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A GENERALIZATION OF CONVEX FUNCTIONS VIA SUPPORT PROPERTIES*

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Abstract

With respect to a given family of functions $\mathscr{F}$, a function is said to be $\mathscr{F}$-convex, if it is supported, at each point, by some member of $\mathscr{F}$. For particular choices of $\mathscr{F}$ one obtains the convex functions and the generalized convex functions in the sense of Beckenbach. $\mathscr{F}$-convex functions are characterized and studied, retaining some essential results of classical convexity.

Introduction

Let $\mathscr{F}$ be a family of functions: $R^n \to R$, depending on $(n+1)$ parameters. A function f: $R^n \to R$ is called *$\mathscr{F}$-convex* if its graph is supported at each point by some member of $\mathscr{F}$. For particular choices of $\mathscr{F}$, the $\mathscr{F}$-convex functions reduce to the ordinary proper convex functions and the sub $\mathscr{F}$-functions of Beckenbach.

In this paper we study the basic properties of $\mathscr{F}$-convex functions.

Sections 2 and 3 contain definitions and examples.

Section 4 gives first order conditions (so called because they involve only first derivatives and "generalized gradients") for $\mathscr{F}$-convexity. For families $\mathscr{F}$ which possess certain uniqueness property, $\mathscr{F}$-convexity is characterized by an analog of the gradient inequality. The remaining results in Section 4 are conditions for $\mathscr{F}$-convexity or strict $\mathscr{F}$-convexity, in terms of the injectivity properties of the "generalized gradients" (termed here "$\mathscr{F}$-gradients").

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Second order conditions for $\mathcal{F}$-convexity and strict $\mathcal{F}$-convexity are given in Section 5. These conditions involve a "generalized Hessian matrix" $H(x)$, depending on f and $\mathcal{F}$, which in the classical case reduces to the Hessian Matrix $\partial^2 f/\partial x_i x_j$.

The main results here are Theorems 5.1 and 5.5. An analog of the differential inequality of Peixoto (1949), characterizing sub-$\mathcal{F}$ functions, is obtained as a special case.

Section 6 deals with the monotonicity properties of the $\mathcal{F}$-gradient x_f^* of an $\mathcal{F}$-convex function, where $\mathcal{F}$ belongs to a certain class of functions. The derivative of x_f^* is computed and the result is used, for the "separable" families to establish that x_f^* is a P_0–function [P–function] if f is $\mathcal{F}$-convex [strictly $\mathcal{F}$-convex].

In a sequel paper we study the corresponding generalizations of conjugacy and duality in the sense of Fenchel (see Rockafellar (1970)). These results involve a conjugate family $\mathcal{F}^*$, and are hidden in the classical case by the fact that there $\mathcal{F} = \mathcal{F}^*$. In future papers the authors intend to apply the theory developed here to derive duality results for nonconvex programs and to furnish a unified theory for various recent results on generalized Lagrangians.

2. $\mathcal{F}$-convex functions: definitions and examples

2.1 Definitions. Let $\mathcal{F}$ be a family of functions: $X \to R$ where $X \subset R^n$ with range

(2.1) $$\Xi \triangleq \cup \{\text{range } F: F \in \mathcal{F}\}.$$

Let f be a function: $R^n \to R$ with domain

(2.2) $$\operatorname{dom} f \subset X$$

and let S be an open subset of dom f. Then f is called *$\mathcal{F}$-convex in S* if for every $x \in S$, there exists an $F \in \mathcal{F}$ such that

(2.3) $$f(x) = F(x) \text{ and } f(z) \geqq F(z) \quad \text{for all } x \neq z \in S,$$

in which case F is called a *support of* $f: S$ *at* x. The function f is called *strictly $\mathcal{F}$-convex in S* if strict inequality holds in (2.4) for all $x \neq z \in s$.

The name $\mathcal{F}$-convex function was used recently (Roberts and Varberg (1973), p. 241) to denote the sub-$\mathcal{F}$ functions, see Example 2.2.

If there is no need to specify S, for example if $S = \operatorname{dom} f$, the above names are abbreviated by omitting S, e.g., *$\mathcal{F}$-convex*, *support of f at x*, etc.

2.2. Example. Let $\mathcal{F}$ be the family of affine functions: $R^n \to R$, i.e.,

(2.4) $$\mathcal{F} = \{F(\cdot) = \langle x^*, \cdot \rangle - \xi^*: x^* \in R^n, \xi^* \in R\}.$$

Then a function $f\colon R^n \to R$ is $\mathcal{F}$-convex if and only if it is a proper convex function, i.e., a convex function whose epigraph is a non empty set containing no vertical lines, ([Rockafellar (1970), §4].

2.3 Example. Let $\mathcal{F}$ be a family of continuous functions: $X \to R$ where X is the open interval (a, b), and such that

(B) For any two distinct points in X, say,

$$a < x_1 < x_2 < b$$

and any two real numbers $\{y_1, y_2\}$, there is a unique $F \in \mathcal{F}$ satisfying

$$F(x_i) = y_i, \qquad (i = 1, 2).$$

We call such an $\mathcal{F}$ a *Beckenbach family in* (a, b). Beckenbach (1937) called a function $f\colon (a, b) \to R$ a *sub-$\mathcal{F}$ function* if for any two points

$$a < x_1 < x_2 < b$$

the member of $\mathcal{F}$, F_{12}, defined by

$$F_{12}(x_i) = f(x_i), \qquad (i = 1, 2), \tag{2.5}$$

satisfies

$$f(x) \leqq F_{12}(x), \qquad x_1 < x < x_2. \tag{2.6}$$

Peixoto (1948, 1949, Theorem 1) showed that if f is a sub-$\mathcal{F}$ function and $a < x_0 < b$ then there exist two functions

$$F_i \in \mathcal{F}, \qquad (i = 1, 2),$$

such that

$$F_i(x_0) = f(x_0), \qquad (i = 1, 2),$$

$$F_2(x) \leqq F_1(x) \leqq f(x), \qquad (a < x < x_0),$$

and

$$F_1(x) \leqq F_2(x) \leqq f(x), \qquad x_0 < x < b.$$

(Furthermore, if the derivatives $f'(x_0)$, $F_1'(x_0)$ and $F_2'(x_0)$ exist, they are equal). Thus both F_1 and F_2 support f at x_0.

