DIOPHANTINE APPROXIMATION AND HOROCYCLIC GROUPS

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1. Introduction. Let ω be an irrational number. It is well known that there exists a positive real number h such that the inequality

$$\left|\omega - \frac{a}{c}\right| < \frac{1}{hc^2} \tag{1}$$

has infinitely many solutions in coprime integers a and c. A theorem of Hurwitz asserts that the set of all such numbers h is a closed set with supremum $\sqrt{5}$. Various proofs of these results are known, among them one by Ford **(1)**, in which he makes use of properties of the modular group. This approach suggests the following generalization.

Let Γ be a (real) zonal horocyclic group. That is, Γ is a Fuchsian group of bilinear transformations of the first kind **(4)** having the real axis as principal circle and ∞ as a parabolic fixed point. A bilinear transformation T is defined by

$$w = Tz = \frac{az + b}{cz + d},$$

where a, b, c and d are real numbers such that $ad - bc = 1$, and w and z are complex numbers. It is convenient to use T to denote the matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

as well as for the transformation, and to assume that Γ is such that if T belongs to Γ, then so does

$$-T = \begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix}.$$

Since $-T$ gives rise to the same value of w as T, we do not distinguish between T and $-T$.

Since Γ has a parabolic fixed point at infinity and is discrete, there exists a least positive λ such that $U^\lambda \in \Gamma$, where

$$U = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad U^\lambda = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}.$$

The number λ has been called by Petersson the *width* of the cusp at ∞. The transformation U^λ generates a cyclic subgroup of Γ which we call Γ_U. In particular, if $T \in \Gamma$ then $TU^{m\lambda} \in \Gamma$ for any integer m, and $TU^{m\lambda}$ has the same

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first and third entries, namely a and c, as T. Thus a and c are functions of, and are determined by, the left cosets of Γ_U in Γ.

In terms of a zonal horocyclic group Γ we can generalize Diophantine approximation to irrational numbers by asking if there exists a positive $h = h(\Gamma)$ such that (1) is soluble for transformations T in infinitely many left cosets of Γ_U in Γ, where ω is any real number not belonging to a certain countable set, to be specified. We can, for this purpose, rewrite (1) as

$$|\omega - T\infty| < \frac{1}{hc^2}, \tag{2}$$

and it is possible to express the right-hand side in terms of T also, although it is not particularly advantageous to do this. The exceptional countable set will always include the set of all points congruent to ∞ with respect to Γ.

If such a number h exists, we can then ask what is the supremum of all such h, and whether this supremum is attained. If (2) holds for infinitely many left cosets of Γ_U in Γ, the numbers $T\infty$ satisfying (2) will approximate to ω as closely as we please since it is known that $|c|$ is less than any given positive number for only a finite number of cosets of Γ_U (**3**, Satz 2).

In the particular case where Γ is the modular group $\Gamma(1)$, this reduces to the approximation by rationals to an arbitrary irrational number ω, as described at the beginning of the paper. Ford's demonstration of Hurwitz's theorem does not, however, generalize immediately to an arbitrary zonal horocyclic group, since the possible existence of parabolic cycles not congruent to ∞ causes difficulties. Also, as might be expected, horocyclic groups of the second kind, which have an infinity of generators, are more difficult to treat since the set of points congruent to ∞ covers the real axis more thinly. In this paper we consider horocyclic groups of the first kind only (see §2 for definition).

Applications of the results obtained to the modular group and some of its subgroups are given in §7 and compared with results obtained by others. Considering the generality of the methods used in §§2-6, it is somewhat surprising that best possible results are nevertheless obtained in some cases. The horocyclic group approach also sheds some light on the connexion between irrationals which cannot be closely approximated to and hyperbolic transformations.

2. The existence of numbers h. We write $z = x + iy$, where x and y are real and use $\mathfrak{H}$ to denote the finite upper half-plane $y > 0$, and $\mathfrak{A}$ for the real axis $y = 0$, excluding the point at infinity. We suppose that Γ is a zonal horocyclic group and that $\mathfrak{D}$ is a fundamental region for Γ constructed by Ford's isometric circle method (**2**, §35) and occupying the strip $\xi < x \leqslant \xi + \lambda$, where λ is the width of the cusp at infinity and ξ is chosen suitably later.

We suppose that Γ is a horocyclic group of the first kind, i.e. that $\mathfrak{D}$ has finite hyperbolic area. This is the case if and only if $\mathfrak{D}$ has a finite number of sides.

Adjacent sides of $\mathfrak{D}$ meet at vertices which are of two kinds. The point at infinity and vertices lying on $\mathfrak{A}$ are called *cusps*. They can be divided into groups of congruent cusps with respect to Γ, called *parabolic cycles*. One such cycle, which may be the only one, consists of the point at infinity alone.

The remaining vertices all lie in $\mathfrak{H}$. They include points at which different isometric circles meet, and possibly elliptic fixed points of period 2 belonging to Γ which lie at a midpoint of a single bounding arc. Such vertices we call *ordinary vertices of the first kind.*

It may happen that some sides of $\mathfrak{D}$ consist of complete semicircles: if this happens it is convenient to regard such a semicircle as two equal sides meeting at a vertex which is the point farthest from $\mathfrak{A}$; such vertices we call *ordinary vertices of the second kind.*

We now choose ξ so that the line $x = \xi$ passes through an ordinary vertex of the first or second kind. Then every vertex of $\mathfrak{D}$ in $\mathfrak{H}$ is of the first or second kind, and $\mathfrak{D}$ is bounded by a finite number of sides which can be grouped in pairs of equal arcs transformable into each other by transformations of Γ. One side of such a pair (if not part of a straight line) is an arc of the isometric circle $\mathfrak{J}(T)$ of a transformation T, and the other is an equal arc of $\mathfrak{J}(T^{-1})$, where $T \in \Gamma$.

