21

From two-dimensional solitons to four-dimensional magnetic monopoles

21.1 Introduction

Solitons play an important role in non-perturbative two-dimensional fields as we have seen in the first part of this book. They are intimately related to non-trivial topology, they are an essential ingredient in integrable models, and they enable the phenomenon of fermion boson duality-bosonization. When passing to four-dimensional field theories the topology may be even richer and thus we would anticipate having topological solitons as static solutions also in four-dimensional space-time. As we have seen in Section 5.3, Derrik's theorem does not permit the existence of solitons of scalar field theory in space dimensions higher than one, however, they are not prohibited in theories that include higher spin fields, in particular in theories of scalar fields coupled to non-abelian gauge fields. Indeed as we will see in this section certain theories of this type that admit spontaneous symmetry breaking, admit soliton solutions. These configurations carry a conserved topological charge which guarantees their stability against decay to the vacuum. As it will turn out this charge is in fact a magnetic charge and hence these solitons are magnetic monopoles, or in the more general case dyons with both magnetic and electric charge. The construction of dyons from static solutions will be the analog process of building up breathers from two-dimensional solitons.

In the next section we present the basics of the Yang–Mills Higgs theory. We then show the relation between magnetic monopoles and topological solitons both for the simplest case of $SU(2)$ (and $SO(3)$) as well as for a general non-abelian gauge group. The next topic is the seminal solution of 't Hooft and Polyakov. Then we discuss zero modes, time-dependent solutions and dyons. In the following section we discuss the very important limit of BPS. We then describe the construction of multi-monopole solutions that was proposed by Nahm. We show its application to the construction of BPS monopoles of charge one and two. The next topic is the moduli space of monopoles. We determine the metric on this space for the case of widely separated monopoles.

The topic of magnetic monopoles and dyons has been covered by several review papers, proceedings of meetings and books, for instance [21], [214], [67], [7], [182] and [193], respectively. Here in this chapter we made use of mainly the former two references.

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Monopoles and dyons play an important role in $\mathcal{N}=2$ SYM. They admit a mathematical non-perturbative structure that can be determined exactly. Since we do not discuss supersymmetry in this book the monopoles and dyons of that theory will not be addressed in this chapter. We refer the reader to [191] and [192].

21.2 The Yang–Mills Higgs theory – basics

Consider the Yang–Mills Higgs system described by the following Lagrangian density,

$$\mathcal{L} = -\frac{1}{2}\mathrm{Tr}[F_{\mu\nu}F^{\mu\nu}] + \mathrm{Tr}[D_\mu\Phi D^\mu\Phi] - V(\Phi), \tag{21.1}$$

where the (non-abelian) gauge group is G, Φ is in the adjoint representation of the group, namely,

$$A_\mu = T^a A^a_\mu \quad \Phi = T^a\Phi^a, \tag{21.2}$$

where T^a are the generators of G (see Section 3.1), the covariant derivative reads,

$$D_\mu = \partial_\mu\Phi + ie[A_\mu, \Phi], \tag{21.3}$$

and $V(\Phi)$ is given by,

$$V(\Phi) = -\mu^2\mathrm{Tr}[\Phi^2] + \lambda(\mathrm{Tr}[\Phi^2])^2, \tag{21.4}$$

where λ is taken to be positive so that the energy is bounded from below, and we also take $\mu^2 > 0$. In general one can discuss a similar system where Φ is in any representation of G but here we consider only the case of the adjoint representation.

Let us start with the simplest case where $G = SU(2)$ and Φ is in the triplet (adjoint) representation. For such a case the vacuum solution can be put in the form,

$$\begin{aligned}\Phi(x) &= v\frac{\sigma_3}{2} \equiv \Phi_0 \quad v \equiv \sqrt{\frac{\mu^2}{\lambda}}\\ A_\mu(x) &= 0.\end{aligned} \tag{21.5}$$

In this case the vacuum expectation value of the Higgs field breaks the $SU(2)$ symmetry down spontaneously to a $U(1)$ symmetry along the $a = 3$ direction. The physical fields will be denoted as follows,

$$\mathcal{A}_\mu = A^3_\mu \quad W_\mu = \frac{A^1_\mu + iA^2_\mu}{\sqrt{2}} \quad \varphi = \Phi^3, \tag{21.6}$$

which associate with the massless “photon”, pair of mesons W, W^* with a mass of eV and charges $\pm e$ and an electrically neutral scalar boson with mass $m_H = \sqrt{2}\mu$, respectively.

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For a general group G which we take to be a simple Lie group of rank r (for the basic definitions see Section 3.1). In this case the expectation value of $\phi = \Phi_0$ can be taken to lie in the Cartan subalgebra of the group G. Using the notation $\vec{H}$ for the r-dimensional vector of the elements of the Cartan subalgebra, Φ_0 is characterized by a vector $\vec{h}$ such that,

$$\Phi_0 = \vec{h} \cdot \vec{H}. \tag{21.7}$$

The generators of the unbroken subgroup are those generators of G that commute with Φ_0. These are all the generators of the Cartan subalgebra together with ladder operators associated with roots orthogonal to $\vec{h}$. If none of the $\vec{\gamma}$ are orthogonal to h the unbroken symmetry is $U(1)^r$, whereas if there are some roots $\vec{\gamma}$ such that $\vec{\gamma} \cdot \vec{h} = 0$ then the unbroken symmetry is $U(1)^{r-r'} \times K$ where K is of rank r' and it has $\vec{\gamma}$ as its root diagram.

