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Spherical Functions on $SO_0(p,q)/SO(p) \times SO(q)$

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Abstract. An integral formula is derived for the spherical functions on the symmetric space $G/K = SO_0(p,q)/SO(p) \times SO(q)$. This formula allows us to state some results about the analytic continuation of the spherical functions to a tubular neighbourhood of the subalgebra $\mathfrak{a}$ of the abelian part in the decomposition $G = KAK$. The corresponding result is then obtained for the heat kernel of the symmetric space $SO_0(p,q)/SO(p) \times SO(q)$ using the Plancherel formula.

In the Conclusion, we discuss how this analytic continuation can be a helpful tool to study the growth of the heat kernel.

1 Introduction

We refer to [6] and [7] for the standard notation and results.

The (elementary) spherical functions on a symmetric space G/K are defined by

$$\phi_\lambda(g) = \int_K e^{(i\lambda-\rho)(H(gk))}\,dk \tag{1}$$

where $H(g)$ is the log of the abelian part of g in the Iwasawa decomposition $G = KAN$. Since $G = KAK$, any function such as ϕ_λ which is left and right invariant under the action of K can be seen as a function on the Lie algebra $\mathfrak{a}$ of A: $H \to \phi_\lambda(e^H)$. The spherical functions are eigenfunctions of the Laplace-Beltrami operator $L_{G/K}$ on the symmetric space. If G/K is of noncompact type, the radial part of $L_{G/K}$ in terms of the "polar coordinates" is given in [7, Proposition 3.9, Chapter II] as

$$\Delta(L_{G/K}) = L_{\mathfrak{a}} + \sum_{\alpha\in\Sigma^+} m_\alpha \coth \alpha H_\alpha. \tag{2}$$

Here Σ^+ is the set of positive roots, $H_\alpha \in \mathfrak{a}$ is defined by the equation $\langle H_\alpha, H\rangle = \alpha(H)$ and is identified with a differential operator on $\mathfrak{a}$ in the usual way (the scalar product on $\mathfrak{a}$ is a fixed multiple of the Killing form). We denote $L_{\mathfrak{a}}$ as the Laplacian on the Euclidean space $\mathfrak{a}$.

It is known (see for instance [7, Proposition 2.2, Chapter IV]) that the spherical functions are real analytic on $\mathfrak{a}$.

This paper discusses the analytic continuation of ϕ_λ to an open neighbourhood of $\mathfrak{a}$ in $\mathfrak{a}_{\mathbf{C}}$ (the complexification of $\mathfrak{a}$).

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This question has been studied before. Based on equation (1), the analytic continuation of the function $g \to H(g)$ is clearly relevant. The analytic continuation of the function $g \to a^2(g) = e^{2H(g)}$ is discussed in [2] by Jean-Louis Clerc. Building on the results of E. P. Van den Ban in [17], Clerc shows that the function $g \to a^2(g)$ can be continued analytically in the open set $K_{\mathbf{C}}A_{\mathbf{C}}N_{\mathbf{C}}$ of $G_{\mathbf{C}}$ (the groups which correspond to the complexification of the Lie algebras K, A N and G).

As mentioned above, we wish to consider ϕ_λ as a function on $\mathfrak{a}$. The question, in this setting, has been discussed in the papers [4, 5, 9, 10, 11, 12] by G. J. Heckman and E. M. Opdam.

The existence of a tubular neighbourhood of $\mathfrak{a}$ in $\mathfrak{a}_{\mathbf{C}}$ is shown in [12, Theorem 3.15]. However, it is not described explicitly.

Given the terms $\coth \alpha$ in (2), one would expect that a natural boundary for that neighbourhood would be $\Im\alpha = \pm\pi$, $\alpha \in \Sigma^+$ (we use $\Re z$ to denote the real part of z and $\Im z$ for the imaginary part of z).

Let us introduce some notation.

Definition 1 For $\eta > 0$, let $\Omega_\eta = \{H \in \mathfrak{a}_{\mathbf{C}} : |\Im\alpha(H)| < \eta \text{ for all } \alpha \in R\}$ where R is the root system.

The discussion above brings us to the following conjecture.

Conjecture 2 The spherical functions can be extended analytically to the tubular neighbourhood Ω_π of $\mathfrak{a}$ in $\mathfrak{a}_{\mathbf{C}}$.

This is easily seen to be valid when G is a complex group. In the rank one case, the conjecture can be verified by checking all the possibilities (see for instance [1]). The conjecture is also valid in the case of $\mathrm{SL}(n, \mathbf{R})/\mathrm{SO}(n)$ (see [16, Theorem 3.1]). A careful analysis of the results of [14] show that it is also valid for the space $\mathrm{SU}^*(2n)/\mathrm{Sp}(n)$. In each of these cases, it is easy to see that no larger neighbourhood of $\mathfrak{a}$ would do.

This paper is organized as follows. In Section 2, we derive the basic Lie algebra structure of $\mathfrak{so}(p, q)$ (see in particular Theorem 5 for the root system of $\mathfrak{so}(p, q)$).

Our goal is to be able to find an analytic expression for the function $g \to H(g)$ found in equation (1). An important step toward that goal is finding the orthogonal matrix S of Theorem 9. Indeed, if $g = ke^{H(g)}n$ is the Iwasawa decomposition of $g \in \mathrm{SO}(p, q)$ then $S^{-1}gS = (S^{-1}kS)e^{S^{-1}H(g)S}(S^{-1}nS)$ is the Iwasawa decomposition of an element of $\mathrm{SL}(p + q, \mathbf{R})$ which is something we know how to handle (see for instance [6, Exercise A.2, p. 434]).