Define h_Γ to be twice the minimum distance of the vertices of $\mathfrak{D}$ in $\mathfrak{H}$ from $\mathfrak{A}$. We are now in a position to state

THEOREM 1. *Let Γ be a zonal horocyclic group of the first kind, and let ω be any real number which is not a parabolic fixed point for Γ. Then there are infinitely many left cosets of Γ_U in Γ whose members T satisfy*

$$\left|\omega - \frac{a}{c}\right| = \left|\omega - T\infty\right| < \frac{1}{h_\Gamma c^2} \qquad (c \neq 0). \tag{3}$$

3. Proof of Theorem 1. Let ω be any real number which is not a parabolic fixed point for Γ and let $\mathfrak{L} = \mathfrak{L}(\omega)$ be the straight line through ω which is perpendicular to $\mathfrak{A}$.

For any $T \in \Gamma$ with $T \notin \Gamma_U$ and for any $h > 0$, let $\mathfrak{S}(T, h)$ be the circle

$$\left|cz - a - \frac{i}{hc}\right| = \frac{1}{hc}, \tag{4}$$

which is of radius $1/(hc^2)$ and touches $\mathfrak{A}$ at a/c, but otherwise lies in $\mathfrak{H}$. If $\mathfrak{L}$ cuts $\mathfrak{S}(T, h)$, we have

$$\left|\omega - \frac{a}{c}\right| < \frac{1}{hc^2},$$

and conversely; i.e. (2) holds.

Let $\mathfrak{L}'(h)$ be the line $y = \frac{1}{2}h$; then

$$T\,\mathfrak{L}'(h) = \mathfrak{S}(T, h)$$

if $T \in \Gamma$, $T \notin \Gamma_U$. For $T \in \Gamma_U$ we define $\mathfrak{S}(T, h)$ to be $\mathfrak{L}'(h)$ and it then follows that

$$S\,\mathfrak{S}(T, h) = \mathfrak{S}(ST, h) = ST\,\mathfrak{L}'(h) = ST\,\mathfrak{S}(E, h)$$

for all S, $T \in \Gamma$, where E is the identical transformation.

Accordingly $\mathfrak{L}$ cuts $\mathfrak{S}(T, h)$ $(T \notin \Gamma_U)$, and so (2) holds, if and only if $T^{-1}\,\mathfrak{L}$ cuts $\mathfrak{L}'(h)$. But $T^{-1}\,\mathfrak{L}$ is a circle centred at a point of $\mathfrak{A}$ and so cuts $\mathfrak{L}'(h)$ if and only if its radius exceeds $\frac{1}{2}h$. Also, if $T^{-1}\,\mathfrak{L}$ cuts $\mathfrak{L}'(h)$ so does $(T')^{-1}\,\mathfrak{L}$ if T' and T are in the same left coset of Γ_U in Γ; for $T^{-1}\,\mathfrak{L}$ and $(T')^{-1}\,\mathfrak{L}$ are equal circles $r\lambda$ apart where r is an integer. Thus the theorem will follow if we show that $\mathfrak{L}'(h_\Gamma)$ is cut by infinitely many circles $T^{-1}\,\mathfrak{L}$ $(T \in \Gamma)$ at points lying in the strip $\xi < x \leqslant \xi + \lambda$.

Since ω is not a parabolic fixed point, the line $\mathfrak{L}$ passes through the interiors of infinitely many fundamental regions $T\mathfrak{D}$ $(T \in \Gamma)$. Let the fundamental regions through whose closures $\mathfrak{L}$ passes be $T_n\mathfrak{D}$ $(n = 1, 2, 3, \ldots)$ taken in order as a point z moves along $\mathfrak{L}$ from ∞ through $\mathfrak{H}$ to $\mathfrak{A}$; if $\mathfrak{L}$ passes through any points congruent to vertices of $\mathfrak{D}$ round which more than two fundamental regions cluster, then some definite ordering of the associated transformations T_n must be determined.

We suppose that Theorem 1 is false. Then there exists an integer N such that $\mathfrak{L}_n = T_n^{-1}\,\mathfrak{L}$ does not cut $\mathfrak{L}'(h_\Gamma)$ for $n \geqslant N$. Let k be the maximum number of images of $\mathfrak{D}$ which are grouped round an ordinary vertex of $\mathfrak{D}$. We show first that $\mathfrak{L}_n$ does not pass through an ordinary vertex of $\mathfrak{D}$ if $n \geqslant p = N + k$.

For suppose that $\mathfrak{L}_n$ passes through the ordinary vertex A of $\mathfrak{D}$ for some $n \geqslant p$. Then A is at a distance $\frac{1}{2}h_\Gamma$ from $\mathfrak{A}$ and is that point in $\mathfrak{H}$ of $\mathfrak{L}_n$ which is farthest from $\mathfrak{A}$, i.e. the *summit* of $\mathfrak{L}_n$. It follows that A is not an ordinary vertex of the second kind since, if it were, then $\mathfrak{L}_n$ would cut $\mathfrak{A}$ in two parabolic fixed points which is impossible since ω is not a parabolic fixed point.

