_n \rangle$$

of F/N_i . Then the map α factors through $\pi_2 Y_i \oplus \pi_2 Z_i \xrightarrow{\beta} \pi_2 Y$ which is therefore surjective: however, by [11] its cokernel is $R_i \cap N_i/[R_i, N_i]$.

In [4, 7] hyper-relative homology groups are considered for any group F relative to a family of normal subgroups $\{N_1, \dots, N_n\}$. We recall from [7] that the first hyper-relative homology group $H_1(F; N_1, \dots, N_n)$ may be defined algebraically as

$$H_1(F; N_1, \dots, N_n) = \frac{N_1 \cap \dots \cap N_n}{\prod_{I \subseteq \{1, \dots, n\}} \left[\bigcap_{i \in I} N_i, \bigcap_{j \notin I} N_j \right]},$$

with $\bigcap_{\emptyset} N_i$ interpreted as R .

PROPOSITION 2.3. Let F be a free group and $\{R_1, \dots, R_n\}$ a family of independent normal subgroups of F , with $n > 1$. Then

- (a) $H_1(F; N_1, \dots, N_n) = 0$,
- (b) for each i with $1 \leq i \leq n$, we have $H_2(R/N_i) = 0 = H_2(R/R_i)$.

Proof. (a) Recall from the proof of Theorem 2.1 that, assuming the independence of the R_i , we have $N_1 \cap \dots \cap N_n = \prod_{i=1}^n [R_i, N_i]$. We shall show that

$$\prod_{i=1}^n [R_i, N_i] \subseteq \prod_{I \subseteq \{1, \dots, n\}} \left[\bigcap_{l \in I} N_l, \bigcap_{j \notin I} N_j \right].$$

Consider a term in the product on the right-hand side with $I = \{i\}$. Then by Lemma 2.2,

$$\left[N_i, \bigcap_{j \neq i} N_j \right] = \left[N_i, \prod_{j \neq i} (R_j \cap N_j) R_i \right]$$

and the right-hand side here clearly contains $[N_i, R_i]$.

- (b) We use Hopf's formula for H_2 : for R/N_i this gives

$$\begin{aligned} H_2(R/N_i) &= H_2(R_i N_i / N_i) \\ &= (N_i \cap [R_i N_i, R_i N_i]) / [N_i, R_i N_i] \\ &= (N_i \cap [R_i, R_i][N_i, N_i][R_i, N_i]) / [N_i, N_i][N_i, R_i]. \end{aligned}$$

But $[N_i, N_i][N_i, R_i] = [N_i, N_i](R_i \cap N_i) = N_i \cap [N_i, N_i]R_i$ which clearly contains $N_i \cap [R_i, R_i][N_i, N_i][R_i, N_i]$. The proof that $H_2(R/R_i) = 0$ proceeds in the same way.

2.1. Examples

2.1.1 Free products

Consider a free product $G = A_1 * \dots * A_n$. Given presentations $\langle \mathbf{x}_i \mid \mathbf{r}_i \rangle$ of the A_i , we obtain a presentation $\langle \mathbf{x} \mid \mathbf{r} \rangle$ of G , where $\mathbf{x} = \mathbf{x}_1 \sqcup \dots \sqcup \mathbf{x}_n$ and $\mathbf{r} = \mathbf{r}_1 \sqcup \dots \sqcup \mathbf{r}_n$. Let F_i be the free group with basis \mathbf{x}_i and F the free group with basis \mathbf{x} : then R_i is the normal closure of \mathbf{r}_i in F . We write Φ_i for the free group $*_{j \neq i} F_j$. Let M_i be the relation module of the presentation $\langle \mathbf{x}_i \mid \mathbf{r}_i \rangle$, that is, the A_i -module $(\langle \langle \mathbf{r}_i \rangle \rangle^{F_i})^{ab}$: then the following three equivalent statements hold.