Theorem 16 of Section 3 is the main calculation of the paper. In that result, we derive an expression for the spherical functions associated with the symmetric space $\mathrm{SO}_0(p, q)/\mathrm{SO}(p) \times \mathrm{SO}(q)$. This expression allows us, in Section 4, to give an explicit continuation of the spherical functions on a tubular neighbourhood of $\mathfrak{a}$. The tubular neighbourhood, $\tilde{\Omega}_{\pi/4}$ of Lemma 18, is smaller than what we predict in our conjecture.

Finally, after we find a crude estimate on $|\phi_\lambda|$ in Proposition 24, we show in Theorem 26 that the heat kernel can be extended analytically in the space variable to the same neighbourhood using the Plancherel formula. In [3], François Golse, Eric Leichtnam and Matthew Stenzel discuss the question of the analytic continuation of the heat kernel for real-analytic compact Riemannian manifolds. They discuss in particular the case where the manifold is locally symmetric and conclude by considering rank one compact symmetric spaces.

The fact that we are dealing with symmetric spaces make the question (relatively) simpler. In the Conclusion, we discuss how such results for other symmetric spaces were useful in the past when dealing with a conjecture Jean-Philippe Anker made in [1] on the growth of the heat kernel.

2 Roots and root vectors

We start by collecting some useful information on the Lie group $\mathrm{SO}_0(p,q)$ and on its Lie algebra $\mathfrak{so}(p,q)$. Our main goal is then to derive a practical method of computing the Iwasawa decomposition. This will be accomplished with the help of the matrix S as described in the Introduction and as given in Theorem 9. To that end, we need to describe the root system of $\mathfrak{so}(p,q)$. This is found in Table 1 of Theorem 5. All of this is accomplished by finding suitable block decompositions for the matrices belonging in $\mathrm{SO}(p,q)$ and in $\mathfrak{so}(p,q)$.

In what follows, E_{ij} is a rectangular matrix of appropriate size with 0's everywhere except at at the position (i, j) where it is 1. We will also assume, without loss of generality, that $p \leq q$.

Recall that $\mathrm{SO}(p,q)$ is the group of matrices $g \in \mathrm{SL}(p+q,\mathbf{R})$ such that $g^T I_{p,q} g = I_{p,q}$ where $I_{p,q} = \left[\begin{smallmatrix} -I_p & 0_{p\times q} \\ 0_{q\times p} & I_q \end{smallmatrix}\right]$. The subscripts in the block decomposition of $I_{p,q}$ indicates the size of the blocks. Unless otherwise specified, all 2×2 block decompositions in this paper follow the same pattern. If $g = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] \in \mathrm{SO}(p,q)$ then

$$\begin{cases} A^TA - C^TC = I_p, \\ D^TD - B^TB = I_q, \\ C^TD - A^TB = 0. \end{cases} \tag{3}$$

The group $\mathrm{SO}_0(p,q)$ is the connected component of $\mathrm{SO}(p,q)$ containing the identity. The Lie algebra $\mathfrak{so}(p,q)$ of $\mathrm{SO}_0(p,q)$ consists of the matrices $\left[\begin{smallmatrix} A & B \\ B^T & D \end{smallmatrix}\right]$ where A and D are skew-symmetric.

The following proposition is a first step toward describing the root system:

Proposition 3 *The Killing form of $\mathfrak{so}(p,q)$ is $B(X,Y) = (q+p-2)\,\mathrm{tr}(XY)$.*

Proof Denote $\tilde{B}$ as the Killing form of the Lie algebra $\mathfrak{so}(p+q,\mathbf{C})$. Since $\mathfrak{so}(p,q)$ is a real form of $\mathfrak{so}(p+q,\mathbf{C})$, we have $B(X,Y) = \tilde{B}(X,Y)$ for X, $Y \in \mathfrak{so}(p,q)$ (see [6, Lemma 6.1, Chapter III]). Since $\tilde{B}(X,Y) = (q+p-2)\,\mathrm{tr}(XY)$ (see [6, (16) p. 189]) the proof is complete. ■

Another necessary element in our investigations is the Cartan decomposition of $\mathfrak{so}(p,q)$.

Proposition 4 *Let K be the subgroup of $\mathrm{SO}(p,q)$ consisting of the matrices $\left[\begin{smallmatrix} A & 0 \\ 0 & D \end{smallmatrix}\right]$ of size $(p+q)\times(p+q)$ such that $A \in \mathrm{SO}(p)$ and $D \in \mathrm{SO}(q)$ (hence $K \simeq \mathrm{SO}(p)\times\mathrm{SO}(p)$). If $\mathfrak{k}$ is*

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the Lie algebra of K and $\mathfrak{p}$ is the set of matrices $\left[\begin{smallmatrix} 0 & B \\ B^T & 0 \end{smallmatrix}\right]$ then the Cartan decomposition is given by $\mathfrak{so}(p,q) = \mathfrak{k} \oplus \mathfrak{p}$ with corresponding Cartan involution $\theta(X) = -X^T$.

Proof This is straightforward if one uses Proposition 3 and [6, Proposition 7.4, Chapter III]. ■

The next result is in itself interesting. It gives an explicit description of the root system for the Lie algebra $\mathfrak{so}(p,q)$.