21.3 Topological solitons and magnetic monopoles

In Section 5.2 when discussing two dimensional solitons, we identified a topological conserved current and an associated topological charge. Configurations that carry a non-trivial value with respect to this charge cannot, due to charge conservation, decay to vacuum. These configurations were shown to be stable solutions of the equations of motion and to have finite energy. Thus we anticipate that also in four-dimensional field theories, and in particular in the Yang–Mills Higgs theory we discuss here, there should be solutions of the equations of motion that associate with non-trivial topological charges and one can determine their existence even without solving the equations of motion. It is easy to verify that for $G = SU(2)$ the following current,

$$k_\mu = \frac{1}{8\pi} \epsilon_{\mu\nu\rho\sigma} \epsilon^{abc} \partial^\nu \hat{\Phi}^a \partial^\rho \hat{\Phi}^b \partial^\sigma \hat{\Phi}^c, \tag{21.8}$$

where $\hat{\Phi}^a = \frac{\Phi^a}{|\Phi|}$, is conserved for any configuration whether it solves the equations of motion or not. It follows trivially from the total anti-symmetry of $\epsilon_{\mu\nu\rho\sigma}$ that $\partial^\mu k_\mu = 0$. The corresponding charge is,

$$\begin{aligned} Q &= \int \mathrm{d}^3 x k_0 = \frac{1}{8\pi} \int \mathrm{d}^3 x \epsilon_{ijk} \epsilon^{abc} \partial^i \hat{\Phi}^a \partial^j \hat{\Phi}^b \partial^k \hat{\Phi}^c \\ &= \frac{1}{8\pi} \int \mathrm{d}^3 x \epsilon_{ijk} \epsilon^{abc} \partial^i (\hat{\Phi}^a \partial^j \hat{\Phi}^b \partial^k \hat{\Phi}^c) = \frac{1}{8\pi} \int \mathrm{d}^2 S_i x \epsilon_{ijk} \epsilon^{abc} \hat{\Phi}^a \partial^j \hat{\Phi}^b \partial^k \hat{\Phi}^c. \end{aligned} \tag{21.9}$$

This topological charge is the winding number associated with the map,

$$\Phi_0(\infty): \quad S_2^{\text{space}} \to S_2^{G/H}, \tag{21.10}$$

where S_2^{space} is the boundary of the space at $r = \infty$ and where the coset space G/H is in our example $G/H = SU(2)/U(1) = S_2$. It is thus an integer charge $Q = n$ and as such it must be invariant under smooth deformations of the surface

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of integration that do not cross any of the zeros of Φ. In fact a configuration with $Q = n$ must have at least $|n|$ zeros of the Higgs field. If we distinguish between a $+$ zero and a $-$ anti-zero then the net number is precisely n. Obviously if we consider Higgs configurations of $Q = n = 1$ there must be one with minimum energy. This cannot be smoothly deformed to a vacuum since the winding number is quantized. Hence such a configuration must be a local minimum of the energy and therefore a static classical solution.

The next question we want to address is what is the connection between these non trivial soliton solutions and magnetic monopoles? The field strength of the abelian gauge field $\mathcal{A}_\mu$ defined in (21.6), $\mathcal{F}_{\mu\nu} = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu$ is the outcome of,

$$\tilde{\mathcal{F}}_{\mu\nu} = \hat{\Phi}^a F^a_{\mu\nu} - \frac{1}{g}\epsilon^{abc}\hat{\Phi}^a D_\mu \hat{\Phi}^b D_\nu \hat{\Phi}^c, \tag{21.11}$$

when we take $\hat{\Phi} = (0,0,1)$, namely $\Phi = \varphi$. What distinguishes $\tilde{\mathcal{F}}_{\mu\nu}$ from an ordinary abelian field strength is that it does not obey the Biachi identity,

$${}^*d\tilde{\mathcal{F}} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\partial^\nu \tilde{\mathcal{F}}^{\rho\sigma} = ek_\mu = \frac{4\pi}{g}k_\mu, \tag{21.12}$$

where g is the magnetic charge which will be shown to be equal to $4\pi/e$. Defining now the magnetic field B_i associated with $\tilde{\mathcal{F}}_{\mu\nu}$ as usual as,

$$B_i \equiv \frac{1}{2}\epsilon_{ijk}\tilde{\mathcal{F}}^{jk}, \tag{21.13}$$

we find that,

$$\nabla \cdot \vec{B} = \frac{4\pi}{g}k_0 \quad Q_M = \frac{1}{g}\int \mathrm{d}^3x k_0 = \frac{4\pi}{e}Q = \frac{4\pi}{e}n. \tag{21.14}$$

We have thus realized that the non-trivial soliton configurations carry a magnetic charge and hence are magnetic monopoles. We can further determine the classical mass of the monopole since the total energy of such a solution is,

$$E = \int \mathrm{d}^3x \left[\mathrm{Tr}[E_i^2] + \mathrm{Tr}[(D_0\Phi)^2] + \mathrm{Tr}[B_i^2] + \mathrm{Tr}[(D_i\Phi)^2] + V(\Phi)\right]. \tag{21.15}$$

For a static configuration that does not carry electric charge, the first two terms are expected to vanish. Then one can show that the form of the mass has to be,

$$M = \frac{4\pi v}{e} f\left(\frac{\lambda}{e^2}\right), \tag{21.16}$$

where $f(\frac{\lambda}{e^2})$ should be of order one.