Thus A must be an ordinary vertex of the first kind, and so lies on an isometric circle $\mathfrak{J}(T)$ of a transformation T of Γ, of which a finite arc γ forms one of the sides of $\mathfrak{D}$. If T is chosen in its right coset of Γ_U in Γ so that TA is also a vertex of $\mathfrak{D}$, then $T\mathfrak{L}_n = TT_n^{-1}\,\mathfrak{L}$ meets the closure of $\mathfrak{D}$ in TA, and so $TT_n^{-1} = T_m^{-1}$ for some $m \neq n$. Moreover $m \geqslant n - k \geqslant N$. Hence $\mathfrak{L}_m = T\,\mathfrak{L}_n$ does not cut $\mathfrak{L}'(h_\Gamma)$ but meets the closure of $\mathfrak{D}$ in TA which must therefore be the summit of $\mathfrak{L}_m$. We thus have two equal circles $\mathfrak{L}_n$ and $T\,\mathfrak{L}_n$ with summits A and TA. Since the transformation T is equivalent to an inversion in $\mathfrak{J}(T)$ followed by a reflexion in the radical axis of $\mathfrak{J}(T)$ and $\mathfrak{J}(T^{-1})$, this is possible only if A lies on $\mathfrak{J}(T)$ and if $\mathfrak{L}_n = \mathfrak{J}(T)$. Hence γ is an arc of $\mathfrak{L}_n$ with one endpoint at its summit A. The other endpoint is either an ordinary vertex of the first kind which is nearer to $\mathfrak{A}$ than A, or is a parabolic fixed point which is congruent to ω since it is not congruent to ∞. In either case we have a contradiction.

Hence $\mathfrak{L}_n$ does not pass through an ordinary vertex of $\mathfrak{D}$ if $n \geqslant p$. Thus, for each $n \geqslant p$, $\mathfrak{L}_n$ passes through the interior of $\mathfrak{D}$ cutting its sides at interior points. This is a contradiction if there are no parabolic cycles other than that containing the point at infinity, and so the theorem is true in this case. Suppose therefore that other parabolic cycles exist. Then, for each $n \geqslant p$, $\mathfrak{K}_n$ contains in its interior a cusp A_n of $\mathfrak{D}$, and consists in part of an arc lying in the interior of $\mathfrak{D}$ whose endpoints B_n, B_n' lie on sides l_n, l_n' of $\mathfrak{D}$ which meet at A_n. Here

$$T_n B_n' = T_{n+1} B_{n+1},$$

and the points B_n, B_n' are interior points of the sides l_n, l_n', respectively. It follows that

$$T_{n+1}^{-1} T_n$$

transforms l_n' into l_{n+1} and therefore maps A_n into A_{n+1}. Hence all the points A_n for $n \geqslant p$ are congruent and so belong to the same parabolic cycle. Accordingly there exists a positive integer q such that

$$A_{p+rq} = A_p \qquad (r = 0, 1, 2, \ldots)$$

and

$$T_{p+rq}^{-1} T_p = P^r, \qquad (r = 0, 1, 2, \ldots)$$

where

$$P = T_{p+q}^{-1} T_p .$$

Here P is a parabolic transformation having A_p as fixed point.

Choose a bilinear transformation S_1 having $\mathfrak{A}$ as fixed circle and such that $S_1 A_p = \infty$. Then $P = S_1^{-1} U^\mu S_1$ for some $\mu \neq 0$, and so

$$S_1 \mathfrak{K}_{p+rq} = S_1 P^r T_p^{-1} \mathfrak{K} = U^{\mu r} S_1 T_p^{-1} \mathfrak{K} = U^{\mu r} S_1 \mathfrak{K}_p. \tag{5}$$

Since $\mathfrak{K}_{p+rq}$ cuts l_p and l_p', $S_1\mathfrak{K}_{p+rq}$ cuts $S_1 l_p$ and $S_1 l_p'$; these are lines perpendicular to $\mathfrak{A}$ since they pass through $S_1 A_p = \infty$, and this holds for $r = 0, 1, 2, \ldots$. But, by (5), the set of circles $S_1\mathfrak{K}_{p+rq}$ consists of the circle $S_1\mathfrak{K}_p$ and its translates through multiples of μ parallel to $\mathfrak{A}$, and so $S_1\mathfrak{K}_{p+rq}$ cannot cut $S_1 l_p$ and $S_1 l_p'$ for every $r \geqslant 0$.

This contradiction shows that no integer N exists, and this completes the proof of Theorem 1.

4. Upper and lower bounds for sup h**.** Let $\mathfrak{E}(E, \Gamma)$ denote the set of all positive numbers h for which (2) holds for transformations T belonging to infinitely many left cosets of Γ_U in Γ, where Γ is, as previously, a zonal horocyclic group of the first kind. The reason for displaying E will appear later. By Theorem 1, $h_\Gamma \in \mathfrak{E}(E, \Gamma)$ and so $h \in \mathfrak{E}(E, \Gamma)$ for all positive $h \leqslant h_\Gamma$. In this section we obtain an upper bound for $\mathfrak{E}(E, \Gamma)$. We prove first

THEOREM 2. *The set $\mathfrak{E}(E, \Gamma)$ is bounded above.*

Proof. The group Γ contains hyperbolic transformations since if $T \in \Gamma$

and $c \neq 0$, then $U^{r\lambda}T \in \Gamma$ for all integers r and is hyperbolic if r is sufficiently large. Let

$$H = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{6}$$

be a hyperbolic transformation belonging to Γ with $C > 0$ and $A + D > 2$: this latter condition can always be satisfied by taking H^{-1} if it is not true for H. Write

$$A + D = 2 \sec \theta \qquad (0 < \theta < \tfrac{1}{2}\pi).$$

Then the fixed points of H are

$$z_1 = \{\tan(\tfrac{1}{4}\pi - \tfrac{1}{2}\theta) - D\}/C, \quad z_2 = \{\tan(\tfrac{1}{4}\pi + \tfrac{1}{2}\theta) - D\}/C.$$

They are the points F_1, F_2 of Fig. 1.