Theorem 5 *We continue with the notation of Proposition 4. Let $Q = \left[\begin{smallmatrix} I_p \\ 0_{(q-p)\times p} \end{smallmatrix}\right]$. Define $\mathfrak{a} \subset \mathfrak{p}$ to be the set of matrices $H = \left[\begin{smallmatrix} 0 & \mathcal{D}Q^T \\ Q\mathcal{D} & 0 \end{smallmatrix}\right]$ where $\mathcal{D} = \operatorname{diag}[H_1, \ldots, H_p]$. Define $\mathfrak{m} \subset \mathfrak{k}$ to be the set of matrices $\left[\begin{smallmatrix} 0_{(2p)\times(2p)} & 0_{(2p)\times(q-p)} \\ 0_{(q-p)\times(2p)} & F_{(q-p)\times(q-p)} \end{smallmatrix}\right]$ where F is skew-symmetric. Let $\mathfrak{h}_{\mathfrak{k}}$ be any maximal abelian subalgebra of $\mathfrak{m}$ and let $\mathfrak{h} = \mathfrak{h}_{\mathfrak{k}} \oplus \mathfrak{a}$. Then $\mathfrak{h}$ is a Cartan subalgebra of the Lie algebra $\mathfrak{so}(p,q)$. The restricted roots and associated root vectors for the Lie algebra $\mathfrak{so}(p,q)$ with respect to $\mathfrak{a}$ are given in Table 1.*

root α	multiplicity	root vectors X_α
$\alpha(H) = \pm H_i$ $1 \le i \le p$	$q-p$	$X_{ir}^{\pm} = E_{i2p+r} + E_{2p+ri} \pm (E_{p+i2p+r} - E_{2p+rp+i})$ $r = 1, \ldots, q-p$
$\alpha(H) = \pm(H_i - H_j)$ $1 \le i,j \le p, i<j$	1	$Y_{ij}^{\pm} = \pm(E_{ij} - E_{ji} + E_{p+ip+j} - E_{p+jp+i})$ $+ E_{ip+j} + E_{p+ji} + E_{jp+i} + E_{p+ij}$
$\alpha(H) = \pm(H_i + H_j)$ $1 \le i,j \le p, i<j$	1	$Z_{ij}^{\pm} = \pm(E_{ij} - E_{ji} - E_{p+ip+j} + E_{p+jp+i})$ $-(E_{ip+j} + E_{p+ji}) + E_{jp+i} + E_{p+ij}$

Table 1: Restricted roots and associated root vectors

Proof If we use the appropriate block decomposition on the elements of $\mathfrak{h}$ and of $\mathfrak{so}(p,q)$, we find that $\mathfrak{h}$ is a maximal abelian subalgebra of $\mathfrak{so}(p,q)$. We know that $[E_{ij}, E_{rl}] = \delta_{jr}E_{il} - \delta_{li}E_{rj}$ and that $H \in \mathfrak{a}$ can be written as $H = \sum_{j=1}^{p} H_j(E_{jp+j} + E_{p+jj})$. It is easy to verify that the matrices given in Table 1 are indeed root vectors; although it is somewhat tedious (this verification can be done using a computer package such as Maple or Mathematica). We also have

$$\begin{aligned}\mathfrak{so}(p,q) &\\ &= \mathfrak{m} \oplus \mathfrak{a} \bigoplus_{\substack{1\le i\le p \\ 1\le r\le q-p}} \mathbf{R}X_{ir}^{+} \bigoplus_{\substack{1\le i\le p \\ 1\le r\le q-p}} \mathbf{R}X_{ir}^{-} \bigoplus_{\substack{1\le i,j\le p \\ i<j}} \mathbf{R}Y_{ij}^{+} \bigoplus_{\substack{1\le i,j\le p \\ i<j}} \mathbf{R}Y_{ij}^{-} \bigoplus_{\substack{1\le i,j\le p \\ i<j}} \mathbf{R}Z_{ij}^{+} \bigoplus_{\substack{1\le i,j\le p \\ i<j}} \mathbf{R}Z_{ij}^{-}\end{aligned}$$

(the sums on the right hand side are direct and the dimensions of both sides agree). ■

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Remark 6 In the rest of the paper, the relationship between $H \in \mathfrak{a}$, $\mathcal{D}$ and $H_1, \dots, H_p$ will be implicitly assumed.

The positive roots can be chosen as $\alpha(H) = H_i - H_j$, $1 \le i < j \le p$, $\alpha(H) = H_i$, $i = 1, \dots, p$ and $\alpha(H) = H_i + H_j$, $1 \le i < j \le p$.

We therefore have $\mathfrak{a}^+ = \{H \in \mathfrak{a} \colon H_1 > H_2 > \cdots > H_p > 0\}$. The simple roots are given by $\alpha(H) = H_i - H_{i+1}$, $i = 1, \dots, p-1$ and $\alpha(H) = H_p$.

We can now describe the Iwasawa decomposition of the Lie algebra:

Corollary 7 *The Iwasawa decomposition of the Lie algebra* $\mathfrak{so}(p,q)$ *is as follows:* $\mathfrak{so}(p,q) = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}$ *where* $\mathfrak{k}$ *and* $\mathfrak{a}$ *are as described before and* $\mathfrak{n} = \bigoplus_{\substack{1 \le i \le p \\ 1 \le r \le q-p}} \mathbf{R}X^+_{ir} \bigoplus_{1 \le i < j \le p} \mathbf{R}Y^+_{ij} \bigoplus_{1 \le i < j \le p} \mathbf{R}Z^+_{ij}$.