So far we have discussed the topological charges of the group $G = SU(2)$ case. Let us now address the general case. Instead of the map (21.10), the asymptotic Higgs field constitutes in the general case a map,

$$\Phi(\infty): \quad \partial\mathcal{M} \to \frac{G}{H}, \tag{21.17}$$

where $\partial\mathcal{M}$ is the boundary of the space which for ordinary flat Minkowski spacetime is S_2 and G/H is the coset of the unbroken symmetry group H and the

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original group G, which in the discussion above was $SU(2)/U(1) = S_2$. These maps from the boundary of space to the coset, fall into equivalent classes which form the homotopy group $\pi_2(G/H)$. For simply connected group G, namely with $\pi_1(G) = 0$ the classification of the maps is in fact done by $\pi_1(H)$ since for this type of G,

$$\pi_2(G/H) = \pi_1(H). \tag{21.18}$$

This follows from the following exact sequence[1]

$$\ldots \to \pi_2(G) \quad \to \quad \pi_2(G/H) \quad \to \quad \pi_1(H) \quad \to \quad \pi_1(G) \quad \to \ldots. \tag{21.19}$$

The image of a given homomorphism equals the kernel of the next one in the sequence. It is well known that for any semi-simple group G, $\pi_2(G) = 0$ and hence,

$$\pi_2(G/H) \cong \mathrm{Ker}[\pi_1(H) \quad \to \pi_1(G)]. \tag{21.20}$$

Now since for a simply connected group $\pi_1(G) = 0$ we find (21.18). Let us describe now several cases of physical interest:

- The 't Hooft–Polyakov solution that will be discussed in the next section, is slightly different since in that case $G = SO(3)$ which is not simply connected $\pi_1(SO(3)) = \mathcal{Z}_2$ and hence only the even elements of $\pi_1(H = U(1))$ are in the kernel of the homomorphism of (21.20). This is the source of the fact that the quantization condition is twice the one given by Dirac (see (21.27)) even though in both cases $H = U(1)$.
- For a simply connected G of rank r and with H which is the full Cartan subalgebra, namely $H = U(1)^r$, the homotopy group that classifies the magnetic monopoles is $\pi_1(H = U(1)^r) = \mathcal{Z}^r$.
- In the spontaneous symmetry breaking of the electro-weak theory we have $G = SU(2) \times U(1)$ and $H = U(1)$ such that $G/H \cong S^3$. Since $\pi_2(S^3) = 0$ magnetic monopoles are excluded in this theory.
- On the other hand a wide class of grand unified theories do admit magnetic monopoles. The most prominent example is the $G = SU(5)$ grand unified model with $H = SU(3) \times SU(2) \times U(1)$. This is an example of the case that $H = U(1) \times K$ where K is a semi-simple and simply connected group. In this case there is only a single component of the magnetic charge that is topologically conserved.
- Another interesting scenario is the case where the group G twice undergoes a spontaneous symmetry breaking namely, $G \to H_1 \subset G \to H_2 \subset H_1$. This is relevant to an evolution of the universe where at as early stage magnetic

[1] The reader who is not familiar with the notion of an exact sequence can refer to any text book on topology or alternatively to the book of Coleman [66] where an elegant proof of this theorem is presented.

monopoles associated with $\pi_2(G/H_1)$ are being created and then the question is what is their fate when the universe undergoes a second phase transition to H_2? This can be determined by an exact sequence similar to (21.19).

21.4 The 't Hooft–Polyakov magnetic monopole solution

Equipped with knowledge based on the topological arguments of above, that magnetic monopoles do exist for theories associated with non-trivial $\pi_2(G/H)$ we want to proceed to the determination of explicit configurations of the non-trivial topological solutions. It turns out that this is a non-trivial task and only for a limited set of cases can it be accomplished analytically. We start with the simplest case where $G = SU(2)$, or $G = SO(3)$ as was done in the original solution of 't Hooft and Polyakov [123], [175]. We will see in the next section an explicit solution using a special limit of the theory. Without this limit one can simplify the procedure by searching for spherically symmetric solutions. This implies that the fields must be invariant under a combination of rotations and a compensating gauge transformation. The latter can be space independent by using the so called "hedgehog" gauge where rotational invariance requires that the fields be invariant under a combined rotation and global internal $SU(2)$ transformation. Stating it differently, one looks for a configuration which is symmetric under a mixed angular momentum,

$$\vec{J} = \vec{L} + \vec{I}, \tag{21.21}$$

where $\vec{L} = -i\vec{r} \times \vec{\nabla}$ is the ordinary spatial part of the angular momentum and $\vec{I}$ are the generators of the $SU(2)$ gauge group. With this definition of $\vec{J}$ the ansatz should obey,

$$[J_i, \Phi] = 0 \quad [J_i, A_j] = i\epsilon_{ijk} A_k. \tag{21.22}$$

Using this gauge 't Hooft and Polyakov suggested the following ansatz for the fields,

$$A_i^a = \epsilon_{iam} \hat{r}^m \left[\frac{1 - u(r)}{er}\right] \qquad \Phi^a = \hat{r}^a h(r). \tag{21.23}$$

To write down the equations that determine $u(r)$ and $h(r)$ one can substitute these expressions into the Lagrangian density (21.1) and vary with respect to $u(r)$ and $h(r)$. This is easier than the usual procedure of substituting (21.23) into the equations of motion derived from (21.1). The resulting equations are,

$$\begin{aligned} u'' - \frac{(u^2 - 1)u}{r^2} + e^2 u h^2 &= 0 \\ h'' + \frac{2}{r} h' - \frac{2u^2 h}{r^2} + \lambda(v^2 - h^2)h &= 0. \end{aligned} \tag{21.24}$$

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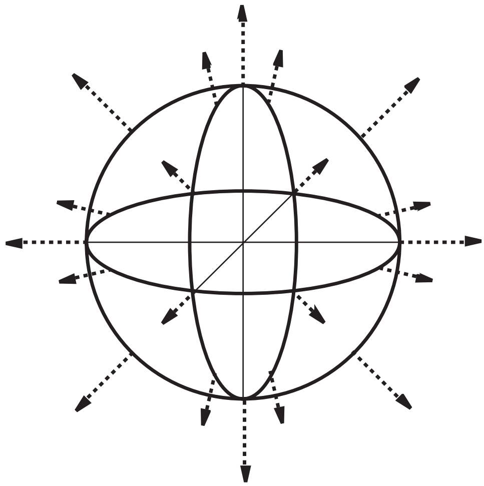

Fig. 21.1. The hedgehog configuration of the Higgs field. The orientation in isospace is aligned with the position vector.