- (a) the normal subgroups R_1, \dots, R_n are independent,
- (b) the inclusions $\mathbf{r}_i \hookrightarrow \mathbf{r}$ induce an isomorphism

$$R^{ab} \cong \bigoplus_{i=1}^n M_i \otimes_{A_i} \mathbb{Z}G$$

of $\mathbb{Z}G$ -modules,

- (c) the inclusions $\mathbf{r}_i \hookrightarrow \mathbf{r}$ induce an isomorphism

$$R^{ab} \cong \bigoplus_{i=1}^n (\langle \langle \mathbf{r}_i \rangle \rangle^F)^{ab} \otimes_{G_i} \mathbb{Z}G$$

of $\mathbb{Z}G$ -modules, where G_i is the free product $\Phi_i * A_i$ with presentation $\langle \mathbf{x} \mid \mathbf{r}_i \rangle$.

The equivalence of (b) and (c) follows from the group isomorphism $R_i \cong *_{t \in T} t^{-1} \langle\langle \mathbf{r}_i \rangle\rangle^{F_i} t$, where T is a transversal for the double cosets $F_i \backslash F / \langle\langle \mathbf{r}_i \rangle\rangle^{F_i}$, and the equivalence of (a) and (c) follows from Theorem 2.1. It is easy to deduce (b) from the relation sequence

$$0 \rightarrow R^{ab} \rightarrow \bigoplus_{\mathbf{x}} \mathbb{Z}G \rightarrow IG \rightarrow 0$$

and the decomposition of the augmentation ideal IG as $IG \cong \bigoplus_{i=1}^n IA_i \otimes_{A_i} \mathbb{Z}G$, see [12].

2.1.2 Efficient factorizations of free groups

Let $\langle \mathbf{x} \mid \mathbf{r} \rangle$ be a balanced finite presentation of the trivial group, and let F be the free group with basis \mathbf{x} . For any partition $\mathbf{r} = \mathbf{r}_1 \sqcup \dots \sqcup \mathbf{r}_n$, we let R_i denote the normal closure in F of \mathbf{r}_i . Bogley [1] calls the product $F = R_1 \dots R_n$ an *efficient factorization* of F and conjectures that if R and S are distinct factors in an efficient factorization then $R \cap S = [R, S]$. Bogley proves this conjecture for $n = 2$, and this case shows that the factors in an efficient factorization of a finitely generated free group are independent. Here we have a good source of examples. If Bogley’s conjecture is true, then the factors of an efficient factorization are an independent set of normal subgroups every subset of which is also independent.

3. Independence and one-relator products. A *one-relator product* is the quotient of a free product by the normal closure of a single element. Let $G = A * B / \langle\langle w \rangle\rangle$ be a one-relator product, where w is cyclically reduced and of length at least 2 in $A * B$. We write $w = u^m$ for $m \geq 1$ as large as possible, so that u is not a proper power in $A * B$. Howie [13, 14, 15] and Duncan and Howie [6] have established relation module decompositions for certain one-relator products. The group G is said to be *exceptional* of type $E(p, q, m)$ if some cyclic permutation of $u \in A * B$ can be written as $aVbV^{-1}$ with $a, b \in A \sqcup B$ and $V \in A * B$ such that $a^p = b^q = 1$ in $A * B$ and $1/p + 1/q + 1/m > 1$. Note that if G is exceptional then there are at least two cyclic permutations of u of the given form, namely $aVbV^{-1}$ and $bV^{-1}aV$. If there are exactly two such cyclic permutations of u then G is *uniquely exceptional*.

We obtain a presentation for G , from given presentations $\langle \mathbf{x}_1 \mid \mathbf{r}_1 \rangle$ and $\langle \mathbf{x}_2 \mid \mathbf{r}_2 \rangle$ for A and B respectively, in the following way. Let F be the free group with basis $\mathbf{x} = \mathbf{x}_1 \sqcup \mathbf{x}_2$. We replace u by a word in F mapping to it, and so obtain the presentation $\langle \mathbf{x} \mid \mathbf{r}_1, \mathbf{r}_2, u^m \rangle$ of G . We let F_i be the free group with basis \mathbf{x}_i and set $G_1 = A * F_2$, $G_2 = F_1 * B$. We set $R_i = \langle\langle \mathbf{r}_i \rangle\rangle^F$, $R_3 = \langle\langle u^m \rangle\rangle^F$ and $R = \langle\langle \mathbf{r}_1, \mathbf{r}_2, u^m \rangle\rangle^F$.