Therefore every sub-$\mathcal{F}$ function is $\mathcal{F}$-convex. We will now prove the converse for Beckenbach families $\mathcal{F}$.

2.4 Proposition. *Let $\mathcal{F}$ be a Beckenbach family in (a, b). Then a function $f\colon (a, b) \to R$ is $\mathcal{F}$-convex in (a, b) if and only if f is a sub-$\mathcal{F}$ function.*

Proof. The proof of the "if" part was cited above.

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To prove "only if" suppose f is not a sub-$\mathscr{F}$ function, i.e., there are three points

$$a < x_1 < x_0 < x_2 < b$$

such that the function $F_{12} \in \mathscr{F}$, defined by (2.5), satisfies

$$F_{12}(x_0) < f(x_0). \tag{2.7}$$

Suppose that $F_0 \in \mathscr{F}$ is a support of f at x_0, i.e.,

$$f(x_0) = F_0(x_0) \text{ and } f(x) \geqq F_0(x), \qquad a < x < b. \tag{2.8}$$

From (2.5), (2.7) and (2.8) it follows that F_{12} and F_0 intersect twice over the interval (a, b), contradicting (B). Therefore f is not $\mathscr{F}$-convex. □

2.5 EXAMPLE. Let $G(x, y, z)$ be a continuous function: $(a, b) \times R \times R \to R$, such that

(P1) For each $\{x_0, y_0, y_0'\} \in (a, b) \times R \times R$, the differential equation

$$y'' = G(x, y, y'), \qquad (a < x < b), \tag{2.9}$$

has a unique solution $y = y(x)$ satisfying

$$y(x_0) = y_0, \qquad y'(x_0) = y_0'. \tag{2.10}$$

(P2) The solution of (2.9) is continuous with respect to the initial values y_0, y_0'.

(P3) For any two points $\{x_i, y_i\} \in (a, b) \times R$ $(i = 1, 2)$ with $x_1 \neq x_2$, there is a unique solution of (2.9) satisfying

$$y(x_i) = y_i, \qquad i = 1, 2. \tag{2.11}$$

Let $\mathscr{F}$ be the Beckenbach family of solutions of (2.9). Peixoto (1949, Theorem 2) showed that a function $f \in C^2(a, b)$ is a sub-$\mathscr{F}$ function if and only if

$$f'' \geqq G(x, f, f'), \qquad a < x < b. \tag{2.12}$$

2.6 EXAMPLE. While sub-$\mathscr{F}$ functions are continuous (Beckenbach (1937), Roberts and Varberg (1973), p. 242), an $\mathscr{F}$-convex function need not be continuous in its domain, even if each $F \in \mathscr{F}$ is continuous:
Let $\mathscr{F}$ be the family of functions: $R \to R$

$$F(x) = \xi^* \cos(x^* x)$$

depending on the two parameters

$$x^* \in R \quad \xi^* \in R.$$

Then the function

$$f(x) = \begin{cases} 1 & \text{if } x \neq 0 \\ -1 & \text{if } x = 0 \end{cases}$$

is $\mathscr{F}$-convex. Indeed, for every $x \neq 0$, the function $F = F(x^*, \xi^*; \cdot)$ defined by

$$x^* = \frac{\pi}{x} \qquad \xi^* = -1$$

supports f at x. Also every $F \in \mathscr{F}$ supports f at 0.

We show now that an $\mathscr{F}$-convex function inherits from $\mathscr{F}$ its lower semi continuity.

2.7 Proposition. *Let $\mathscr{F}$ be a family of l.s.c. (= lower semi continuous) functions, and let f be $\mathscr{F}$-convex in dom f. Then f is l.s.c. in dom f.*

Proof. Suppose f is not l.s.c.. Then there exists an $x \in \operatorname{dom} f$ such that

$$f(x) > \liminf_{y \to x} f(y). \tag{2.13}$$

Let $F \in \mathscr{F}$ support f at x. Then

$$F(x) = f(x) > \liminf_{y \to x} f(y) \geqq \liminf_{y \to x} F(y),$$

by (2.3) and (2.13), contradicting the lower semi continuity of F. □

2.8 Notes. For further generalizations of convexity see the surveys in Beckenbach and Bellman (1965), Chapter 4, and Roberts and Varberg (1973), Chapter VIII.

For functions of several variables, the analogs of the sub $\mathscr{F}$-functions are the subfunctions in particular the subharmonic functions; see Beckenbach and Bellman (1965), p. 146, Beckenbach and Jackson (1953) and Jackson (1968), where applications to second order differential inequalities are surveyed.

3. Requirements on $\mathscr{F}$

3.1 General. With Examples 2.2, 2.3 and 2.5 as our motivation, we consider from now on only families $\mathscr{F}$ of functions $F: R^n \to R$ depending continuously on $n + 1$ parameters

$$\{x^*, \xi^*\} \in X^* \times \Xi^*$$

where the sets of parameters X^* and Ξ^* are given subsets of R^n and R respectively. The general member of $\mathscr{F}$ is thus denoted by

$$F(\cdot) = F(x^*, \xi^*; \cdot), \quad (x^* \in X^*, \xi^* \in \Xi^*), \tag{3.1}$$

with function values

$$F(x) = F(x^*, \xi^*; x), \qquad x \in X. \tag{3.2}$$

We assume that the mapping: $\{x^*, \xi^*\} \to F(x^*, \xi^*; \cdot)$ is one to one on $X^* \times \Theta^*$.

3.2 THE CLASS $\boldsymbol{A}$. Let $D^k(X)$ denote the functions: $R^n \to R$ which are k times differentiable in X. If $\mathscr{F} \subset D(X) \overset{\Delta}{=} D^1(x)$ we define the set

$$Z \overset{\Delta}{=} \cup \left\{\text{range}\begin{bmatrix} F \\ F_x \end{bmatrix} : F \in \mathscr{F}\right\} \subset R \times R^n \tag{3.3}$$

where F_x is the gradient of F with respect to x.

A family $\mathscr{F}$ of differentiable functions is said to be in *class* $\boldsymbol{A}$, denoted by $\mathscr{F} \in \boldsymbol{A}$, if for every $x \in X$ and $\begin{bmatrix} \xi \\ y \end{bmatrix} \in Z$, the system

$$\xi = F(x^*, \xi^*; x) \tag{3.4}$$

$$y = F_x(x^*, \xi^*; x) \tag{3.5}$$

has a unique solution $\{x^*, \xi^*\} \in X^* \times \Xi^*$.