Proof This proof is straightforward when we refer to Proposition 4, to Table 1 and to our choice of positive roots in Remark 6. ■

The Iwasawa decomposition of the Lie group $\mathrm{SO}_0(p,q)$ is immediate consequence of Corollary 7.

Corollary 8 *If* $A = \exp(\mathfrak{a})$ *and* $N = \exp(\mathfrak{n})$ *then* $\mathrm{SO}_0(p,q) = KAN$.

In the next result, we introduce the matrix S which allows us to diagonalize simultaneously all the elements of $\mathfrak{a}$ and to express the elements of $\mathfrak{n}$ as upper triangular matrices with 0's on the diagonal. This will enable us to compute the function $g \to H(g)$ in Proposition 14.

Theorem 9 *Let*

$$S = \begin{bmatrix} \frac{\sqrt{2}}{2} I_p & 0_{p \times (q-p)} & \frac{\sqrt{2}}{2} J_p \\ \frac{\sqrt{2}}{2} I_p & 0_{p \times (q-p)} & -\frac{\sqrt{2}}{2} J_p \\ 0_{(q-p) \times p} & I_{q-p} & 0_{(q-p) \times p} \end{bmatrix}$$

where $J_p = (\delta_{i,p+1-i})$ *is a matrix of size* $p \times p$. *If* $X \in \mathfrak{n}$ *then* $S^T X S$ *is an upper triangular matrix with 0's on the diagonal. Moreover, if* $H = \left[\begin{smallmatrix} 0 & \mathcal{D}Q^T \\ Q\mathcal{D} & 0 \end{smallmatrix}\right]$ *with* $\mathcal{D} = \mathrm{diag}[H_1, \dots, H_p]$ *then* $S^T H S = \mathrm{diag}[H_1, \dots, H_p, \overbrace{0, \dots, 0}^{q-p}, -H_p, \dots, -H_1]$.

Proof Refer to Table 1. We need only compute $S^T X^+_{ir} S$, $S^T Y^+_{ij} S$, $S^T Z^+_{ij} S$ (in each case with $i < j$) and $S^T H S$. The verification is simpler if we make use of block decomposition: $X^+_{ir} = \left[\begin{smallmatrix} 0 & 0 & E_{ir} \\ 0 & 0 & E_{ir} \\ E_{ri} & -E_{ri} & 0 \end{smallmatrix}\right]$, $Y^+_{ij} = \left[\begin{smallmatrix} E_{ij}-E_{ji} & E_{ij}+E_{ji} & 0 \\ E_{ij}+E_{ji} & E_{ij}-E_{ji} & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right]$, $Z^+_{ij} = \left[\begin{smallmatrix} E_{ij}-E_{ji} & -E_{ij}+E_{ji} & 0 \\ E_{ij}-E_{ji} & E_{ji}-E_{ij} & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right]$ and $H = \left[\begin{smallmatrix} 0 & \mathcal{D} & 0 \\ \mathcal{D} & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right]$ (these block decompositions are to make use of the block decomposition of S). Using block multiplication, the fact that $J_p E_{ij} = E_{p+1-ij}$ and $E_{ij} J_p = E_{ip+1-j}$, the rest of the proof is straightforward. ■

Naturally, we can also state the "group" version of this result.

Corollary 10 *Suppose S is as in the above theorem. If $H = \left[\begin{smallmatrix} 0 & \mathcal{D}Q^T \\ Q\mathcal{D} & 0 \end{smallmatrix}\right]$ where $\mathcal{D} = \operatorname{diag}[H_1, \dots, H_p]$ then $S^T e^H S = \operatorname{diag}[e^{H_1}, \dots, e^{H_p}, \overbrace{1, \dots, 1}^{q-p}, e^{-H_p}, \dots, e^{-H_1}]$ and if $n \in N$ then $S^T nS$ is an upper triangular matrix with 1's on the diagonal.*

Proof Since $S^T = S^{-1}$ and $\exp(S^{-1}XS) = S^{-1}\exp(X)S$, this follows directly from Theorem 9. ■

Corollary 11 *If $\rho = \frac{1}{2}\sum_{\alpha>0} m_\alpha \alpha$ (the half-sum of the positive roots counting multiplicities) then $\rho(H) = \sum_{i=1}^{p}(\frac{p+q}{2} - i)H_i$.*

Proof $2\rho(H) = (q-p)\sum_{i=1}^{p} H_i + \sum_{i<j}(H_i - H_j) + \sum_{i<j}(H_i + H_j) = (q-p)\sum_{i=1}^{p} H_i + 2\sum_{i<j} H_i = (q-p)\sum_{i=1}^{p} H_i + 2\sum_{i=1}^{p}(p-i)H_i = \sum_{i=1}^{p}\big(q - p + 2(p-i)\big)H_i = \sum_{i=1}^{p}(q + p - 2i)H_i$. ■

The Weyl group is another important object relating to the root system of $\mathfrak{so}(p,q)$.