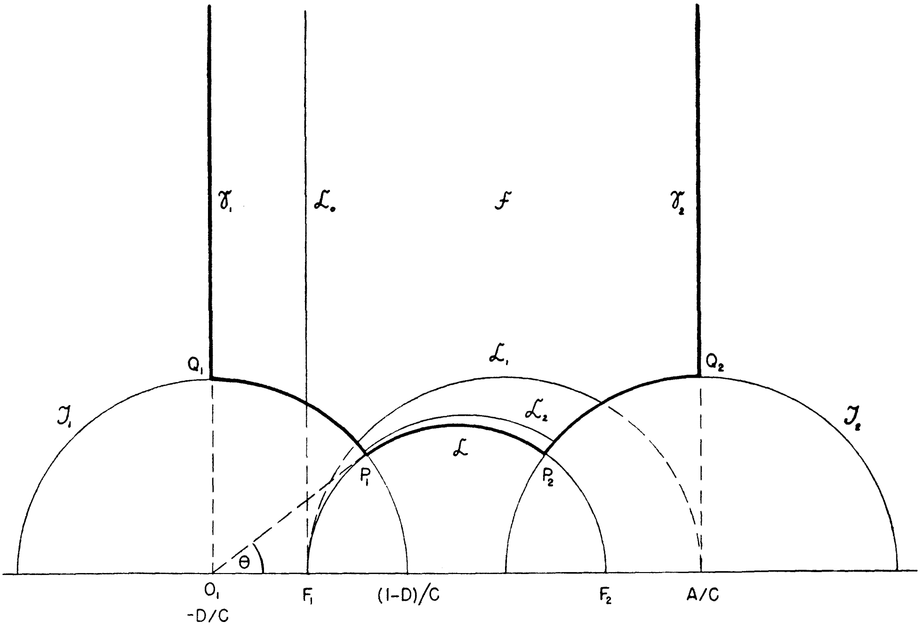

Fig. 1

A fundamental region $\mathfrak{D}_H$ in $\mathfrak{H}$ for the subgroup Γ_H of Γ generated by H is the region exterior to the circles

$$\mathfrak{J}_1 = \mathfrak{J}(H): \quad |Cz + D| = 1,$$

and

$$\mathfrak{J}_2 = \mathfrak{J}(H^{-1}): \quad |Cz - A| = 1.$$

The circles $H^n\mathfrak{J}_1$ ($n = 0, \pm 1, \pm 2, \ldots$) bound the images of $\mathfrak{D}_H$ by Γ_H, and consist of a set of non-intersecting coaxial circles with limit points F_1 and F_2. For $n > 1$, these circles are interior to $H\mathfrak{J}_1 = \mathfrak{J}_2$, and all contain F_2; for $n < 0$, these circles are interior to $\mathfrak{J}_1$ and all contain F_1.

Let $\mathfrak{C}$ be the semicircle in $\mathfrak{H}$ on F_1F_2 as diameter. $\mathfrak{C}$ has radius $(\tan\theta)/C$ and cuts $\mathfrak{J}_1$ and $\mathfrak{J}_2$ orthogonally at the points P_1 and P_2, say; θ is the angle $P_1\,O_1\,F_1$, where O_1 is the centre of $\mathfrak{J}_1$.

Suppose that m is the least positive value of $|c|$ for all transformations $T\in\Gamma$; by considering $T^{-1}\,U^\lambda\,T$ we see that $m\geqslant 1/\lambda$. Take any number h satisfying

$$h > \max\left(\frac{2\tan\theta}{C},\ \frac{C}{m^2}\cot\tfrac{1}{2}\theta\right)$$
$$= \frac{[(A+D)^2-4]^{\frac{1}{2}}}{|C|}\max\left\{1,\ \frac{C^2}{m^2(|A+D|-2)}\right\} \tag{7}$$
$$= \rho_H,$$

say. Also take $\omega = z_1$. We show that $\mathfrak{L}(\omega)$ cuts only a finite number of the circles or straight lines $\mathfrak{S}(T,h)$ for $T\in\Gamma$.

If $\mathfrak{L}$ cuts $\mathfrak{S}(T,h)$ at a point of $\mathfrak{H}$ lying in $H^{-n}\,\mathfrak{D}_H$, then $H^n\,\mathfrak{L}$ will cut

$$H^n\,\mathfrak{S}(T,h) = \mathfrak{S}(H^n\,T,h)$$

in a point of $\mathfrak{H}$ lying in $\mathfrak{D}_H$. But for $n<0$, $H^n\,\mathfrak{L}$ lies entirely within $\mathfrak{J}_1$, and so does not meet $\mathfrak{D}_H$; thus we need only consider $n\geqslant 0$. Let $\mathfrak{C}_n$ be the closure of the part of $H^n\,\mathfrak{L}$ which lies in $\mathfrak{D}_H$. Then it suffices to show that the arcs $\mathfrak{C}_n$ $(n=0,1,2,\ldots)$ have only a finite number of intersections with the curves $\mathfrak{S}\,(T,h)$ for $T\in\Gamma$.