If $\mathscr{F} \subset D(X)$ and if f and S are a function: $R^n \to R$ and an open subset of dom f respectively, we denote by

$$f \overset{S}{\approx} \mathscr{F} \tag{3.6}$$

the facts

$$S \subset \text{dom } f \subset X \tag{D1}$$

$$f \in D(S) \tag{D2}$$

$$\text{range}\left\{\begin{bmatrix} f(x) \\ f_x(x) \end{bmatrix} : x \in S\right\} \subset Z. \tag{D3}$$

We abbreviate $f \overset{\text{dom} f}{\approx} \mathscr{F}$ by $f \approx \mathscr{F}$.

If $\mathscr{F} \in \boldsymbol{A}$, $f \approx \mathscr{F}$ and $x \in \text{dom} f$ we denote by

$$(x_f^*(x), \xi_f^*(x)) \tag{3.7}$$

the unique solution of

$$f(x) = F(x^*, \xi^*; x) \tag{3.8}$$

$$f_x(x) = F_x(x^*, \xi^*; x). \tag{3.9}$$

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3.3 THE CLASS C. A family $\mathscr{F}$ is said to be in *class* C, denoted by $\mathscr{F} \in C$, if for every $\{x^*, x\} \in X^* \times X$ the function $F(x^*, \cdot\,; x)$ is a strictly decreasing function of ξ^*, $\xi^* \in \Xi^*$. In this case, we denote by $F^I(x, \cdot\,; x^*)$ the *inverse function* of $F(x^*, \cdot\,; x)$. It satisfies the identity

$$\xi = F(x^*, F^I(x, \xi; x^*); x), \qquad \xi \in \Xi. \tag{3.10}$$

If $\mathscr{F} \in A \cap C$, $f \approx \mathscr{F}$ and $x \in \operatorname{dom} f$, then (3.8) gives

$$\xi^* = F^I(x, f(x); x^*) \tag{3.11}$$

which, substituted in (3.9), gives

$$f_x(x) = F_x(x^*, F^I(x, f(x); x^*); x). \tag{3.12}$$

The unique solution of (3.12) is then called the *$\mathscr{F}$-gradient of f at x*, and is denoted by $x^*_f(x)$.

3.4 EXAMPLE. Let $\mathscr{F}$ be the family (2.4) of affine functions: $R^n \to R$. Then

(a) $\mathscr{F} \subset D(R^n)$, $F_x(x^*, \xi^*; x) = x^*$ for every $F \in \mathscr{F}$ and $x \in R^n$, and (3.3) gives $Z = R \times R^n$.

(b) $\mathscr{F} \in A$. For every $x \in R^n$ and $\begin{bmatrix} \xi \\ y \end{bmatrix} \in R \times R^n$, the unique solution of (3.4)–(3.5) is

$$x^* = y, \qquad \xi^* = \langle y, x \rangle - \xi.$$

(c) $\mathscr{F} \in C$.

(d) $f \approx \mathscr{F}$ means that $f \in D(\operatorname{dom} f)$.

(e) If $f \approx \mathscr{F}$ then for every $x \in \operatorname{dom} f$

$$x^*_f(x) = f_x(x), \quad \xi^*_f(x) = \langle f_x(x), x \rangle - f(x). \tag{3.13}$$

Thus the $\mathscr{F}$-gradient of f, x^*_f, coincides here with its ordinary gradient f_x.

3.5 EXAMPLE. Let ϕ be a given function: $X^* \times X \to R$ and let the family $\mathscr{F}$ consist of the functions $F(x^*, \xi^*; \cdot)$, $\{x^*, \xi^*\} \in X^* \times \Xi^*$, with values

$$F(x^*, \xi^*; x) = \phi(x^*, x) - \xi^*, \qquad x \in X. \tag{3.14}$$

Then: $\mathscr{F} \in C$ and

(a) $\mathscr{F} \in A$ if and only if the following two conditions hold:

(a1) $\phi(x^*, \cdot) \in D(X)$ for every $x^* \in X^*$.

(a2) For every $x \in X$, $y \in \cup_{x^*}$ range $\phi_x(x^*, \cdot)$, the system $y = \phi_x(x^*, x)$ has a unique solution x^*.

The $\mathcal{F}$-gradient of f at x, $x_f^*(x)$, is the unique solution x^* of

$$f_x(x) = \phi_x(x^*, x). \tag{3.15}$$

A concrete example is the following family $\mathcal{F}$ given by

$$F(x^*, \xi^*; x) = \frac{\langle x^*, x\rangle}{\langle a, x\rangle + \beta} - \xi^* \tag{3.16}$$

with domain $X = \{x : \langle a, x\rangle + \beta > 0\}$, and $X^* = R^n$. Here a is a given vector in R^n and β is a given nonzero scalar. Note that, for the special case $a = 0$, $\beta = 1$, the family (3.16) reduces to the family (2.4) of affine function. Here condition (a1) clearly holds, while condition (a2) holds if and only if the linear system of equations

$$y_i = \sum_{j \neq i} \left\{ \frac{-x_j a_i}{(\langle a, x\rangle + \beta)^2} x_j^* \right\} + \frac{\langle a, x\rangle + \beta - a_i x_i}{(\langle a, x\rangle + \beta)^2} x_i^*, \qquad i = 1, 2, \cdots, n$$

has a unique solution. This occurs of course if and only if det $G \neq 0$ where G is the $n \times n$ matrix

$$G = \begin{bmatrix} \langle a, x\rangle + \beta - a_1 x_1, & -a_2 x_2, & \cdots, & -a_n x_1 \\ -a_1 x_2, & \langle a, x\rangle + \beta - a_2 x_2, & & \\ -a_1 x_n, & -a_2 x_n, & \cdots, & \langle a, x\rangle + \beta - a_n x_n \end{bmatrix}$$

Using Sylvester's identity (Gantmacher (1959), section II.3) it can be shown, by induction, that $\det G = [\langle a, x\rangle + \beta]^{n-1}\beta$, and since $\beta \neq 0$ it follows that $\det G \neq 0$ and hence $\mathcal{F} \in A$. For a $\mathcal{F}$-convex function f, the $\mathcal{F}$-gradient is given by

$$x_f^*(x) = G^{-1}(\nabla f(x)).$$

4. First order conditions for $\mathcal{F}$-convexity

In this section we give first order conditions (so-called because they involve only first derivatives and the "gradients" $\{x_f^*, \xi_f^*\}$ of f, see (3.7)) for $\mathcal{F}$-convexity, for families $\mathcal{F}$ in class **A**. These conditions use the extremal property of the supports implied by the inequality (2.3). First we require

4.1 LEMMA. *Let $\mathcal{F} \in \mathbf{A}$, $f \colon R^n \to R$, and let $f \overset{S}{\approx} \mathcal{F}$. If $f \colon S$ is supported (by some $F \in \mathcal{F}$) at a point $x \in S$, then*

$$F(x_f^*(x),\ \xi_f^*(x);\ \cdot\ ) \tag{4.1}$$

is the unique support of f at x.