Proposition 12 *The Weyl group of $\mathfrak{so}(p,q)$ is $W = \{-1,1\}^p \times S_p$ where S_p is the symmetric group on p letters. The action is as follows: if $H \in \mathfrak{a}$ then $\big((\epsilon_1, \dots, \epsilon_p)\big) \times \sigma) \cdot H = \tilde{H}$ with $\tilde{H}_r = \epsilon_r H_{\sigma(r)}$.*

Proof This is straightforward if we use the definition given in [6, p. 284] and the structure of the Lie group $\mathrm{SO}_0(p,q)$ as described above. ■

3 Spherical functions

The goal of this section is to achieve an analytic expression for the spherical functions defined by equation (1) for the space $\mathrm{SO}_0(p,q)/\mathrm{SO}(p) \times \mathrm{SO}(q)$. In Proposition 14, we give an explicit method of computing the function $g \to H(g)$ of (1). An analytic expression for the spherical functions is achieved in Theorem 23.

Definition 13 If $A = (a_{ij})$ is a square matrix, the r-th *Gram determinant* of A is $\Delta_r(A) = \det((a_{ij})_{1\le i,j\le r})$. We will also write $\Delta_0(A) = 1$.

T. S. Bhanu Murti uses the Gram determinant explicitly in [8] to describe the Plancherel measure on $\mathrm{SL}(n,\mathbf{R})/\mathrm{SO}(n)$. The reader may also wish to look at [6, Exercise A.2, page 434] and at [15].

Proposition 14 *If $g = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] \in \mathrm{SO}_0(p,q)$ is written as $g = k\exp(H)n$ (the Iwasawa decomposition) then for $r = 1, \dots, p$ we have*

$$H_r = \frac{1}{2}\log\frac{\Delta_r\big((A+BQ)^T(A+BQ)\big)}{\Delta_{r-1}\big((A+BQ)^T(A+BQ)\big)}. \tag{4}$$

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Proof This is a consequence of Corollary 10. Indeed, if $g = k\exp(H)n$ then

$$\Delta_r(S^T g^T gS) = \Delta_r\big(S^T n^T \exp(2H) nS\big) = \Delta_r\Big((S^T nS)^T \big(S^T \exp(2H) S\big)(S^T nS)\Big).$$

Since $S^T \exp(2H)S$ is a diagonal matrix and $S^T nS$ is an upper triangular matrix with 1's on the diagonal, $\Delta_r(S^T g^T gS)$ is the product of the first r diagonal entries of $S^T \exp(2H)S$. It then remains to verify that the upper right $p \times p$ block of the matrix $S^T g^T gS$ is indeed $(A + BQ)^T(A + BQ)$. This is straightforward using block multiplication, the fact that $S = \begin{bmatrix} \frac{\sqrt{2}}{2} I_p & * \\ \frac{\sqrt{2}}{2} Q & * \end{bmatrix}$ and the relations in (3). ■

This next lemma allows us to compute explicitly e^H when $H \in \mathfrak{a}$.

Lemma 15 *We have* $\exp\left(\begin{bmatrix} 0 & Y \\ Y^T & 0 \end{bmatrix}\right) = \begin{bmatrix} \cosh(\sqrt{YY^T}) & \frac{\sinh(\sqrt{YY^T})}{\sqrt{YY^T}} Y \\ Y^T \frac{\sinh(\sqrt{YY^T})}{\sqrt{YY^T}} & \cosh(\sqrt{Y^T Y}) \end{bmatrix}$.

Proof This proof is straightforward since $\begin{bmatrix} 0 & Y \\ Y^T & 0 \end{bmatrix}^{2r} = \begin{bmatrix} (YY^T)^r & 0 \\ 0 & (YY^T)^r \end{bmatrix}$ and $\begin{bmatrix} 0 & Y \\ Y^T & 0 \end{bmatrix}^{2r+1} = \begin{bmatrix} 0 & (YY^T)^r Y \\ Y^T (YY^T)^r & 0 \end{bmatrix}$. ■

In the next result and in the corollary that follows, we give an explicit expression for the spherical functions. These results are the heart of this paper since they will allow us to give an analytic continuation of the spherical functions and of the heat kernel to a larger domain.

Theorem 16 *Suppose λ is a complex-valued linear functional on $\mathfrak{a}$: $\lambda(H) = \sum_{r=1}^{p} a_r H_r$. If $g = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathrm{SO}_0(p,q)$ then*

$$\phi_\lambda(g) = \int_{\mathrm{SO}(q)} \int_{\mathrm{SO}(p)} \Delta_\lambda\big(U^T (A + BVQ)^T (A + BVQ) U\big)\, dU\, dV \tag{5}$$

where $\Delta_\lambda(W) = \prod_{r=1}^{p-1} \Delta_r(W)^{(i(a_r - a_{r+1})-1)/2} \Delta_p(W)^{(ia_p - (q-p)/2)/2}$.

Proof Referring to (1), we find as an immediate consequence of Proposition 14 that

$$\phi_\lambda(g) = \int_{\mathrm{SO}(q)} \int_{\mathrm{SO}(p)} \Delta_\lambda\big((AU + BVQ)^T (AU + BVQ)\big)\, dU\, dV.$$

In what follows, we write $\tilde{U} = \begin{bmatrix} U & 0_{p\times(q-p)} \\ 0_{(q-p)\times p} & I_{q-p} \end{bmatrix} \in \mathrm{SO}(q)$. We have

$$\begin{aligned} \phi_\lambda(g) &= \int_{\mathrm{SO}(q)} \int_{\mathrm{SO}(p)} \Delta_\lambda\big(U^T (A + BVQU^T)^T (A + BVQU^T) U\big)\, dU\, dV \\ &= \int_{\mathrm{SO}(q)} \int_{\mathrm{SO}(p)} \Delta_\lambda\big(U^T (A + BV\tilde{U}^T Q)^T (A + BV\tilde{U}^T Q) U\big)\, dU\, dV \\ &= \int_{\mathrm{SO}(q)} \int_{\mathrm{SO}(p)} \Delta_\lambda\big(U^T (A + BVQ)^T (A + BVQ) U\big)\, dU\, dV \end{aligned}$$