Suppose that either

- (1) $m \geq 4$ and that G is not exceptional [14], or
- (2) that $m \geq 3$ and $u \in A * B$ contains no letter of order 2 [6], or
- (3) A and B are locally indicable [13].

Then the relation module R^{ab} decomposes as a G -module as

$$(M_A \otimes_A \mathbb{Z}G) \oplus (M_B \otimes_B \mathbb{Z}G) \oplus \mathbb{Z}G / (1 - u)$$

where $M_A = (\langle\langle \mathbf{r}_1 \rangle\rangle^{F_1})^{ab}$ and $M_B = (\langle\langle \mathbf{r}_2 \rangle\rangle^{F_2})^{ab}$. It follows that R^{ab} is isomorphic to

$$(R_1^{ab} \otimes_{G_1} \mathbb{Z}G) \oplus (R_2^{ab} \otimes_{G_2} \mathbb{Z}G) \oplus (R_3^{ab} \otimes_{G_3} \mathbb{Z}G)$$

where G_3 is the one-relator group $\langle \mathbf{x} \mid u^m \rangle$, and by Lyndon's Identity Theorem, $R_3^{ab} \cong \mathbb{Z}G_3/(1-u)$. From Theorem 2.1 we deduce the following result.

THEOREM 3.1. *Suppose that one of the three conditions (1) to (3) above holds. Then the family $\{R_1, R_2, R_3\}$ is independent.*

If $u \notin [F, F]$ then $H_i(F/N) = H_i(A) \oplus H_i(B)$, ($i = 1, 2, 3, 4$), $H_2(F/M) = 0 = H_4(F/M)$ and $H_3(F/M) = \mathbb{Z}_m$. It then follows from the calculations of $H_i(G)$ in [6, 14, 15] that in Theorem 1.1 (in which we set $\Gamma = F$, $M = R_3$ and $N = R_1R_2$) the terms V_2 and V_3 are zero. If $u \in [F, F]$ then $H_2(F/M) = \mathbb{Z}$. However, as pointed out to us by Bogley, it is still true that $V_2 = 0 = V_3$. For the calculations of second homotopy groups in [6, 14, 15] imply that the map

$$H_2(A) \oplus H_2(B) \oplus H_2(F/M) \rightarrow H_2(G)$$

is injective and that for $i > 2$ we have $H_i(F/M) = H_i(\mathbb{Z}_m)$. From these facts it follows that $V_2 = 0 = V_3$.

In the case where G is a uniquely exceptional group and $m \geq 4$ it is still possible to determine the structure of $R_3 \cap N_3/[R_3, N_3]$, although $\{R_1, R_2, R_3\}$ is not independent. Recall from [11] that $R_3 \cap N_3/[R_3, N_3] \cong \text{coker}(\beta)$, where for suitably chosen complexes Y_3, Z_3 and Y , we have $\beta: \pi_2 Y_3 \otimes \pi_2 Z_3 \rightarrow \pi_2 Y$. In fact, as before, we can take Y_3, Z_3 and Y to be the standard 2-complex models of $\langle \mathbf{x} \mid u^m \rangle$ and $\langle \mathbf{x} \mid \mathbf{r}_1, \mathbf{r}_2 \rangle$ and $\langle \mathbf{x} \mid \mathbf{r}_1, \mathbf{r}_2, u^m \rangle$, respectively. If G is uniquely exceptional with $m \geq 4$, then it follows from Section 4 of [14] that $R_3 \cap N_3/[R_3, N_3]$ is generated as a G -module by a single element.