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PROOF. Let $F(x_0^*, \xi_0^*; \cdot) \in \mathscr{F}$ support $f \colon S$ at x, i.e.,

$$h(z) \stackrel{\Delta}{=} f(z) - F(x_0^*, \xi_0^*; z) \geqq 0, \qquad \forall z \in S, \tag{4.2}$$

and

$$h(x) = f(x) - F(x_0^*, \xi_0^*; x) = 0. \tag{4.3}$$

Therefore $h(z)$ is minimized, in S, by $z = x$. Since S is open, this implies that x is a critical point of h, i.e.,

$$h_z(x) = f_x(x) - F_x(x_0^*, \xi_0^*; x) = 0. \tag{4.4}$$

Since $\mathscr{F} \in \mathbf{A}$, a comparison of (4.3)–(4.4) and (3.8)–(3.9) shows that

$$\{x_0^*, \xi_0^*\} = \{x_f^*(x), \xi_f^*(x)\}$$

proving that (4.1) is the unique support at x. □

4.2 THEOREM. Let $\mathscr{F} \in \mathbf{A}$, $f \colon R^n \to R$, and $f \stackrel{S}{\approx} \mathscr{F}$. Then f is $\mathscr{F}$-convex in S if and only if for every $x \in S$

$$f(z) \geqq F(x_f^*(x), \xi_f^*(x); z), \qquad \forall x \neq z \in S. \tag{4.5}$$

Furthermore, f is strictly $\mathscr{F}$-convex in S if and only if for every $x \in S$

$$f(z) > F(x_f^*(x), \xi_f^*(x); z), \qquad \forall x \neq z \in S. \tag{4.6}$$

PROOF. *If.* From (4.5) and (3.8) it follows, for any $x \in S$, that the function (4.1) supports $f \colon S$ at x. Moreover, x is the unique point of support if (4.6) holds.

Only if. Let f be $\mathscr{F}$-convex in S. Then, by Lemma 4.1, for any $x \in S$, the function (4.1) is the unique support of $f \colon S$ at x. The inequality (4.5) then follows from (2.3). Similarly (4.6) follows from the strict $\mathscr{F}$-convexity of f. □

4.3 EXAMPLE. Let $\mathscr{F}$ be the family (2.4) of affine functions: $R^n \to R$,

$$\mathscr{F} = \{F(x^*, \xi^*; \cdot) = \langle x^*, \cdot \rangle - \xi^* \colon x^* \in R^n, \xi^* \in R\}.$$

Then, using (3.13), the inequality (4.5) reduces to

$$f(z) \geqq \langle f_x(x), z - x \rangle + f(x), \qquad \forall x \neq z \in S,$$

the classical gradient inequality.

4.4 COROLLARY. (a) *Let $\mathscr{F} \in \mathbf{A}$, and let $f \colon R^n \to R$, $f \stackrel{S}{\approx} \mathscr{F}$, be $\mathscr{F}$-convex in*

S. Then f is strictly $\mathcal{F}$-convex in S if and only if the mapping

$$x \to \{x_f^*(x),\ \xi_f^*(x)\} \tag{4.7}$$

is one to one on S.

(b) *Let, in addition, $\mathcal{F} \in C$. Then f is strictly $\mathcal{F}$-convex in S if and only if the mapping*

$$x \to x_f^*(x) \tag{4.8}$$

is one to one on S.

PROOF. From Lemma 4.1 it follows, for every $x \in S$, that the function (4.1) is the unique support of f: S at x. By definition, f is strictly $\mathcal{F}$-convex in S if, and only if, every support of f: S supports f at exactly one point of S. This is equivalent to the mapping (4.7) being one to one on S.

To prove the last part, note that the additional hypothesis $\mathcal{F} \in C$ implies

$$\left[x \overset{1:1}{\to} x_f^*(x) \text{ on } S\right] \Leftrightarrow \left[x \overset{1:1}{\to} \{x_f^*(x),\ \xi_f^*(x)\} \text{ on } S\right]. \tag{4.9}$$

Indeed, the implication $\Rightarrow$ is always true. Conversely, suppose that x_f^* is not one to one on S, i.e., there exist $x_1, x_2 \in S$, $x_1 \neq x_2$, such that

$$x_f^*(x_1) = x_f^*(x_2) \overset{\Delta}{=} x_0^*. \tag{4.10}$$

Let $\xi_i^* \overset{\Delta}{=} \xi_f^*(x_i) = F^I(x_i, f(x_i); x_0^*)$ and let

$$F^i(\cdot) \overset{\Delta}{=} F(x_0^*, \xi_i^*;\ \cdot), \qquad (i = 1, 2).$$

Then

$$F^i(x_i) = f(x_i), \quad F_x^i(x_i) = f_x(x_i), \qquad i = 1, 2.$$

Hence by Theorem 4.2, F^i supports f at x_i. If $\xi_1^* = \xi_2^*$, then this and (4.10) contradicts the fact that (x_f^*, ξ_f^*) is $1:1$, established earlier. Thus suppose that $\xi_1^* > \xi_2^*$. This implies, since $\mathcal{F} \in C$, that $F^1(z) < F^2(z)$ $\forall z \in S$. In particular

$$F^2(x_1) > F^1(x_1) = f(x_1)$$

contradicting the fact that F^2 is a support. □

4.5 THEOREM. *Let $\mathcal{F} \in \mathbf{A} \cap \mathbf{C}$, f: $R^n \to R$, and $f \overset{S}{\approx} \mathcal{F}$. Then f is strictly $\mathcal{F}$-convex in S if the following two conditions hold.*

(a) *The mapping x_f^* is one to one on S.*

(b) *For every $x \in S$ and for every sequence $\{z_k\} \subset S$ which either converges* *to a point $y \in \operatorname{bdry} S$ or $\|z_k\| \to \infty$ there exists an $\hat{x} \in S$ such that*

$$\limsup_{k \to \infty} \{F^I(z_k, f(z_k), x_f^*(x)) - F^I(\hat{x}, f(\hat{x}), x_f^*(x))\} \leqq 0 \tag{4.11}$$

where F^I is defined in §3.3.