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using the fact that dV is the Haar measure on $\mathrm{SO}(q)$. ■

Stating the result in terms of $g = e^H \in A$:

Corollary 17 *With the same notation as in Theorem 16, if* $H = \begin{bmatrix} 0 & \mathcal{D}Q^T \\ Q\mathcal{D} & 0 \end{bmatrix} \in \mathfrak{a}$ *then*

(6)

$$\phi_\lambda(e^H) = \int_{\mathrm{SO}(q)} \int_{\mathrm{SO}(p)} \Delta_\lambda\Big(U^T\big(\cosh\mathcal{D} + (\sinh\mathcal{D})Q^T V Q\big)^T \big(\cosh\mathcal{D} + (\sinh\mathcal{D})Q^T V Q\big)U\Big)\, dU\, dV.$$

Proof If we write $e^H = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ then $A = \cosh\mathcal{D}$ and $B = (\sinh\mathcal{D})Q^T$ (see Lemma 15). ■

It is not difficult to see that the spherical functions on $\mathrm{SO}_0(p,q)/\,\mathrm{SO}(p)\times\mathrm{SO}(q)$ can be expressed in terms of the spherical functions on $\mathrm{SL}(p,\mathbf{R})/\,\mathrm{SO}(p)$ (see for instance [15]).

4 Analytic Continuation

The difficulty in extending $\phi_\lambda(e^H)$, as given in (6), to a complex domain comes essentially from the need to take the log of the determinant of a complex matrix. We will first consider a domain where taking the log of the determinant can be done analytically, the domain Λ of Definition 19, and then show that for H chosen in an appropriate complex neighbourhood of $\mathfrak{a}$, $U^T(\cosh\mathcal{D} + (\sinh\mathcal{D})Q^T V Q)^T(\cosh\mathcal{D} + (\sinh\mathcal{D})Q^T V Q)U$ of (6) will fall in that domain.

In Definition 1 of the Introduction, we define the domain $\Omega_\eta = \{H \in \mathfrak{a}_{\mathbf{C}} : |\Im\alpha(H)| < \eta$ for all $\alpha \in R\}$ where R is the root system. We now describe that domain in terms of the root system given in Table 1 of Theorem 5.

Lemma 18 *For* $\eta > 0$, *we have* $\Omega_\eta = \{H \in \mathfrak{a}_{\mathbf{C}} : |\Im(H_r \pm H_s)| < \eta$ *for all* $r, s\}$ *and if* $\tilde{\Omega}_\eta = \{H \in \mathfrak{a}_{\mathbf{C}} : |\Im H_r| < \eta$ *for all* $r\}$ *then* $\Omega_\eta \subset \tilde{\Omega}_\eta \subset \Omega_{2\eta}$.

Proof The roots are given in Table 1 of Theorem 5. Note that if $|\Im(H_r \pm H_s)| < \eta$ for all r and s then $-2\eta < \Im(H_r - H_s) + \Im(H_r + H_s) < 2\eta$ *i.e.* $|\Im H_r| < 2\eta$. If $H \in \tilde{\Omega}_\eta$ then $|\Im(H_r \pm H_s)| \le |\Im H_r| + |\Im H_s| < \eta + \eta = 2\eta$. ■

Definition 19 Let $\Lambda = \Lambda_p$ be the set of complex symmetric matrices of size $p \times p$ with positive definite real part. For $P + iQ \in \Lambda$, let

$$g(P + iQ) = \log(\det P) + \sum_{j=1}^{p} \log(1 + i\, d_j) \tag{7}$$

where $d_1, \dots, d_p$ are the (real) eigenvalues of $P^{-1/2}QP^{-1/2}$ and log represents the principal branch of the logarithm.

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The function g is well defined although the eigenvalues $d_1, \dots, d_p$ of $P^{-1/2}QP^{-1/2}$ are only defined up to their order. The domain Λ and the function g of Definition 19 were suggested to us by Professor Robert L. Bryant of Duke University.

Proposition 20 *The function g is analytic on $\Lambda = \Lambda_p$ and* $\exp\bigl(g(P+iQ)\bigr) = \det(P+iQ)$.

Proof We first note that

$$\begin{aligned}
\exp\bigl(g(P+iQ)\bigr) &= \exp\Bigl(\log(\det P) + \sum_{j=1}^{p} \log(1+i\,d_j)\Bigr)\\
&= \exp\bigl(\log(\det P)\bigr) \prod_{j=1}^{p} \exp\bigl(\log(1+i\,d_j)\bigr)\\
&= (\det P) \prod_{j=1}^{p} (1+i\,d_j) = (\det P)\det(I + iP^{-1/2}QP^{-1/2})\\
&= (\det P^{1/2})\det(I + iP^{-1/2}QP^{-1/2})(\det P^{1/2}) = \det(P+iQ).
\end{aligned}$$

Since the function $\exp(g)$ is analytic, it suffices to show that g is continuous. Since the functions $P+iQ \to \det P$ and $P+iQ \to P^{-1/2}QP^{-1/2}$ are continuous, it remains to show that the map which sends the eigenvalues $d_1, \dots, d_p$ of a real symmetric matrix to $\sum_{j=1}^{p} \log(1+i\,d_j)$ is continuous. Since the roots of a polynomial, in this case the roots of the characteristic polynomial of a symmetric matrix, depend continuously on the coefficients of the polynomial, the result follows. ■

We need to extend this result keeping in mind equation (4).