The detailed argument involves the representation of elements of $\pi_2 Y$ by *spherical pictures* over G , and the generator for $R_3 \cap N_3/[R_3, N_3]$ is obtained from a picture of type $\Gamma(p, q, m)$ over G , see [14]. In terms of pictures, the quotient map $\pi_2 Y \rightarrow R_3 \cap N_3/[R_3, N_3]$ amounts to choosing an *equator* [2] on a spherical picture Γ . To construct an equator we choose a minimal subtree T of Γ (regarding vertices as 0-dimensional) and then choose a regular neighbourhood N of $T \cup \{\text{vertices of } \Gamma\}$ such that no region of Γ is contained in N . The boundary of the closure of N is then an equator on Γ . The equator bounds a picture over G on the disc, the boundary label of which is an element of $R_3 \cap N_3$.

The word labelling the boundary can be of considerable complexity, which we illustrate with G uniquely exceptional of type $E(2, 3, 4)$. Here $u = aVbV^{-1}$ with $a^2 = b^3 = 1$. Let $c = V^{-1}aV$: then $(R_3 \cap N_3)/[R_3, N_3]$ is generated over G by

$$c((bc)^3b)((bc)^2b)b((bc)^2b)b(bc)^4(c(bc)^3)(c(bc))(c(bc)^3)(c(bc)^3).$$

This is the label on an equator of the $\Gamma(2, 3, 4)$ picture shown in Figure 3.1.

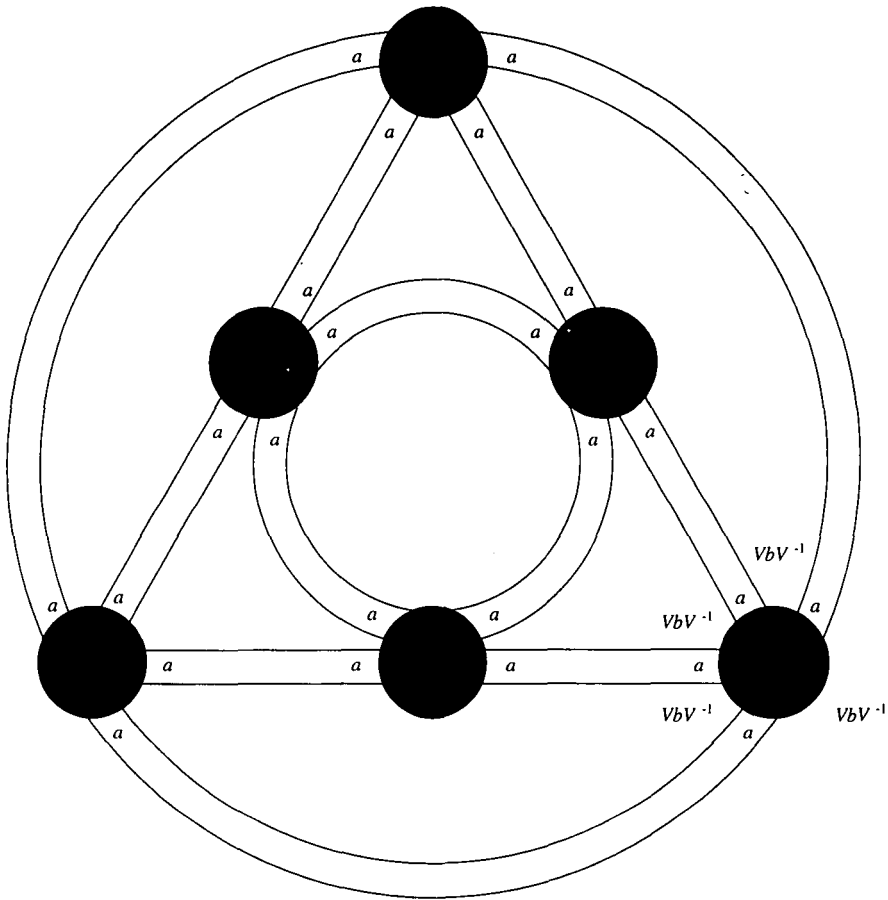


Figure 3.1

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