PROOF. For any $x \in S$ consider the function

$$T(z) \stackrel{\Delta}{=} F^I(z, f(z); x_f^*(x)). \tag{4.12}$$

We show first that $z = x$ is a critical point of T. Differentiating the identity

$$F(x^*, F^I(y, f(y); x^*); y) - f(y) = 0 \tag{4.13}$$

with respect to y we get

$$F_x(\cdot, \cdot; \cdot) + F_{\xi^*}(\cdot, \cdot; \cdot)[F_x^I(y, f(y); x^*) + F_\xi^I(y, f(y); x^*) f_x(y)] - f_x(y) = 0 \tag{4.14}$$

where

$$(\cdot, \cdot; \cdot) = (x^*, F^I(y, f(y); x^*); y).$$

Now $F_{\xi^*} \neq 0$, since $\mathscr{F} \in C$. Therefore, for $y = x$ and $x^* = x_f^*(x)$, it follows from (4.14) and (3.12) that

$$F_x^I(x, f(x); x_f^*(x)) + F_\xi^I(x, f(x); x_f^*(x)) f_x(x) = 0 \tag{4.15}$$

which, by (4.12), is the same as $T_z(x) = 0$, proving that $z = x$ is critical.

Moreover, $z = x$ is the unique critical point of T in S. For suppose that $x \neq x' \in S$ is another critical point of T, i.e.

$$T_z(x') = F_x^I(x', f(x'); x_f^*(x)) + F_\xi^I(x', f(x'); x_f^*(x)) f_x(x') = 0$$

implying that for $y = x'$ and $x^* = x_f^*(x)$, (4.14) reduces to

$$F_x(x_f^*(x), F^I(x', f(x'); x_f^*(x)); x') - f_x(x') = 0$$

which, together with (3.12), implies that

$$x_f^*(x') = x_f^*(x),$$

contradicting (a).

We show next that

$$\sup\{T(z) : z \in S\} = T(x). \tag{4.16}$$

Indeed if this supremum occurs at some $z = y \in \operatorname{bdry} S$ or if a supremizing sequence $\{z_k\}$ is such that $\|z_k\| \to \infty$ then the supremum is also attained at $\hat{x} \in S$, by (4.11). Therefore $z = \hat{x}$ is a critical point of T, proving that $\hat{x} = x$, since the latter is the unique critical point in S, and therefore (4.16) becomes

$$F^I(x, f(x); x_f^*(x)) > F^I(z, f(z); x_f^*(x)), \qquad \forall x \neq z \in S,$$

which is the same as

$$f(z) > F(x_f^*(x), F^I(x, f(x); x_f^*(x)); z), \qquad \forall x \neq z \in S,$$

proving that f is strictly $\mathscr{F}$-convex in S, by theorem 4.2. $\square$

4.6 EXAMPLE. Consider the family

$$\mathscr{F} = \{\phi(x^*, \cdot) - \xi^*: x^* \in X^*, \xi^* \in \Xi^*\},$$

of Example 3.5 and let $\mathscr{F} \in \mathbf{A}$, f: $R^n \to R$, and $f \overset{s}{\approx} \mathscr{F}$. Then condition (b) of Theorem 4.5 follows from

(b1) For every $x^* \in$ range $\{x_f^*(x): x \in S\}$ and every sequence $\{z_k\}$ as in Theorem 4.5(b),

$$\liminf_{k \to \infty} \{f(z_k) - \phi(x^*, z_k)\} = +\infty. \tag{4.17}$$ $\square$

In particular, if

$$S = \operatorname{dom} f = X = R^n$$

and

$$\limsup_{\|x\| \to \infty} \frac{\phi(x^*, x)}{\|x\|} < \infty, \qquad \forall x^* \in \text{range } x_f^*. \tag{4.18}$$

then condition (b) of Theorem 4.5 is satisfied if

$$\lim_{\|x\| \to \infty} \frac{f(x)}{\|x\|} = \infty.$$

Note that (4.18) is trivially satisfied by the family $\mathscr{F}$ of affine functions. Hence, a differentiable function f: $R^n \to R$ is strictly convex if the following two conditions hold.

(a) The mapping

$$x \to f_x(x)$$

is one to one on R^n.

(b)
$$\lim_{\|x\| \to \infty} \frac{f(x)}{\|x\|} = \infty.$$

As a concrete example of condition (b1) let $\mathscr{F}$ be the family of functions: $R^2 \to R$ given by

$$F(x^*, \xi^*; x) = x_1^* e^{-x_1} + x_2^* x_2 e^{-x_1} - \xi^* \tag{4.19}$$

with $X = X^* = R^2$, $\Xi^* = R$.

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Consider the function $f\colon R^2 \to R$

$$f(x) = \frac{1}{2}e^{2x_1} + \frac{1}{2}x_2^2 e^{-x_1} \tag{4.20}$$

with $\operatorname{dom} f = R^2$. Then f is $\mathcal{F}$-convex in R^2 since:

(a) The $\mathcal{F}$-gradient

$$x_f^*(x) = \begin{bmatrix} -e^{3x_1} - \frac{1}{2}x_2^2 \\ \\ x_2 \end{bmatrix}$$

is one-to-one, and

$$\operatorname{range} x_f^* = \left\{(x_1^*, x_2^*) \in R^2 \colon x_1^* + \frac{1}{2}x_2^{*2} < 0\right\}. \tag{4.21}$$

(b) $$f(z) - \phi(x^*, z) = \left(\frac{1}{2}e^{2z_1} + \frac{1}{2}z_2^2 e^{-z_1}\right) - (x_1^* e^{-z_1} + x_2^* z_2 e^{-z_1})$$

$$= \frac{1}{2}e^{2z_1} - \left(x_1^* + \frac{1}{2}x_2^{*2}\right)e^{-z_1} + \frac{1}{2}(z_2 - x_2^*)^2 e^{-z_1}$$

by (4.21) the coefficients of all exponentials are positive and hence

$$\lim_{\|z\| \to \infty} [f(z) - \phi(x^*, z)] = \infty \qquad \forall x^* \in \operatorname{range} x_f^*.$$

5. Second order conditions for $\mathcal{F}$-convexity

In this section we collect second order conditions (involving second derivatives) for $\mathcal{F}$-convexity.