For $n\geqslant 1$, $H^n\,\mathfrak{L}$ is a circle orthogonal to $\mathfrak{A}$ passing through the fixed point ω and through $\zeta_n = H^n\,\infty$. The point ζ_n lies within $H^n\,\mathfrak{J}_1$, and so within $\mathfrak{J}_2$, and $z_2<\zeta_n\leqslant\zeta_1 = A/C$. Hence the arcs $\mathfrak{C}_n$ for $n\geqslant 0$ all lie in the closed region $\mathfrak{F}$ of $\mathfrak{H}$ lying between the abscissae $x=-D/C$, $x=A/C$ and outside $\mathfrak{J}_1$, $\mathfrak{J}_2$ and $\mathfrak{C}$. Let Q_1, Q_2 be the summits of $\mathfrak{J}_1$ and $\mathfrak{J}_2$ respectively so that the boundary of $\mathfrak{F}$ consists of five sides (taken closed) joining the vertices ∞, Q_1, P_1, P_2, Q_2 and ∞. We write $\mathfrak{S}_1$ and $\mathfrak{S}_2$ for the sides joining ∞ to Q_1 and Q_2 respectively.

We divide the proof into five cases.

(i) The line $\mathfrak{L}'(h) = \mathfrak{S}(E,h)$ cuts only a finite number of the arcs $\mathfrak{C}_n$, since $\mathfrak{C}_n$ tends to $\mathfrak{C}$ as $n\to\infty$ and $\frac{1}{2}h$ exceeds $(\tan\theta)/C$ which is the radius of $\mathfrak{C}$.

We now show that no circle $\mathfrak{S}(T,h)$, for $T\in\Gamma$, $T\notin\Gamma_U$, has any points in common with $\mathfrak{F}$.

Suppose (ii) that $\mathfrak{S}\,(T,h)$ meets $\mathfrak{S}_1$ but contains no point of the arc Q_1P_1. Then its radius exceeds $1/C$, so that

$$C > hc^2 \geqslant hm^2 > C\cot\tfrac{1}{2}\theta,$$

which is a contradiction since $\theta<\frac{1}{2}\pi$. Similarly $\mathfrak{S}\,(T,h)$ cannot meet $\mathfrak{S}_2$ and contain no point of Q_2P_2.

Write $T_1 = HT$, where

$$T_1 = \begin{pmatrix} a_1 & b_1\\ c_1 & d_1\end{pmatrix}, \tag{8}$$

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and suppose next (iii) that $c_1 = Ca + Dc = 0$, so that $\mathfrak{S}\,(T, h)$ touches $\mathfrak{A}$ at O_1. Then $T_1 \in \Gamma_U$, so that we may take $c = C$, $a = -D$. Then if $\mathfrak{S}\,(T, h)$ meets $\mathfrak{F}$, $2/(hc^2) \geqslant 1/C$ so that, by (7),

$$\tan\theta < 1, \quad C^2 < 2m^2 \tan \tfrac{1}{2}\theta.$$

This gives a contradiction since $2 \tan \frac{1}{2}\theta < 2(\sqrt{2} - 1) < 1$ and $C^2 \geqslant m^2$. In the same way we can show that $\mathfrak{S}\,(T, h)$ cannot touch $\mathfrak{A}$ at A/C.

We now suppose (iv) that $c_1 \neq 0$ and that $\mathfrak{S}\,(T, h)$ contains in its interior or on its perimeter a point

$$z = (e^{i\phi} - D)/C$$

of Q_1P_1, where $\theta \leqslant \phi \leqslant \frac{1}{2}\pi$. Then, by (4),

$$\left| cz - a - \frac{i}{hc} \right| = \frac{1}{C} \left| (c \cos\phi - c_1) + i\left(c \sin\phi - \frac{C}{hc}\right) \right| \leqslant \frac{1}{hc}.$$

Hence

$$\begin{aligned}\frac{2C}{h} \sin\phi &\geqslant (c\cos\phi - c_1)^2 + c^2 \sin^2\phi \\ &= (c + c_1)^2 \sin^2 \tfrac{1}{2}\phi + (c - c_1)^2 \cos^2 \tfrac{1}{2}\phi.\end{aligned}$$

Since $T_1 \in \Gamma$ and $c_1 \neq 0$, either $|c + c_1| \geqslant 2m$ or $|c - c_1| \geqslant 2m$. Also $\sin^2 \frac{1}{2}\phi \leqslant \cos^2 \frac{1}{2}\phi$, so that

$$\frac{2C}{h} \sin\phi \geqslant 4m^2 \sin^2 \tfrac{1}{2}\phi.$$

Thus

$$h \leqslant \frac{C}{m^2} \cot \tfrac{1}{2}\phi \leqslant \frac{C}{m^2} \cot \tfrac{1}{2}\theta,$$

which contradicts (7). We can also show that $\mathfrak{S}\,(T, h)$ cannot contain a point of Q_2P_2; for, if it did, $\mathfrak{S}\,(H^{-1}\,T, h)$ would contain a point of Q_1P_1.

Finally (v) suppose that $\mathfrak{S}\,(T, h)$ contains a point of P_1P_2 but not any point of Q_1P_1 or Q_2P_2. Then the radius of $\mathfrak{S}\,(T, h)$ is greater than that of the circle which touches $\mathfrak{A}$ and touches $\mathfrak{C}$ internally at P_1. This circle has radius $(\tan \frac{1}{2}\theta)/C$, so that

$$1/(hc^2) > \tan \tfrac{1}{2}\theta/C.$$

Thus

$$h < (C/c^2) \cot \tfrac{1}{2}\theta \leqslant (C/m^2) \cot \tfrac{1}{2}\theta,$$

which contradicts (7).

We have therefore shown that no circle $\mathfrak{S}\,(T, h)$ with $T \in \Gamma$, $T \notin \Gamma_U$, has any point in common with $\mathfrak{F}$, so that $\mathfrak{L}$ cuts at most a finite number of the circles $\mathfrak{S}\,(T, h)$ and therefore $h \notin \mathfrak{C}(E, \Gamma)$. This proves Theorem 2.