The primes denote derivatives with respect to r. Finiteness of the energy associated with the solution requires that,

$$\begin{aligned} \Phi_0(\infty) = 0 \quad &\rightarrow \quad u(\infty) = 0 \\ D_i\Phi_0(\infty) = 0 \quad &\rightarrow \quad h(\infty) = v. \end{aligned} \tag{21.25}$$

Similarly requiring that the solutions are non-singular at the origin implies that,

$$u(0) = 1 \quad h(0) = 0. \tag{21.26}$$

Qualitatively, the profile of the Higgs field is that of a hedgehog, as can be seen in Fig. 21.1. The orientation in isospace is aligned with the position vector.

Analytic solutions of these equations will be derived in the next section using a special (BPS) limit. In general one has to solve these equations numerically. The physical picture that comes out from these calculations is that there is a central core of radius $R_{\text{core}} \sim \frac{1}{ev}$, outside of which $u(r)$ and $|h-u|(r)$ decrease exponentially. The mass of the monopole takes the form $M = \frac{4\pi v}{e} f(\frac{\lambda}{e^2})$ where $f(0) = 1$ and $f(\infty) = 1.787$.

21.5 Charge quantization

In his seminal paper on magnetic monopoles in quantum mechanics [78] Dirac found out that the magnetic charge g and the electric charge e must be related via the famous charge quantization condition,

$$eg = 2\pi n, \tag{21.27}$$

where n is an integer. This implies that the existence of a magnetic monopole explains the observed quantization of all electric charges. He proposed a solution of a magnetic potential of the following form,

$$e\vec{A} = \frac{eg}{4\pi} \frac{\hat{r} \times \hat{n}}{r(1 - \hat{r} \cdot \hat{n})}, \tag{21.28}$$

which has in addition to the singularity at the origin also a singularity extending from the origin out along the $\hat{n}$ direction. The quantization condition follows from the requirement that physical charges should not be able to detect the string.

Yang–Mills Higgs models with spontaneous symmetry breaking such that $\pi_2(G/H) \neq 0$, as we have seen above, admit magnetic monopole solutions associated with each element of the Cartan subalgebra of the unbroken group H. For large distance the corresponding magnetic fields take the form,

$$eA_i = \frac{eg}{4\pi}(1 + \cos\theta)\partial_i\phi + \mathcal{O}\left(\frac{1}{r^2}\right). \tag{21.29}$$

The generalization of the quantization argument of Dirac to the non-abelian case is straightforward. This follows from the demand that the electrically charged fields of the theory are single valued if acted upon by a group element e^{eg}. This quantization condition can be solved in terms of the simple roots $\vec{\gamma}_i$ where $i = 1, \ldots, r$ where r is the rank of H, of the root system of H. From the simple roots one constructs a convenient basis (H_i) for the elements of the Cartan subalgebra with the property that each element has half-integer eigenvalues when acting on the basis vector of any representation. This is achieved by taking,

$$H_i \equiv \frac{\vec{\gamma}_i}{|\vec{\gamma}_i|^2}\vec{H}. \tag{21.30}$$

In this basis the solution of the quantization condition is,

$$\frac{eg}{4\pi} = \sum_{i=1}^{r} n_i H_i. \tag{21.31}$$

This solution can be represented as an r-dimensional lattice dual to the weight lattice of the group (see Section 3.1). For the simple example of $SO(3)$ the rank is one and thus one gets $eg = 4\pi n$ twice as the condition of Dirac due, as was explained above, to the fact that $SO(3)$ is not simply connected. For the group $SU(3)$ which is of rank $r = 2$ the charge lattice is drawn in Fig. 21.2. In general it was shown that the charge lattice is the weight lattice of the dual gauge group.

21.6 Zero modes, time-dependent solutions and dyons

From the static $SU(2)$ monopole solution discussed above we can generate obviously (infinitely) more solutions by applying gauge transformations. This of course will be avoided by fixing a gauge. However even in that case there is

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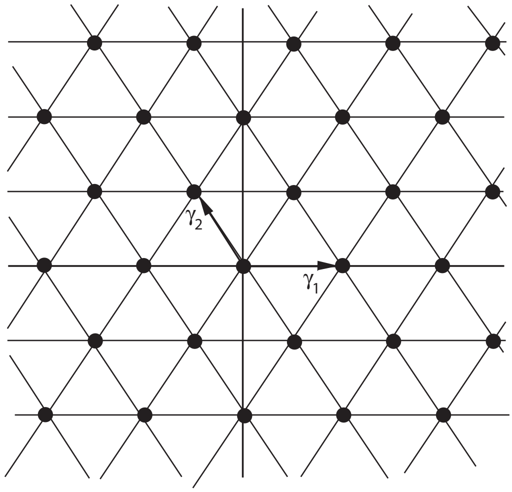

Fig. 21.2. The charge lattice of the $SU(3)$ monopole.

one unfixed global gauge $U(1)$ phase. This adds up to the parameters associated with the translation of the monopole. Infinitesimal transformations of this set of four parameters generate field variations δA_i and $\delta\Phi$ that preserve the equations of motion and leave the energy unchanged. These variations in general will be referred to as *zero modes*.

Time-dependent excitations of the translational zero modes can be derived by substituting $\vec{r} \to \vec{r} - \vec{v}t$ into the static solution. This has to be done together with ensuring the Gauss law constraint,

$$D_i E^i = ie[\Phi, D^0\Phi], \tag{21.32}$$

which also implies that for most choices of gauge $A^0 \neq 0$. Substituting the solutions of the Gauss law into the expression of the energy (21.15) one can show that the change of energy for non-relativistic velocities is as one expects, given by,

$$\Delta E = \frac{1}{2}M|\vec{v}|^2. \tag{21.33}$$

Next we want to describe the excitation of the zero mode associated with the fourth parameter, that of the global gauge phase. The Noether charge associated with this transformation is the electric charge that corresponds to the unbroken $U(1)$ gauge symmetry. In terms of the physical variables defined in (21.6) this takes the form,

$$Q_E = -ie\int \mathrm{d}^3x[W^{j\,*}\mathcal{D}_0(W_j - \mathcal{D}_jW_0) - W^j\mathcal{D}_0(W_j^* - \mathcal{D}_jW_0^*)], \tag{21.34}$$

where the $U(1)$ covariant derivative is defined by $\mathcal{D}_\mu W_\nu = (\partial_\mu - ie\mathcal{A}_\mu)W_\nu$ and where a string gauge is used where the Higgs field direction is uniform.