Corollary 21 *Let $(P+iQ)_r$ stand for the $r \times r$ principal minor of $P+iQ$. For $r \le p$, the maps $g_r \colon \Lambda_p \to \mathbf{C}$ defined by $g_r(P+iQ) = g\bigl((P+iQ)_r\bigr)$ are analytic. Moreover, $|\Im g_r| < (\pi/2)r$ for each r.*

Proof It suffices to point out that $(P+iQ)_r \in \Lambda_r$ and to consider the expression (7) for the function g which involves the logarithm of a complex number with positive real part. ■

The next lemma takes the complicated portion of the expression in (6) and shows that it has an analytic continuation.

Lemma 22 *We use the notation of Corollary 17. Fix $V \in \mathrm{SO}(q)$, let $A = A_V = Q^T V Q$ and let $\beta(H) = \beta_A(H) = \bigl(\cosh \mathcal{D} + (\sinh \mathcal{D})A\bigr)^T(\cosh \mathcal{D} + \sinh \mathcal{D} A)$. Then on $\tilde{\Omega}_{\pi/4}$, β is a Λ_p-valued analytic map.*

Proof Let $\mathcal{D} = \mathcal{D}_1 + i\mathcal{D}_2$ where $\mathcal{D}_1$, $\mathcal{D}_2$ are real diagonal matrices. Using $\sinh u = (e^u - e^{-u})/2$ and $\cosh u = (e^u + e^{-u})/2$, we find that

$$\begin{aligned}4\Re\beta(H) &= (I+A)^T e^{2\mathcal{D}_1}\cos(2\mathcal{D}_2)(I+A) + (I-A)^T e^{-2\mathcal{D}_1}\cos(2\mathcal{D}_2)(I-A) + 2(I-A^TA)\\ &= (I+A)^T(e^{2\mathcal{D}_1} - e^{-2|\mathcal{D}_1|})\cos(2\mathcal{D}_2)(I+A)\\ &\quad + (I-A)^T(e^{-2\mathcal{D}_1} - e^{-2|\mathcal{D}_1|})\cos(2\mathcal{D}_2)(I-A)\\ &\quad + 2A^Te^{-2|\mathcal{D}_1|}\cos(2\mathcal{D}_2)A + 2(I-A^TA) + 2e^{-2|\mathcal{D}_1|}\cos(2\mathcal{D}_2).\end{aligned}$$

Note that since A is the $p \times p$ principal minor of an orthogonal matrix, it is clear that $I - A^TA$ is a positive semidefinite matrix.

As long as the absolute values of the entries of $\mathcal{D}_2$ are strictly less than $\pi/4$, $\Re\beta(H)$ is a sum of four positive semidefinite matrices, the last one being a positive definite matrix. ∎

If we refer to Proposition 14, the whole point of introducing the functions $g_1, \ldots, g_r$ is that, for a symmetric positive definite matrix W, $\Delta_r(W) = \exp\bigl(g_r(W)\bigr)$. This gives us an analytic continuation of the function $g \to H(g)$.

We can now state one of the main result of this paper.

Theorem 23 *$\phi_\lambda(e^H)$ has an analytic continuation on $\hat{\Omega}_{\pi/4}$. On that domain, $\phi_\lambda(e^{w\cdot H}) = \phi_\lambda(e^H)$ for every $w \in W$, the Weyl group.*

Proof If we refer to Lemma 22, to equation (6) and write $A_V = Q^TVQ$ then we have

$$\phi_\lambda(e^H) = \int_{\mathrm{SO}(q)}\int_{\mathrm{SO}(p)} \Delta_\lambda\bigl(U^T\beta_{A_V}(H)U\bigr)\,dU\,dV. \tag{8}$$

It is clear from Lemma 22 that $U^T\beta_{A_V}(H)U \in \Lambda = \Lambda_p$. Now,

$$\begin{aligned}\Delta_\lambda(W) &= \prod_{r=1}^{p-1}\Delta_r(W)^{(i(a_r-a_{r+1})-1)/2}\Delta_p(W)^{(ia_p-(q-p)/2)/2}\\ &= \prod_{r=1}^{p-1}\exp\bigl(g_r(W)\bigr)^{(i(a_r-a_{r+1})-1)/2}\exp\bigl(g_p(W)\bigr)^{(ia_p-(q-p)/2)/2}\\ &= \prod_{r=1}^{p-1}\exp\Bigl(\frac{i(a_r-a_{r+1})-1}{2}g_r(W)\Bigr)\exp\Bigl(\frac{ia_p-(q-p)/2}{2}g_p(W)\Bigr)\end{aligned}$$

which is analytic by Proposition 20. As the integration in (8) is over a compact set and $\Delta_\lambda\bigl(U^T\beta_{A_V}(H)U\bigr)$ depends continuously on U and V, the first part of the result follows. The second part follows from the fact that for a given $w \in W$, $\phi_\lambda(e^{w\cdot H})$ is also an analytic continuation of ϕ_λ and the two are the same on $\mathfrak{a}$. ∎

Subject to appropriate regularity and growth conditions, a function f which is left and right invariant under the action of K can be expressed in terms of the spherical functions via its spherical transform $\tilde{f}$ and the Plancherel formula. Therefore, if we know $\tilde{f}$ (as in the case of the heat kernel) we can use the analytic continuation of the spherical functions to look for an analytic continuation of the function f. To start with, we need a rough estimate on the growth of ϕ_λ when λ is real-valued.