If we define

$$h(E, \Gamma) = \sup h \qquad (h \in \mathfrak{C}(E, \Gamma)),$$

and write

(9) $$h_\Gamma' = \inf \rho_H \qquad (H \text{ hyperbolic in } \Gamma),$$

we have proved

THEOREM 3. $h_\Gamma \leqslant h(E, \Gamma) \leqslant h_\Gamma'$.

5. Approximation to ω by $ST\infty$ for fixed S. Let $S \in \Omega_R$, where Ω_R is the continuous group of all bilinear transformations with $\mathfrak{A}$ as principal circle, and

(10) $$S = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}.$$

We consider whether it is possible to approximate to a real number ω by $ST\infty$ where T belongs to a zonal horocyclic group Γ of the first kind. This is not quite the same as approximating to $S^{-1}\omega$ by $T\infty$, since instead of using $1/c^2$ as a measure of approximation we shall use $1/c_1^2$ where $T_1 = ST$ and is given by (8).

Let $\mathscr{R}(\Gamma_U, \Gamma)$ be a set of representatives of the left cosets of Γ_U in Γ arranged in such an order that $|c| \to \infty$ as T runs through $\mathscr{R}(\Gamma_U, \Gamma)$. We prove the

LEMMA. *Let $\mathscr{R}'$ be any subset of $\mathscr{R}(\Gamma_U, \Gamma)$ consisting of transformations T for which $T_1\infty = ST\infty$ is bounded. Then $|c_1| < M$ for only a finite number of $T \in \mathscr{R}'$ where M is any given positive number.*

Proof. Suppose that $|T_1\infty| \leqslant K$ for all $T \in \mathscr{R}'$. Then

$$|c_1| = |\gamma a + \delta c| = |c||\gamma T\infty + \delta| = |c|/|\gamma T_1\infty - \alpha|$$
$$\geqslant |c|/\{|\gamma|K + |\alpha|\}.$$

Since $|c| < M\{|\gamma|K + |\alpha|\}$ for only a finite number of $T \in \mathscr{R}(\Gamma_U, \Gamma)$, the result follows.

Now suppose that ω is any finite real number such that $S^{-1}\omega$ is not a parabolic point of Γ. Then, for any $h \in \mathfrak{E}(E, \Gamma)$,

(11) $$\left| S^{-1}\omega - T\infty \right| < \frac{1}{hc^2} \qquad (c \neq 0),$$

for infinitely many $T \in \mathscr{R}(\Gamma_U, \Gamma)$. Let $\mathscr{R}'$ be an infinite subset of $\mathscr{R}(\Gamma_U, \Gamma)$ for which (11) holds. Since $|c| \to \infty$ as T runs through $\mathscr{R}(\Gamma_U, \Gamma)$, we may suppose that $\mathscr{R}'$ is taken so that the points $T\infty$, for $T \in \mathscr{R}'$, lie in as small an interval as we please round $S^{-1}\omega$, and hence that the points $ST\infty$ lie in a bounded region containing ω. It follows from the lemma that $|c_1| < M$ for a finite number of $T \in \mathscr{R}'$ only, and we may therefore arrange the transformations T in $\mathscr{R}'$ in such an order that both $|c|$ and $|c_1|$ tend to infinity as T runs through $\mathscr{R}'$.

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For any $T \in \mathscr{R}'$ we have, from (11),

$$|\omega - ST\infty| = |(S^{-1}\omega - T\infty)\{(\gamma S^{-1}\omega + \delta)(\gamma T\infty + \delta)\}^{-1}|$$
$$< \frac{1}{h}\left|\frac{\gamma T\infty + \delta}{\gamma S^{-1}\omega + \delta}\right| \; |c(\gamma T\infty + \delta)|^{-2}$$
$$= \frac{1}{hc_1^2}\left|\frac{\gamma T\infty + \delta}{\gamma S^{-1}\omega + \delta}\right|.$$

As T runs through $\mathscr{R}'$,

$$\left|\frac{\gamma T\infty + \delta}{\gamma S^{-1}\omega + \delta}\right| \to 1,$$

and so, if h' is any positive number less than h, we have

$$|\omega - ST\infty| < \frac{1}{h'c_1^2}$$

for infinitely many $T \in \mathscr{R}(\Gamma_U, \Gamma)$. We have therefore proved

THEOREM 4. *If $h < h(E, \Gamma)$, $S \in \Omega_R$, and ω is any finite real number such that $S^{-1}\omega$ is not a parabolic fixed point of Γ, then*

$$|\omega - ST\infty| = |\omega - T_1\infty| < \frac{1}{hc_1^2}$$

for all T belonging to an infinite subset $\mathscr{R}'$ of $\mathscr{R}(\Gamma_U, \Gamma)$ which may be arranged so that $|c_1| \to \infty$ as T runs through $\mathscr{R}'$.

Let $\mathfrak{E}(S, \Gamma)$ be the set of all positive h for which the conclusions of Theorem 4 hold with the conditions there stated on S, ω and $\mathscr{R}'$, and let $h(S, \Gamma)$ be its supremum. Then, by Theorem 4,

$$h(S, \Gamma) \geqslant h(E, \Gamma).$$

But we have, conversely,

$$\left|S^{-1}\omega - T\infty\right| < \frac{1}{hc^2}\left|\frac{\gamma S^{-1}\omega + \delta}{\gamma T\infty + \delta}\right|$$

for every h in $\mathfrak{E}(S, \Gamma)$. Since $ST\infty \to \omega$ as T runs through $\mathscr{R}'$, $T\infty \to S^{-1}\omega$ as T runs through $\mathscr{R}'$ and so

$$h(E, \Gamma) \geqslant h(S, \Gamma).$$

We have therefore proved

THEOREM 5. *If $S \in \Omega_R$, then $h(E, \Gamma) = h(S, \Gamma)$.*

Hence all the sets $\mathfrak{E}(S, \Gamma)$ are identical, with the possible exception of one point.