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A dyon [134] is defined to be a configuration that carries both magnetic as well as electric charges. To construct such a solution we start with a static solution in the string gauge and multiply the W field by a uniformly varying $U(1)$ phase factor e^{iwt}. In analogy to the magnetic charge found above, the electric charge has the form,

$$Q_E = \int \mathrm{d}^2 S_i \tilde{\mathcal{F}}_{0i} = \int \mathrm{d}^2 S_i \hat{\Phi}^i E_i^a. \tag{21.35}$$

The time-dependent configuration can be transformed into a static solution by a $U(1)$ transformation of the form,

$$W_i \to e^{-iwt} W_i \quad \mathcal{A}_i \to \mathcal{A}_i \quad \mathcal{A}_0 \to \bar{\mathcal{A}}_0 = \mathcal{A}_0 - \frac{w}{e}. \tag{21.36}$$

In this static form we can transform the solution into the non-singular hedgehog gauge with A_i^a and Φ^a given by (21.23) and,

$$A_0^a = \hat{r}^a j(r) = \hat{r}^a \bar{\mathcal{A}}_0. \tag{21.37}$$

For this case the static field equations, which are the analog of (21.24) become,

$$\begin{aligned} h'' + \frac{2}{r} h' - \frac{2u^2 h}{r^2} + \lambda(v^2 - h^2)h &= 0, \\ u'' - \frac{(u^2-1)u}{r^2} - e^2 u(h^2 - j^2) &= 0, \\ j'' + \frac{2}{r} j' - \frac{2u^2 j}{r^2} &= 0, \end{aligned} \tag{21.38}$$

where the first equation is identical to the one in (21.24), in the second there is a $2e^2 uj^2$ addition and the third equation is the Gauss law.

The dependence of Q_E on w can be determined by substituting the ansatz into (21.34) and recalling that $\mathcal{W}_0 = 0$ we find,

$$Q_E = \frac{8\pi w}{e} \int \mathrm{d}r u(r)^2 \frac{j(r)}{j(\infty)} \equiv I w. \tag{21.39}$$

The integral can be estimated since $u(r)$ falls off exponentially outside a region of radius $\sim 1/v$ so that,

$$I = \frac{4\pi}{e^2 v} \bar{I}, \tag{21.40}$$

where $\bar{I}$ is of order unity. In analogy to (21.33) one can show that the correction to the energy is,

$$\Delta E = \frac{Q_E^2}{2eI} = \left(\frac{e}{4\pi}\right)^2 \frac{Q_E^2 M}{2\bar{I} f} \sim \frac{Q_E^2}{2Q_M^2} M. \tag{21.41}$$

21.7 BPS monopoles and dyons

A special limit of the monopole configuration occurs when one takes the limit of,

$$\lambda \to 0 \quad \mu^2 \to 0 \quad v^2 = \frac{\mu^2}{\lambda} \text{ fixed.} \tag{21.42}$$

In this limit the energy of the system,

$$\begin{aligned} E &= \int d^3x \mathrm{Tr}[\vec{E}\cdot\vec{E} + \vec{B}\cdot\vec{B} + (D_0\Phi)^2 + (\vec{D}\Phi)^2] \\ &= \int d^3x \mathrm{Tr}[(\vec{E} \pm \sin\alpha\vec{D}\Phi)^2 + (\vec{B} \pm \cos\alpha\vec{D}\Phi)^2 + (D_0\Phi)^2] \\ &\pm 2\int d^3x[\cos\alpha\mathrm{Tr}[\vec{B}\cdot\vec{D}\Phi] + \sin\alpha\mathrm{Tr}[\vec{E}\cdot\vec{D}]]. \end{aligned} \tag{21.43}$$

The passage from the first line to the rest is of course an identity for arbitrary α. Next we perform an integration by parts in the last line and use the Gauss law $\vec{D}\cdot E - ie[\Phi, D^0\Phi] = 0$ and $\vec{D}\cdot\vec{B} = 0$ we find,

$$\begin{aligned} E = &\int d^3x \mathrm{Tr}[(\vec{E} \pm \sin\alpha\vec{D}\Phi)^2 + (\vec{B} \pm \cos\alpha\vec{D}\Phi)^2 + (D_0\Phi)^2 \\ &\pm \cos\alpha\mathcal{Q}_M \pm \sin\alpha\mathcal{Q}_E \\ E \quad &\geq \pm\cos\alpha\mathcal{Q}_M \pm \sin\alpha\mathcal{Q}_E, \end{aligned} \tag{21.44}$$

where the magnetic and electric charges $\mathcal{Q}_\mathrm{M} = vQ_\mathrm{M}$ and $\mathcal{Q}_\mathrm{E} = vQ_\mathrm{E}$ are given by (21.14) and (21.35)

$$\mathcal{Q}_\mathrm{M} = 2\int d^2\vec{S}\cdot\mathrm{Tr}[\Phi\vec{B}] \quad \mathcal{Q}_\mathrm{E} = 2\int d^2\vec{S}\cdot\mathrm{Tr}[\Phi\vec{E}]. \tag{21.45}$$

Recall that so far α is arbitrary. The most stringent bound is achieved when one takes $\tan\alpha = \frac{\mathcal{Q}_\mathrm{E}}{\mathcal{Q}_\mathrm{M}}$, for which the bound reads,

$$E \geq \sqrt{\mathcal{Q}_\mathrm{M}^2 + \mathcal{Q}_\mathrm{E}^2}. \tag{21.46}$$