5.1 Theorem. *Let $\mathcal{F} \in \mathbf{A} \cap D^2(X)$, $f\colon R^n \to R$, $f \overset{S}{\approx} \mathcal{F}$ and $f \in D^2(S)$. Then*:

(a) *f is $\mathcal{F}$-convex in S only if, for every $x \in S$, the matrix*

$$H(x) \overset{\Delta}{=} f_{xx}(x) - F_{xx}(x_f^*(x), \xi_f^*(x); x) \tag{5.1}$$

is positive semi definite. (A matrix $H \in R^{n \times n}$ is called here positive semi definite if

$$\langle Hz, z\rangle \geqq 0, \qquad \forall z \in R^n.$$

We do not mean by this that H is symmetric.)

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(b) *Let S be convex and let f and each $F \in \mathcal{F}$ be twice continuously differentiable in S. Then f is $\mathcal{F}$-convex in S if*

$$\left\langle y, \int_0^1 (f_{xx}(x+sy) - F_{xx}(x_f^*(x), \xi_f^*(x); x+sy))y\,ds \right\rangle \geqq 0, \tag{5.2}$$

for every $x \in S$ and $y \in S - x$.
If strict inequality holds in (5.2), F is strictly $\mathcal{F}$-convex in S.

PROOF. (a) Let f be $\mathcal{F}$-convex in S. Then, for any $x \in S$, the function

$$h(z) \overset{\Delta}{=} f(z) - F(x_f^*(x), \xi_f^*(x); z) \tag{5.3}$$

satisfies

$$h(x) = 0, \quad h_z(x) = 0, \quad \text{by } (3.8)-(3.9), \tag{5.4}$$

and

$$h(z) \geqq 0, \qquad \forall z \in S, \quad \text{by Theorem 4.2.}$$

Therefore $z = x$ minimizes h in S. Since S is an open set, it follows that

$$h_{zz}(x) = H(x)$$

is positive semi-definite.

(b) The function h of (5.3) satisfies

$$h(z) = h(z) - h(x) - \langle h_z(x), z - x\rangle, \quad \text{by (5.4)},$$

$$= \langle h_z(x + t(z-x)) - h_z(x), z - x\rangle, \quad \text{for some } 0 < t < 1,$$

by a mean value theorem (Ortega and Rheinboldt (1970), Theorem 3.2.2).

$$= \left\langle z - x, \left(\int_0^1 h_{zz}(x + st(z-x))ds\right) t(z-x)\right\rangle,$$

by a mean value theorem (Ortega and Rheinboldt (1970), Theorem 3.2.7),

$$= \frac{1}{t}\left\langle y, \int_0^1 (f_{xx}(x+sy) - F_{xx}(x_f^*(x), \xi_x^*(x); x+sy))\,y\,ds\right\rangle,$$

where $y = t(z-x)$.

Thus, (5.2) implies that

$$h(z) \geqq 0, \qquad \forall z \in S, \tag{5.5}$$

proving that f is $\mathcal{F}$-convex in S, by Theorem 4.2.

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Similarly, strict inequality in (5.2) implies strict inequality in (5.5), hence strict $\mathscr{F}$-convexity. □

5.2 EXAMPLE. Let $\mathscr{F}$ be the family (2.4) of affine functions: $R^n \to R$. Then the matrix $H(x)$ of (5.1) reduces to the Hessian of f

$$H(x) = f_{xx}(x)$$

and Theorem 5.1 gives the classical conditions for convexity in terms of the Hessian.

5.3 EXAMPLE. Let $\mathscr{F}$ be the Beckenbach family of solutions of the second order differential equation

$$y'' = G(x, y, y'), \qquad (a < x < b), \tag{2.9}$$

discussed in Example 2.5. Then (5.1) becomes

$$H = f'' - G(x, f, f').$$

Now, suppose that $\mathscr{F} \subset C^2(X)$, $f \in C^2(S)$, then $H(x) > 0$ implies $H(x + sy) > 0$ for $0 < s < 1$ and y sufficiently close to x. Thus (5.2) is a strict inequality in some neighborhood of x, and we conclude that f is, locally, strictly $\mathscr{F}$-convex. By Proposition 2.4 this implies that f is locally strictly sub-$\mathscr{F}$, which by (Beckenbach (1937), Theorem 7) implies that f is sub-$\mathscr{F}$ globally in (a, b). This result is the analog of (Peixoto (1949), Theorem 3). To get the analogous result of (Peixoto (1949), Theorem 1), we need the implication $H(x) \geqq 0 \Rightarrow H(x + sy) \geqq 0$, for $0 < s < 1$ and y sufficiently close to x, for which Peixoto's additional requirement, (P2) of Example 2.5, is needed (see the proof of Lemma 1 in Peixoto (1949)).

5.4 DEFINITION. *A mapping $T: R^n \to R^n$ is called an extended one to one mapping on R^n if*

(a) $x, y \in R^n$, $x \neq y \Rightarrow T(x) \neq T(y)$.

(b) *The inverse images $T^{-1}(B)$ of bounded sets $B \subset R^n$ are bounded.*

5.5 THEOREM. *Let $\mathscr{F} \in \mathbf{A} \cap \mathbf{C} \cap C^2(R^n)$, $f: R^n \to R$, $f \in C^2(R^n)$ and $f \approx \mathscr{F}$. Then f is strictly $\mathscr{F}$-convex in R^n if the following two conditions hold*

(a) *x_f^* is an extended one to one mapping on R^n.*

(b) *For every $x \in R^n$, the matrix*

$$H(x) = f_{xx}(x) - F_{xx}(x_f^*(x), F^I(x, f(x); x_f^*(x)); x) \tag{5.6}$$

is positive definite. Conversely, if f is strictly $\mathscr{F}$-convex in R^n then x_f^* is one to one and the matrix $H(x)$ is positive semi-definite for every $x \in R^n$.

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PROOF. First we note, by (3.11), that (5.6) and (5.1) are the same.