We can also prove

THEOREM 6. $h_\Gamma \in \mathfrak{E}(S, \Gamma)$ *if* $S \in \Omega_R$.

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To prove this we carry through the argument of §3 with certain modifications. We wish to show that $S^{-1}\mathfrak{L}$ cuts $\mathfrak{S}(T, h)$ for T in infinitely many left cosets of Γ_U in Γ, when $S^{-1}\omega$ is not a parabolic fixed point. The demonstration proceeds as in §4 with $\mathfrak{L}$ replaced by $S^{-1}\mathfrak{L}$ wherever it occurs after (4). Where before we used the fact that ω was not a parabolic fixed point we now require that $S^{-1}\omega$ is not a parabolic fixed point. As before, $S^{-1}\mathfrak{L}$ will pass through the closures of infinitely many fundamental regions $T_n\mathfrak{D}$ $(n = 1, 2, 3, \ldots)$. We now choose $T_1(\mathfrak{D})$ to be the fundamental region containing the summit of $S^{-1}\mathfrak{L}$ and proceed along $S^{-1}\mathfrak{L}$ towards $S^{-1}\omega$. The case where the other endpoint of γ lies on $\mathfrak{A}$ can now be excluded since, because of this choice this endpoint would have to be $S^{-1}\omega$ which is impossible as $S^{-1}\omega$ is not a cusp.

6. Another upper bound for h. Suppose that Γ^* is a zonal horocyclic group of the first kind and that Γ is a subgroup of finite index in Γ^*, and so also a zonal horocyclic group of the first kind. Let μ be the width of the cusp at infinity for Γ^*, so that $\lambda = k\mu$, where k is a positive integer, and let $\Gamma_U{}^*$ be the subgroup of Γ^* generated by U^μ.

The parabolic fixed points of Γ^* are parabolic fixed points of Γ, and conversely. We suppose that ω is not a parabolic fixed point for Γ. Then, by Theorem 5,

$$h(E, \Gamma) = h(S, \Gamma) \text{ for all } S \in \Gamma^*. \tag{12}$$

Take a fixed $S \in \Gamma^*$ and a fixed hyperbolic H in Γ^* with entries given by (6), such that

$$S \notin \Gamma_H^* \Gamma_U^* \Gamma, \tag{13}$$

where $\Gamma_H{}^*$ is the subgroup of Γ^* generated by H. Further, let m_{HS} be the least value of $|c|$ for all $T \in \Gamma_H{}^* S\Gamma$, so that $m_{HS} > 0$, by (13). Also take any h with

$$h > \frac{C}{m_{HS}^2}\cot \tfrac{1}{2}\theta = \frac{C}{m_{HS}^2}\left[\frac{|A + D| + 2}{|A + D| - 2}\right]^{\frac{1}{2}} = \rho'_{HS}, \tag{14}$$

where the notation is that used in §4.

Take $\omega = z_1$. We apply the method of §4 to show that $\mathfrak{L}$ does not intersect any $\mathfrak{S}(ST', h)$ for $T' \in \Gamma$. This is the same as showing that no circle[1] $\mathfrak{S}(T, h)$ intersects any arc $\mathfrak{C}_n$ $(n \geqslant 0)$ where $T = H^nST'$.

Case (i) of §4 does not arise, and the proof of case (ii) is similar, with m replaced by m_{HS}. Case (iii) also does not arise, since if $c_1 = 0$, where $T_1 = HT$, we should have $T_1 = H^{n+1}ST' \in \Gamma_U{}^*$, which contradicts (13). Similarly if $Ca - Ac = 0$. The proof of cases (iv) and (v) is similar with m replaced by m_{HS}. Thus we have shown that if h satisfies (14) and S satisfies (13) then $h \notin \mathfrak{C}(S, \Gamma)$.

Define

$$h''_\Gamma = \inf \rho'_{HS}$$

[1] $\mathfrak{S}(T, h)$ cannot be a straight line by (13).

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for all hyperbolic transformations $H \in \Gamma^*$ and $S \in \Gamma^*$ for which (13) holds; if no such H and S exist define $h_\Gamma'' = \infty$. It follows from (12) and what we have just proved that the following result holds.

THEOREM 7. *$h(S^*, \Gamma) \leqslant h_\Gamma''$ for all $S^* \in \Gamma^*$.*

7. Applications.

7.1. If we take Γ to be the full modular group $\Gamma(1)$ and ω irrational, we obtain $h_\Gamma = \sqrt{3}$. By taking

$$H = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

in §4 we see that if $h > \sqrt{5}$, then $h \notin \mathfrak{C}\,(E, \Gamma(1))$. Thus

$$\sqrt{3} \leqslant h(E, \Gamma(1)) \leqslant \sqrt{5}.$$

In fact $h(E, \Gamma) = \sqrt{5}$ as can be obtained by a more detailed study of the modular configuration, as shown by Ford **(1)**.

7.2. If we take $\Gamma = \Gamma(2)$, the principal congruence group of level 2 consisting of matrices $T \equiv E \pmod 2$, we obtain by Ford's method the fundamental region $\mathfrak{D}$ consisting of points in the strip $-\frac{1}{2} \leqslant x \leqslant 3/2$, $y > 0$ which lie outside the circles $|2z \pm 1| = 1$, $|2z - 3| = 1$. Thus $h_\Gamma = 1$.