It is easy to realize that the bound is saturated if,

$$\vec{E} = \cos\alpha\vec{D}\Phi \quad \vec{B} = \sin\alpha\vec{D}\Phi \quad D_0\Phi = 0. \tag{21.47}$$

These first-order equations are referred to as the Bogomolny Prasad Sommerfeld equations or BPS equations ([41] and [180]). The configurations that obey these equations have the minimal value of energy,

$$E = \sqrt{\mathcal{Q}_\mathrm{M}^2 + \mathcal{Q}_\mathrm{E}^2}, \tag{21.48}$$

for given magnetic and electric charges $\mathcal{Q}_\mathrm{M}, \mathcal{Q}_\mathrm{E}$ respectively and hence are also solutions of the (second-order) equations of motion of the system. In particular

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the magnetic monopole which carries no electric charge and is static and hence $\mathcal{Q}_{\rm E} = 0$ and $D_0\Phi = 0$ (in the $A_0 = 0$ gauge), obeys the Bogomolny equation,

$$\vec{B} = \vec{D}\Phi. \tag{21.49}$$

The BPS limit seems to be unnatural and artificial since we have introduced a potential to give a non-trivial expectation value to the field Φ and then we tuned the potential to zero keeping the expectation value. It turns out that in certain suspersymmetric models the BPS equations follow from the requirement of the invariance under supersymmetry. Since supersymmetry is not discussed in this book we refer the reader to the literature, for instance [214].

If we go back to the equations of motion and substitute $\lambda = 0$ the equations take the form,

$$\begin{aligned} h'' + \frac{2}{r}h' - 2\frac{u^2 h}{r^2} &= 0, \\ u'' - \frac{u(u^2-1)}{r^2} - e^2 u h^2 &= 0. \end{aligned} \tag{21.50}$$

The solution of this set of equations is given by,

$$u(r) = \frac{evr}{\sinh(evr)} \qquad h(r) = v\coth(evr) - \frac{1}{er} \tag{21.51}$$

The fact that Φ falls off as $1/r$ and not exponentially is due to the fact that in the BPS limit it has a vanishing mass and hence associates with a long range force.

The BPS equations can also be solved for the case of a dyon, namely a configuration that carries both a magnetic as well as an electric charge. In that case the solution reads,

$$\begin{aligned} u(r) &= \frac{e\tilde{v}r}{\sinh(e\tilde{v}r)} \\ h(r) &= \frac{\sqrt{Q_M^2 + Q_E^2}}{Q_M}\left[\tilde{v}\coth(e\tilde{v}r) - \frac{1}{er}\right] \\ j(r) &= -\frac{Q_M}{Q_E}\left[\tilde{v}\coth(e\tilde{v}r) - \frac{1}{er}\right]. \end{aligned} \tag{21.52}$$

21.8 Montonen Olive duality

In two dimensions we have seen an equivalence between a soliton configuration and an elementary field. This was the essence of bosonization manifested for instance in the equivalence between the sine-Gordon theory and the Thirring model (see Section 6.2). Montonen and Olive [163] conjectured an analogous duality between the spectrum of states created from the elementary field and from those created by the solitons. Since the former are electrically charged and the latter are magnetically charged, the duality is in a sense a non-abelian

Table 21.1. *The spectrum of the $SU(2)$ YM Higgs theory in the BPS limit*

	Mass	Q_E	Q_M
photon	0	0	0
ϕ	0	0	0
$W^{\pm}$	ev	$\pm e$	0
Monopole	$\frac{4\pi v}{e}$	0	$\pm\frac{4\pi}{e}$

generalization of electric–magnetic duality of the Maxwell equations. To understand this duality notice that under the following operation,

$$Q_{\mathrm{E}} \leftrightarrow Q_{\mathrm{M}} \quad e \leftrightarrow \frac{4\pi}{e}, \tag{21.53}$$

the entries associated with the W mesons and those of the magnetic monopoles in Table 21.1 are interchanged.

It turns out that on top of the self-duality of the spectrum, there is a similar duality also in the low energy scattering. It is a well-known property of BPS states in general and in particular the magnetic monopoles that there is no net force between them. This follows up from an exact cancellation between the magnetic repulsion and the attraction due to an exchange of a Higgs scalar. The zero velocity limit of the scattering amplitude of two W bosons also admits a no force behavior. The exchange of a single photon is cancelled out by the exchange of a Higgs boson.

In the non-supersymmetric YM Higgs theory the duality of the spectrum and the scattering amplitudes cannot be lifted to a duality of the full theory. A simple indication of this is the fact that the W boson carries a spin one, whereas the quantum state built from a spherical monopole carries spin 0. In supersymmetric analogs of the YM Higgs theory this difficulty may be overcome since both the W bosons as well as the magnetic monopoles are members of supersymmetric multiplets that contain states with several different spins. It turns out that the YM theory with 16 supercharges, the so-called $\mathcal{N} = 4$ SYM admits a complete invariance under the Olive Montonen duality. Since we do not discuss supersymmetry in the book we refer the interested reader to the review papers mentioned above, for instance [214].

21.9 Nahm construction of multimonopole solutions

This construction [167] maps the Bogomolny equation in three variables into a nonlinear equation in one variable. We present the construction for the $SU(2)$ case. To simplify the notations we set the coupling constant $e = 1$. It can be restored when needed.