Proposition 24 *Suppose λ is a real-valued linear functional on $\mathfrak{a}$: $\lambda(H) = \sum_{r=1}^{p} a_r H_r$ with $a_r \in \mathbf{R}$ for each r.*

Suppose ϕ_λ is the analytic continuation guaranteed by Theorem 23. There there exists a continuous function F independent of λ defined on $\hat{\Omega}_{\pi/4}$ such that on that domain we have

$$|\phi_\lambda(e^H)| \leq \prod_{r=1}^{p-1} \exp\left(\frac{r|a_r - a_{r+1}|\pi}{4}\right) \exp\left(\frac{p|a_p|\pi}{4}\right) F(H).$$

Proof Using the fact that λ is real and the proof of Theorem 23 (in particular equation (8)), we have

$$|\phi_\lambda(e^H)| \leq \int_{\mathrm{SO}(q)} \int_{\mathrm{SO}(p)} \prod_{r=1}^{p-1} \left| \exp\left(\frac{i(a_r - a_{r+1}) - 1}{2} g_r\big(U^T \beta_{A_V}(H) U\big)\right)\right| \cdot \left|\exp\left(\frac{ia_p - (q-p)/2}{2} g_p\big(U^T \beta_{A_V}(H) U\big)\right)\right| dU\, dV$$

Corollary 21 allows us to conclude. ∎

Remark 25 If we refer to [12, Proposition 6.1], we find that

$$|\phi_\lambda(e^H)| \leq |W|^{1/2} e^{-\min_w \Im\left(w\cdot\lambda(\Im H)\right) + \max_w w\cdot\rho(\Im H) + \max \Re(w\cdot\lambda(\Re H))}$$

$(w \in W)$ if $H \in \Omega_{\pi/2}$ (some adjustments were necessary due to a different notation). Opdam's result applies to a greater category of spherical functions. The drawback of his result is that in the given domain, the spherical function is defined as a multi-valued function as opposed to our situation.

The fundamental solution for the heat equation on a symmetric space of noncompact type G/K can be given as a function of the time and of $gK \in G/K$. Every other solution of moderate growth can be written in terms of that fundamental solution, or heat kernel, using the convolution over G/K (refer to [7, Chapter II, Section 5, 1.]). The heat kernel being left and right invariant under the action of K can be seen as a function on the Lie algebra $\mathfrak{a}$ of A: $H \to P_t(e^H)$.

This leads to:

Theorem 26 *The heat kernel $P_t(e^H)$ on the symmetric space $\mathcal{M} \simeq \mathrm{SO}(p,q)/\mathrm{SO}(p) \times \mathrm{SO}(q)$ has an analytic continuation on $\hat{\Omega}_{\pi/4}$.*

Proof Using the Plancherel formula, one finds that

$$P_t(e^H) = Ce^{-\langle\rho,\rho\rangle t} \int_{\mathfrak{a}^*} e^{-\langle\lambda,\lambda\rangle t} \phi_\lambda(e^H) |c(\lambda)|^{-2} \, d\lambda$$

where $\mathfrak{a}^*$ is the set of real-valued linear functionals on $\mathfrak{a}$, C is a constant and $\langle\cdot,\cdot\rangle$ is the inner product on $\mathfrak{a}^*$ induced by the one on $\mathfrak{a}$ (which corresponds to the Killing form). Since $|c(\lambda)|^{-2}$ is of polynomial growth [7, Proposition 7.2, Chapter IV] and using the bound on ϕ_λ given in Proposition 24 (naturally, we could also use Opdam's bound as given in Remark 25), the result follows. ∎

Conclusion

When we consider the heat kernel $P_t(e^H)$ as a function of $H \in \mathfrak{a}$, we could restrict ourselves to $H \in \mathfrak{a}^+$ since the heat kernel is invariant under the action of the Weyl group W.

In our efforts (see for instance [13, 14, 16]) to prove a conjecture by Anker (see [1]) on the growth of the heat kernel, we had some serious practical difficulties in estimating $P_t(e^H)$ when H is close to the boundary of $\mathfrak{a}^+$ even though we know that $P_t(e^H)$ is quite well behaved there. We feel that this paper provides a tool to avoid this difficulty in the case of the symmetric space $\mathrm{SO}_0(p,q)/\mathrm{SO}(p) \times \mathrm{SO}(q)$.

Indeed, for any given value $\eta > 0$, it is possible to cover Ω_η by polydisks of bounded radius whose boundary is at a distance bounded below by a positive number from the "problem set" $\{H \in \Omega_\eta \colon \alpha(H) = 0, \alpha \in \Sigma^+\}$. The maximum modulus then comes into play. As long as the radius of the polydisks remain bounded, our estimates are not skewed by the fact that we are estimating on the boundary of the polydisks rather than inside. This idea is used successfully in [16] (see for instance Proposition 2.5, Proposition 3.9 and ultimately Theorem 3.10 of that paper). This approach was in turn inspired by Anker's work in [1].

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