Write

$$V = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{15}$$

and take $\Gamma^* = \Gamma(1)$ in §6. Let

$$S = UVU = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad H = H_n = \begin{pmatrix} 2n & 4n^2 - 1 \\ 1 & 2n \end{pmatrix},$$

for any positive integer n. Then (13) states that $S \not\equiv E, V, U, VU \pmod 2$ which is correct, and $m_{HS} = 1$. Hence

$$\rho'_{HS} = \left(\frac{2n+1}{2n-1}\right)^{\frac{1}{2}} \rightarrow 1$$

as $n \rightarrow \infty$. Thus $h_\Gamma'' \leqslant 1$, and so, by Theorems 3, 5, 6 and 7,

$$h(S^*, \Gamma(2)) = 1, \quad 1 \in \mathfrak{C}\,(S^*, \Gamma(2)),$$

for all $S^* \in \Gamma\ (1)$. By taking (i) $S^* = E$, (ii) $S^* = UVU$, (iii) $S^* = V$, we deduce that (1) is soluble for every irrational ω for an infinity of coprime integers a, c in each of the three cases (i) a odd, c even, (ii) a, c both odd, (iii) a even, c odd, if and only if $h \leqslant 1$. These results are due to Scott **(5)**, who also obtained them by considering $\Gamma(2)$.

7.3. Take Γ to be $\Gamma_V(2)$, the group of matrices T congruent to E or V modulo 2. A fundamental region for $\Gamma_V(2)$ is given by

$$0 \leqslant x \leqslant 2, \quad y \geqslant 0, \quad |z| \geqslant 1, \quad |z-2| \geqslant 1,$$

so that $h_\Gamma = 2$.

As before, take $\Gamma^* = \Gamma(1)$ and

$$S = UVU, \quad H = H_n' = \begin{pmatrix} 2n+1 & 2n(n+1) \\ 2 & 2n+1 \end{pmatrix}$$

for any positive integer n. Then (13) states that $S \not\equiv E, V, U, UV \pmod 2$, which is correct, and $m_{HS} = 1$. Thus $\rho_{HS}' \to 2$ as $n \to \infty$, so that $h_\Gamma'' \leqslant 2$. Hence, by Theorems 3, 5, 6 and 7,

$$h(S^*, \Gamma_V(2)) = 2, \quad 2 \in \mathfrak{S}\,(S^*, \Gamma_V(2))$$

for all $S^* \in \Gamma(1)$.

By taking S^* to be (a) U, (b) E, (c) UVU, we deduce that (1) is soluble for irrational ω for an infinity of coprime integers a, c if and only if $h \leqslant 2$, in each of the following three cases (a) (i) or (ii), (b) (i) or (iii), (c) (ii) or (iii) (see §7.2). These results are also due to Scott **(5)**.

7.4. Let Γ be the principal (inhomogeneous) congruence group $\bar\Gamma(N)$ of level $N > 2$. This is a self-conjugate subgroup of $\bar\Gamma(1)$ of index $\mu(N)$, say. From a set of $\mu(N)$ representatives of the cosets of $\bar\Gamma(N)$ in $\bar\Gamma(1)$ we can choose a set G_N of $\mu(N)/N$ matrices S^* with different α, γ modulo N and $(\alpha, \gamma, N) = 1$. Let S be such a matrix with

$$\alpha \equiv 0 \pmod N, \quad \gamma = [\tfrac{1}{2}(N-1)]$$

(integral part) so that $(\gamma, N) = 1$. Also take $\Gamma^* = \bar\Gamma(1)$ and

$$H = H_n'' = \begin{pmatrix} Nn+1 & Nn(Nn+2) \\ 1 & Nn+1 \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \pmod N.$$

Then

$$H^r \equiv \begin{pmatrix} 1 & 0 \\ r & 1 \end{pmatrix}, \quad H^r U^s \equiv \begin{pmatrix} 1 & s \\ r & rs+1 \end{pmatrix} \pmod N$$

and $S \not\equiv H^r U^s \pmod N$, showing that (13) is satisfied. Also, if $T \in \Gamma_H{}^* S \Gamma$, $c \equiv \gamma \pmod N$, so that $m_{HS} = \gamma$. We have, by (14), $\rho_{HS}' \to 1/\gamma^2$ as $n \to \infty$, and so, by Theorem 7,

$$h(S^*, \bar\Gamma(N)) \leqslant \{[\tfrac{1}{2}(N-1)]\}^{-2} \sim 4N^{-2}$$

for large N and $S^* \in G_N$.

Lower bounds for $h(S^*, \bar\Gamma(N))$ may be obtained from Theorem 3, by estimating h_Γ, for example by **(4**, Theorem 10**)**, but this does not yield good results, at any rate for large N.

Each $S^* \in G_N$ yields a result on the approximation to irrational ω by fractions a/c with $a \equiv \alpha^*$, $c \equiv \gamma^* \pmod N$, where a and c need not be positive integers. A considerable amount of work has been done by Descombes and Poitou, Hartman, Koksma and Tornheim **(6)** on this and related problems using different methods such as continued fractions. See **(6)** for references. In particular, it is known that $h(S^*, \bar{\Gamma}(N)) \geqslant 4/N^2$ and that $h(S^*, \bar{\Gamma}(N)) \sim 4/N^2$ for large N. For $N = 3, 4$, $h(S^*, \bar{\Gamma}(N)) = \sqrt{(5/3)}$ and $\frac{1}{2}$, respectively.

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