The construction of an $SU(2)$ k monopole is built from three steps:

1. We look for a quartet of Hermitian $k \times k$ matrices $T_\mu(s)$, where $\mu = 0, i = 1, 2, 3$ which satisfy the Nahm equation,
$$\frac{\mathrm{d}T_i}{\mathrm{d}s} + i[T_0, T_i] + \frac{i}{2}\epsilon_{ijk}[T_j, T_k] = 0, \tag{21.54}$$
where s is an auxiliary variable that takes its value in the interval $-v/2 \leq s \leq v/2$ and where v is the vacuum expectation value of the Higgs field. For $k = 1$ since the commutators vanish we get that T_i are constants. In fact due to the ordinary gauge invariance one can choose a gauge where $T_0 = 0$ and hence the equation reads,
$$\frac{\mathrm{d}T_i}{\mathrm{d}s} + \frac{i}{2}\epsilon_{ijk}[T_j, T_k] = 0. \tag{21.55}$$
The boundary condition that one should impose for the multimonople case is that the $T_i(s)$ have poles at the boundaries of the form,
$$T_i(s) = -\frac{L_i^\pm}{s \mp \frac{v}{2}} + O(1). \tag{21.56}$$
The Nahm equations implies that the $L_i^\pm$ form a k-dimension representation of the $SU(2)$ algebra,
$$[L_i^\pm, L_j^\pm] = i\epsilon_{ijk}L_k^\pm. \tag{21.57}$$
These representations should be irreducible, namely, must be equivalent to the $(k-1)/2$ representation.
2. The next step is to solve the construction equation for the $2k$ component vector $w(s, \vec{r})$,
$$\Delta^\dagger w(s, \vec{r}) \equiv \left[-\frac{\mathrm{d}}{\mathrm{d}s} - T_i \otimes \sigma_i + r_i I_k \otimes \sigma_i\right] w(s, \vec{r}) = 0. \tag{21.58}$$
We denote by $w_a(s, \vec{r})$ a completely linearly independent set of normalizable solutions that obey the orthonormality condition,
$$\int_{-v/2}^{v/2} \mathrm{d}s w_a^\dagger(s, \vec{r}) w_b(s, \vec{r}) = \delta_{ab}. \tag{21.59}$$
3. It can be proved that for the $SU(2)$ case there are only two normalizable solutions $w_a(s, \vec{r})$. The space-time fields are given in terms of these as follows,
$$\Phi^{ab}(\vec{r}) = \int_{-v/2}^{v/2} \mathrm{d}s w_a^\dagger(s, \vec{r}) w_b(s, \vec{r}),$$
$$A_j^{ab}(\vec{r}) = \int_{-v/2}^{v/2} \mathrm{d}s w_a^\dagger(s, \vec{r}) \partial_j w_b(s, \vec{r}). \tag{21.60}$$

We do not bring here the the details of the proof of this construction (see for instance [71]) we just mention that it includes the following elements: (i) Using

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the expressions for Φ^{ab} and A_i^{ab} given above in (21.60) one can show that B_i^{ab} and $D_i\Phi^{ab}$ are identical and hence the Bogomolny equation is obeyed. (ii) Showing that the solutions lie in $SU(2)$ and (iii) computing the long-range behavior and showing that the solutions indeed have a magnetic charge equal to k. We refer the interested reader to, for instance, [214] for the detailed proof. Instead we proceed now to a demonstration of the application of the method both for $k = 1$ and $k = 2$. As was mentioned above for the former case the T_i are constants independent of s. In fact these constant values of $\vec{T}$ enter the construction equation as $(\vec{r} - \vec{T}) \cdot \vec{\sigma}$ namely, they are the coordinates of the center of mass of the monopole and thus by shifting to a frame of coordinates which is centered at the monopole center $\vec{T} = 0$. The construction equation (21.58) takes the form,

$$\frac{\mathrm{d}w}{\mathrm{d}s} = \vec{r} \cdot \vec{\sigma} w. \tag{21.61}$$

The two normalizable solutions of this equation are,

$$w_a(s, \vec{r}) = \mathcal{N}(r) e^{s\vec{r}\cdot\sigma} \eta_a, \tag{21.62}$$

where $\mathcal{N}$ is a normalization factor and η_a are orthonormal constant vectors. From the orthonormality condition we find that,

$$\mathcal{N}(r) = \sqrt{\frac{r}{\sinh(vr)}}, \tag{21.63}$$

where we have made use of the first of the following integrals,

$$\begin{aligned}
\int_{-v/2}^{v/2} \mathrm{e}^{2s\vec{r}\cdot\vec{\sigma}} \mathrm{d}s &= \frac{\sinh(vr)}{r} I_2 \\
\int_{-v/2}^{v/2} s\mathrm{e}^{2s\vec{r}\cdot\vec{\sigma}} \mathrm{d}s &= \frac{\vec{r}\cdot\vec{\sigma}}{r^3}[vr\cosh(vr) - \sinh(vr)].
\end{aligned} \tag{21.64}$$

Using the integral we find that the Higgs field and the corresponding gauge field are given by,

$$\begin{aligned}
\Phi_{ab} &= \frac{1}{2}\left(v\coth(vr) - \frac{1}{r}\right)\eta_a^\dagger \frac{\vec{r}\cdot\sigma}{r}\eta_b, \\
\vec{A}_{ab} &= -i\eta_a^\dagger \partial_i \eta_b - i\epsilon_{ijk}\hat{r}_j \eta_a \sigma_k \eta_b \left(\frac{1}{2r} - \frac{v}{2\sinh(vr)}\right).
\end{aligned} \tag{21.65}$$

Upon setting $\eta_1^t = (1, 0)$ and $\eta_2^t = (0, 1)$ we retrieve the hedgehog solution of 21.51).