For any $x \in R^n$ consider now the function (4.12)

$$T(z) = F^I(z, f(z); x_f^*(x)).$$

As in the proof of Theorem 4.5 it follows from (a) that $z = x$ is the unique critical point of T in R^n.

Differentiating the identity (4.13) twice with respect to y we get, by using (4.15) and (3.12),

$$T_{zz}(x) = \frac{1}{F_{\xi^*}} H(x) \tag{5.7}$$

(where F_{ξ^*} is evaluated at $\{x_f^*(x), F^I(x, f(x); x\})$. From (5.7), (b) and $F \in \boldsymbol{C}$ it follows that $T_{zz}(x)$ is negative definite. Therefore $z = x$ is an isolated local maximizer of T, and its unique critical point in R^n.

Thus, by Leighton's Theorem (Leighton (1966), see also Szegö (1968)), $z = x$ is the global maximizer of T, i.e.,

$$F^I(x, f(x); x_f^*(x)) > F^I(z, f(z); x_f^*(x)), \qquad \forall x \neq z \in R^n,$$

which is the same as

$$f(z) > F(x_f^*(x), F^I(x, f(z); x_f^*(x)); z), \qquad \forall x \neq z \in R^n,$$

proving that f is strictly $\mathcal{F}$-convex in R^n by Theorem 4.2.

If f is strictly $\mathcal{F}$-convex in R^n then (a) and (b) follow from Corollary 4.4 and Theorem 5.1(a) respectively. □

5.6 EXAMPLE. Let $\mathcal{F}$ and f be given by (4.19) and (4.20) respectively. Then the matrix (5.1) is positive definite

$$H(x) = \begin{bmatrix} 3e^{2x_1} & 0 \\ 0 & e^{-x_1} \end{bmatrix}$$

and f is strictly $\mathcal{F}$-convex in R^2, by Theorem 5.5.

6. Monotonicity of $\mathcal{F}$-gradients

In this section we prove monotonicity results for the $\mathcal{F}$-gradient x_f^* of an $\mathcal{F}$-convex function. We recall that a mapping $g: R^n \to R^n$ is a *P-function* [*P_0-function*] if for every $x, y \in \operatorname{dom} g$, $x \neq y$, there is an index $k = k(x, y) \in \{1, 2, \cdots, n\}$ such that

$$(x_k - y_k)(g_k(x) - g_k(y)) > 0 \quad [(x_k - y_k)(g_k(x) - g_k(y)) \geqq 0 \text{ and } x_k \neq y_k],$$

see Moré and Rheinboldt (1973). In particular, a mapping $g: R^n \to R^n$ is *monotone* (*strictly monotone*) if for every $x, y \in \operatorname{dom} g$, $x \neq y$, we have

$$\langle x - y, g(x) - g(y)\rangle \geqq 0 \qquad [\langle x - y, g(x) - g(y)\rangle > 0].$$

We also require the following

6.1 Definitions. *A family $\mathcal{F}$ is said to be in class $\mathbf{A}_1$, denoted by $\mathcal{F} \in \mathbf{A}_1$, if $\mathcal{F} \in \mathbf{A}$ and for every $\{x^*, \xi^*; x\} \in X^* \times \Xi^* \times X$ the derivatives in* (6.1) *are continuous and the matrix*

$$J(x^*, \xi^*; x) = \begin{bmatrix} F_{\xi^*}(x^*, \xi^*; x) & F^T_{x^*}(x^*, \xi^*; x) \\ \\ F_{\xi^* x}(x^*, \xi^*; x) & F_{x^* x}(x^*, \xi^*; x) \end{bmatrix} \tag{6.1}$$

is nonsingular, say

$$\det J(x^*, \xi^*; x) < 0. \tag{6.2}$$

This matrix is the Jacobian matrix of the function

$$\begin{bmatrix} f(\cdot, \cdot; x) \\ F_x(\cdot, \cdot; x) \end{bmatrix},$$

see (3.4) − (3.5).

A family $\mathcal{F}$ is said to be in class $\mathbf{A}_2$, denoted by $\mathcal{F} \in \mathbf{A}_2$, if $\mathcal{F} \in \mathbf{A}_1$ and for every $x \in X$ the matrix

$$J_0(x) \stackrel{\Delta}{=} \frac{1}{F_{\xi^*}}[F_{\xi^*} F_{x^* x} - F_{\xi^* x} F^T_{x^*}], \tag{6.3}$$

where all derivatives are evaluated at $\{x^*_f(x), \xi^*_f(x); x\}$, is positive definite.

6.2 Lemma. *Let $\mathcal{F} \in \mathbf{A}_1 \cap C$, $f: R^n \to R$, $f \overset{S}{\approx} \mathcal{F}$ and let f and each $F \in \mathcal{F}$ be twice continuously differentiable in S. Then, for every $x \in S$,*

$$D_x x^*_f(x) = J_0(x)^{-1} H(x) \tag{6.4}$$

where $D_x x^*_f(x)$ denotes the derivative of x^*_f at x and J_0 and H are given by (6.3) *and* (5.1) *respectively.*

Proof. For any $x \in S$ consider the system

$$F(x^*, \xi^*; x) - f(x) = 0 \tag{3.8}$$

$$F_x(x^*, \xi^*; x) - f_x(x) = 0 \tag{3.9}$$

which, since $\mathcal{F} \in \mathbf{A}$, has a unique solution $\{x^*_f(x), \xi^*_f(x)\}$. The implicit function

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theorem, applicable since $\mathscr{F} \in \boldsymbol{A}_1$, then gives

$$\begin{bmatrix} D_x \xi_f^*(x) \\ D_x x_f^*(x) \end{bmatrix} = \begin{bmatrix} F_{\xi^*} & F_{x^*}^T \\ F_{\xi^* x} & F_{x^* x} \end{bmatrix}^{-1} \begin{bmatrix} f_x(x) - F_x(x_f^*(x), \xi_f^*(x); x) \\ f_{xx}(x) - F_{xx}(x_f^*(x), \xi_f^*(x); x) \end{bmatrix} \tag{6.5}$$

where the derivatives

$$\begin{bmatrix} F_{\xi^*} & F_{x^*}^T \\ F_{\xi^* x} & F_{x^* x} \end{bmatrix}$$

are evaluated at $\{x_f^*(x), \xi_f^*(x); x\}$.

