Canad. Math. Bull. Vol. **18** (5), 1975

THE RANGE OF A GAP SERIES

BY
J. S. HWANG†

THEOREM. *Let* $f(z)=\sum_{k=1}^{\infty} (a_{n_k-p}z^{n_k-p}+\ldots+a_{n_k}z^{n_k})$ *be a function holomorphic in the disk, where p is a natural number and* $n_{k+1}/n_k \geq \lambda > 1$ $(k=1,2,\ldots)$. *If* $\lim_{k\to\infty} \sup |a_{n_k-p}| = a$, $0<a<\infty$, *then* $f(z)$ *assumes every complex value infinitely often in every sector* $\Delta(\alpha, \beta)=\{z:\alpha<\arg z<\beta, |z|<1\}$.

The purpose of this note is to prove the above result. To do this, we first observe that from the condition $a<\infty$, we can easily show that the derivative $f'(z)$ satisfying

$$|f'(z)| < C(\lambda)/(1-r), \quad \text{for} \quad |z| = r < 1,$$

where $C(\lambda)$ is a positive constant depending only on λ.
It follows that the function f is normal [2, p. 87].

Next, we want to show that f has no finite radial limit. To see this, we need only apply a theorem of Hardy and Littlewood [4, Theorem 1].

According to a theorem of Bagemihl and Seidel [1, Theorem 3], f must have the Fatou value ∞ on a dense subset of the unit circle. We then can choose two numbers α' and β' with $\alpha<\alpha'<\beta'<\beta$ such that

$$(*) \qquad \lim_{r\to 1} |f(r \exp(i\alpha'))| = \lim_{r\to 1} |f(r \exp(i\beta'))| = \infty.$$

Now, suppose that f assumes a value v only finitely often in $\Delta(\alpha, \beta)$, then there is an $r>0$ such that the function $g(z)=1/(f(z)-v)$ is holomorphic in $\Delta_r(\alpha, \beta)$, where $\Delta_r(\alpha, \beta)=\{z:\alpha<\arg z<\beta, r<|z|<1\}$. Thus g is also holomorphic in $\Delta_r(\alpha', \beta')$. If g is bounded in $\Delta_r(\alpha', \beta')$, then by virtue of the extension of Fatou's theorem [1, Theorem 2], g will have radial limits almost everywhere on the arc $A(\alpha', \beta')=\{\exp(i\theta):\alpha'\leq\theta\leq\beta'\}$ and so will f. This however is impossible. Hence g must be unbounded in $\Delta_r(\alpha', \beta')$.

From equation (*) we can see that g is bounded on these two segments $R=\{z:\arg z=\alpha', r\leq|z|<1\}$ and $R'=\{z:\arg z=\beta', r\leq|z|<1\}$. Moreover, g is also bounded on the circular part $C_r(\alpha', \beta')=\{r \exp(i\theta):\alpha'\leq\theta\leq\beta'\}$. Hence g is bounded on the union $R \cup R' \cup C_r(\alpha', \beta')$.

Since g is unbounded in $\Delta_r(\alpha', \beta')$, it follows from a theorem of Gross and Iversen [2, Theorem 5.8] that $g(z)$ tends to infinity along a curve Γ lying in $\Delta_r(\alpha', \beta')$. By the normality of f, we can see the curve Γ must end at a boundary

† This research was supported by NRC of Canada: Grant A-5597.

point, say, at $z=1$ [1, Theorem 1]. It then follows from a theorem of Lehto and Virtanen [5, Theorem 5] that g has the angular limit ∞ at $z=1$. This in turn implies that f has the radial limit v at $z=1$, which is a contradiction. The proof is complete.

REMARK. 1. The result of ours is similar to that of Fuchs [3]. We restrict the coefficients to be bounded while we generalize the single gap series to be a union of finite number of them.

2. The same method can give an alternative proof of a theorem of Pommerenke [6], provided the coefficients are bounded.

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DÉPARTEMENT DE MATHÉMATIQUES,
UNIVERSITÉ DE MONTRÉAL,
QUÉBEC, CANADA