DYNAMICAL THEORY OF COLLISIONLESS RELAXATION

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In his classical paper on violent relaxation, Lynden-Bell (1967) gave convincing but qualitative arguments for the approach of collisionless systems towards a stationary state and, applying the methods of equilibrium statistical mechanics to a coarse-grained description of the system, derived the equilibrium distribution attained. Here we report on a dynamical derivation of Lynden-Bell equilibrium, proceeding from the coarse-graining of the collisionless Boltzmann equation.

The coarse-graining is introduced by subdividing phase space into macrocells of volume Γ corresponding to the finite resolving power of observations. The coarse-grained phase density $F(\underset{\sim}{r},\underset{\sim}{v},t) = \langle\langle f(\underset{\sim}{r},\underset{\sim}{v},t)\rangle\rangle$ is coupled to its fluctuation $\delta f = f - F$ by the fluctuation of the mean field $\delta\underset{\sim}{a} = \underset{\sim}{a} - \langle\langle\underset{\sim}{a}\rangle\rangle$

$$(\partial/\partial t + L)F = \langle\langle\delta\underset{\sim}{a}.\partial\delta f/\partial\underset{\sim}{v}\rangle\rangle \quad (1a)$$

$$(\partial/\partial t + L)\delta f = -\delta\underset{\sim}{a}.\partial F/\partial\underset{\sim}{v} - \delta\underset{\sim}{a}.\partial\delta f/\partial\underset{\sim}{v} + \langle\langle\partial\underset{\sim}{a}.\partial\delta f/\partial\underset{\sim}{v}\rangle\rangle \quad (1b)$$

Here $L = \underset{\sim}{v}.\partial/\partial\underset{\sim}{r} + \langle\langle\underset{\sim}{a}\rangle\rangle.\partial/\partial\underset{\sim}{v}$.

The fluctuations δf and $\delta\underset{\sim}{a}$ (which is a functional of δf) can be expressed in terms of the 2-point correlation function $G_{12} = \langle\langle\delta f_1\delta f_2\rangle\rangle$. Assuming that G_{12} vanishes unless both fluctuations lie in the same macrocell, and for the case where f is piecewise uniform ($f(r,v) = \eta$ or $= 0$), one can approximate $G_{12} \simeq F_1(\eta-F_1)\,\Gamma\delta(\underset{\sim}{r}_1-\underset{\sim}{r}_2)\delta(\underset{\sim}{v}_1-\underset{\sim}{v}_2)$, where $\delta(x)$ is the Dirac delta function and $F_1 = F(\underset{\sim}{r}_1,\underset{\sim}{v}_1,t)$.

The system (1) can then be solved using standard methods and approximations of statistical mechanics. A close analogy can in fact be exhibited on the one hand with the theory of weak plasma turbulence, and on the other, with the theory of collisional systems. The final result, involving essentially a linearisation of Eq. (1b), a weak-coupling approximation, and a long time

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approximation ($t \gg t_c$ = crossing time), is a generalized Fokker-Planck equation for F_1:

$$(\partial/\partial t + L_1)F = \Gamma\int dx_2 \delta(r_1 - r_2)\Psi F_1 F_2 (\eta - F_2) \quad (2)$$

where, up to a constant factor, Ψ is the Fokker-Planck operator of the collisional theory. The "collision" integral on the right hand side has previously been obtained in the rather different context of homogeneous plasma turbulence, by Kadomtsev and Pogutse (1970).

The properties of Ψ ensure that one can establish an H-theorem, showing that F attains monotonously the Lynden-Bell equilibrium form $\eta[1+\exp\beta\{\varepsilon(x)-\mu\}]^{-1}$. The corresponding relaxation time is reduced with respect to its value for collisional systems by a factor of the order $\eta\Gamma/m$, the ratio of the mass of a full macrocell to the mass of a star. This ratio can be very large but the validity of the theory is restricted to $\eta\Gamma \ll M/26$, where M is the total mass of the system.

The theory has been generalized to the case of a density spectrum $\{\eta_i\}$, as considered by Lynden-Bell. While it is only marginally applicable to the violent relaxation process itself - in particular, by reason of the restriction to long times - the theory does provide a quantitative description of its final, quiescent phases.

REFERENCES

Lynden-Bell, D.: 1967, *Mon. Not. Roy. Astron. Soc.* 136, 101.
Kadomtsev, B. and Pogutse, O.: 1970, *Phys. Rev. Letters* 25, 1155.