Using (3.9) and (5.1), we rewrite (6.5) as

$$F_{\xi^*} D_x \xi_f^*(x) + F_{x^*}^T D_x x_f^*(x) = 0 \tag{6.6}$$

$$F_{\xi^* x} D_x \xi_f^*(x) + F_{x^* x} D_x x_f^*(x) = H(x). \tag{6.7}$$

Now $F_{\xi^*} \neq 0$ since $\mathscr{F} \in \boldsymbol{C}$. Eliminating $D_x \xi_f^*(x)$ from (6.6) and substituting in (6.7) gives

$$H(x) = \frac{1}{F_{\xi^*}} [F_{\xi^*} F_{x^* x} - F_{\xi^* x} F_{x^*}^T] D_x x_f^*(x).$$

The proof is completed by showing that the matrix

$$[F_{\xi^*} F_{x^* x} - F_{\xi^* x} F_{x^*}^T]$$

is nonsingular, which follows since

$$\det [F_{\xi^*} F_{x^* x} - F_{\xi^* x} F_{x^*}^T] = F_{\xi^*}^{n-1} \det \begin{bmatrix} F_{\xi^*} & F_{x^*}^T \\ F_{\xi^* x} & F_{x^* x} \end{bmatrix}, \tag{6.8}$$

by Sylvester's identity (Gantmacher (1959), Section II.3), $\neq 0$, since $\mathscr{F} \in \boldsymbol{C} \cap \boldsymbol{A}_1$.

□

6.3 Example. Let $\mathscr{F}$ be the family (2.4) of affine functions: $\boldsymbol{R}^n \to \boldsymbol{R}$. Then

$$x_f^*(x) = f_x(x), \quad \text{by Example 3.4},$$

$$J_0(x) = I \quad \text{by (6.3) since} \quad F_{x^* x} = I, \quad F_{\xi^* x} = 0$$

and (6.4) reduces to the obvious

$$D_x f_x(x) = f_{xx}(x). \tag{6.9}$$

If f is a convex [strictly convex] differentiable function, then its gradient f_x is monotone [strictly monotone] in dom f. This is an immediate consequence of the gradient inequality (Example 4.3), and Theorem 4.2. Alternatively and less directly, the monotonicity of f_x can be shown to follow from (6.9) and the fact that f_{xx} is positive semi definite, see, e.g. Ortega and Rheinboldt (1970), Theorem 5.4.3. Two other cases in which the factorization (6.4) is used to establish a monotonicity property of the $\mathcal{F}$-gradient x_f^*, will now be given.

6.4 THEOREM. *Let $\mathcal{F} \in \mathbf{A}_2 \cap C^2(X)$ where $X = I_1 \times I_2 \times \cdots \times I_n$ is the product of open intervals $I_i \subset R$, $(i = 1, \cdots, n)$. Let each $F \in \mathcal{F}$ be of the form*

$$F(x^*, \xi^*; x) = \sum_{i=1}^{n} F^i(x_i^*, x_i) - \xi^* \tag{6.10}$$

where $F^i(x_i^*, \cdot)$: $I_i \to R$ $(i = 1, 2, \cdots, n)$. Let f: $R^n \to R$ be $\mathcal{F}$-convex [strictly $\mathcal{F}$-convex] with $\operatorname{dom} f \supset X$ and $f \in C^2(X)$. Then x_f^* is a P_0–function [P-function] in X.

PROOF. From (6.10), (6.3) and $\mathcal{F} \in \mathbf{A}_2$ it follows that

$$J_0(x) = F_{x^*x}$$

is a diagonal, positive definite matrix. From (6.4) and Theorem 5.1(a) it therefore follows, for an $\mathcal{F}$-convex function f, that $D_x x_f^*(x)$ is a P_0–matrix, (see Fiedler and Pták (1962, 1966), proving that x_f^* is a P_0–function, by Moré and Rheinboldt (1973), Corollary 5.3.

If f is strictly $\mathcal{F}$-convex, then, by Corollary 4.4(b) (applicable since $\mathcal{F} \in \mathbf{C}$), it follows for any $x, y \in X$, $x \neq y$, that there is a $k = k(x, y) \in \{1, 2, \cdots, n\}$ such that

$$x_k \neq y_k \quad \text{and} \quad x_f^*(x)_k \neq x_f^*(y)_k,$$

proving that x_f^* is a P-function. □

A special case of Theorem 6.4 is the following, one dimensional result:

6.5 COROLLARY. *Let $\mathcal{F} \in \mathbf{A}_1 \cap \mathbf{C}$ be a family of functions: $R \to R$, let f: $R \to R$, S an open subset of $\operatorname{dom} f$, and let f and each $F \in \mathcal{F}$ be twice continuously differentiable in S. If f is $\mathcal{F}$-convex in S then x_f^* is a nondecreasing function in S.*

PROOF. Using (6.3), (6.8) and (6.1) we write

$$J_0(x) = \frac{1}{F_{\xi^*}} \det J(x_f^*(x), \xi_f^*(x); x)$$

$$> 0, \quad \text{by (6.2) and } \mathcal{F} \in \mathbf{C}.$$

Therefore

$$\frac{d}{dx} x_f^*(x) \geqq 0, \quad \text{by (6.4) and Theorem 5.1(a).}$$ □

6.6 COROLLARY. *Let $\mathscr{F}$, f and S be as in Corollary 6.5, where S is an interval (a, b). If*

$$f''(x) > F_{xx}(x_f^*(x), \xi_f^*(x), x), \quad x \in S,$$

then f is strictly $\mathscr{F}$-convex.

PROOF. From (6.4) and (6.11) we infer that x_f^* is $1:1$ on (a, b). As in the proof of Theorem 5.6 this implies that $z = x$ is a local minimizer of $h(z) \stackrel{\Delta}{=} f(z) - F(x_f^*(x), \xi_f^*(x); z)$ and that no other critical point exists in (a, b). Hence $z = x$ is the unique global minimizer of $h(z)$, which was previously shown to be equivalent to the strict $\mathscr{F}$-convexity of f. □

6.7 COROLLARY. *Let $\mathscr{F}$ be as in Theorem 6.4, with $X = R^n$. A function $f: R^n \to R$ with $\operatorname{dom} f = R^n$, $f \in C^2(R^n)$, $f \approx \mathscr{F}$ is strictly $\mathscr{F}$-convex, if the matrix $H(x)$ is positive definite.*

PROOF. Follows from (6.4) and Theorem 5.5. □

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