21.9.1 SU(2) two-monopole solutions

The $k = 2$ solutions are characterized by a priori eight parameters out of which one corresponds to the $U(1)$ phase and as for the case of $k = 1$ does not enter the Nahm data. Three parameters relate to the translation and three to the rotations of the solution. Thus we end up with one non-trivial parameter. Intuitively this parameter should relate to the separation distance between the two centers of the solution. Let us see if the Nahm construction verifies this intuition and to what extent one can write an explicit solution for this case. The $T_i(s)$ which now are not constants can be decomposed into the following form,

$$T_i(s) = \frac{1}{2}\vec{T}_i^v(s)\cdot\vec{\sigma} + T_i^s(s)I_2. \tag{21.66}$$

Substituting this into the Nahm equation implies that the $\vec{T}_i^v(s)$ have to obey,

$$\frac{\mathrm{d}\vec{T}_i^v(s)}{\mathrm{d}s} = \frac{1}{2}\epsilon_{ijk}\vec{T}_j^v(s)\times\vec{T}_k^v(s). \tag{21.67}$$

It is easy to realize that there is a relation between the $\vec{T}_k^v(s)$, namely the following matrix,

$$T_{ij} = \vec{T}_i^v(s)\cdot\vec{T}_j^v(s) - \frac{1}{3}\delta_{ij}\vec{T}_k^v(s)\cdot\vec{T}_k^v(s), \tag{21.68}$$

is s independent. After some tedious algebra one can show that the most general form of $\vec{T}_k^v(s)$ takes the form,

$$\vec{T}_i^v(s) = \frac{1}{2}\sum_i A_{ij}f_j(s+v/2,\kappa,d)\tau_j + T_i^s I_2, \tag{21.69}$$

where A_{ij} is an orthogonal matrix of constants that diagonalize T_{ij}, $f(s+v/2,\kappa,d)$ are the Euler top functions[2] and the parameter d can be shown for widely separated monopoles to be the inter-monopole distance as the original intuition taught us.

21.10 Moduli space of monopoles

As we have seen above the monopole configurations are parameterized by certain moduli. One defines a space of these moduli, the moduli space. The properties of this space are intimately related to the low energy behavior of monopoles and dyons. In Section 22.3 we will describe the moduli space of YM instantons. Denoting the collective coordinates that parameterize a monopole configuration

[2] The Euler top function can be expressed in terms of the elliptic functions $SN_\kappa(x), CN_\kappa$ and $DN_\kappa(x)$ as follows:

$$\begin{aligned} f_1(x,\kappa,D) &= -D\frac{CN_\kappa(Dx)}{SN_\kappa(Dx)}\\ f_2(x,\kappa,D) &= -D\frac{DN_\kappa(Dx)}{SN_\kappa(Dx)}\\ f_3(x,\kappa,D) &= -D\frac{1}{SN_\kappa(Dx)} \end{aligned} \tag{21.70}$$

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by z_r, the Lagrangian of the system is approximated by,

$$\mathcal{L} = -(\text{total rest mass of monopoles}) + \frac{1}{2} g_{rs}(z) \dot{z}_r \dot{z}_s. \tag{21.71}$$

The metric on the moduli space $g_{rs}(z)$ can be determined from the background zero models of the gauge fields as follows,

$$g_{rs}(z) = 2 \int \mathrm{d}^3 x \mathrm{Tr}[\delta_r A_i \delta_s A_i + \delta_r \Phi \delta_s \Phi] = 2 \int \mathrm{d}^3 x \mathrm{Tr}[\delta_r A_a \delta_s A_a], \tag{21.72}$$

where a takes the values $a = 1, \ldots, 4$ with $A_4 = \Phi$, and where,

$$\delta_r A^a = \frac{\partial (A^{cl})^a}{\partial z_r} - D^a \epsilon_r, \tag{21.73}$$

and where ϵ_r is defined via $A_0 = \dot{z}_r \epsilon_r$ which follows from the Gauss law, $D_a F^{a0} = 0$.

In the case of a single monopole, as we have seen above, there are four zero modes associated with the location of the center of the monopole $\vec{r}_{cm}$ and the global $U(1)$ phase so that,

$$g_{rs}(z) \dot{z}_r \dot{z}_s = M (\dot{\vec{r}_{cm}})^2 + \frac{I}{e} \dot{\alpha}^2, \tag{21.74}$$

where M is the mass of the monopole and I is defined via (21.39) $Q_E = Iw$.

The moduli space of BPS monopoles is hyper-Kähler. This property that implies that there are three almost complex structures with correspondingly three closed Kähler forms will be discussed in detail in Chapter 22, so we will not discuss it here for the BPS monopoles.

An important part of the structure of the moduli space is encoded in its isometries, namely the symmetries that preserve the form of the metric. Naturally since the underlying space where the monopoles reside is an R^3 the isometries include three translations of the center of mass of the collective coordinates. The same applies to the spatial rotation of the monopole. It takes a monopole solution to another monopole solution and hence it maps one point in the moduli space into another one. Another type of isometries is those associated with the unbroken $U(1)$ gauge groups. These isometries, unlike the rotational isometry preserve the complex structure. One can choose a coordinate basis where the gauge transformations act as translations of the angular variables ξ^A, with the corresponding Killing vectors being $K_A = \frac{\partial}{\partial \xi^A}$. Denoting by y^p the rest of the coordinates, the Lagrangian associated with the moduli space approximation can be written as,

$$L = \frac{1}{2} g_{pq}(y) \dot{y}^p \dot{y}^q + \frac{1}{2} \tilde{g}_{AB}(y) [\dot{\xi}^A + \dot{y}^p w_p^A(y)][\dot{\xi}^B + \dot{y}^q w_q^B(y)]. \tag{21.75}$$

Thus the coordinates ξ^A are cyclic coordinates and their conjugate momenta are conserved. In fact the latter are the electric charges of the dyonic cores.

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For monopoles that are separated by large distances the task of determining the metric of the moduli is much easier. Consider a system of N fundamental monopoles all well separated from each other. In such a layout only abelian interactions are relevant and there is an enhanced gauge symmetry. Instead of having a conserved electric charge for each unbroken $U(1)$, there is an effective conserved charge for each monopole core. The moduli space is spanned in this case by $3N$ coordinates of the positions of the cores and N angles $\vec{\xi}_j$, $j = 1, \ldots, N$. The enhance symmetry is the translation along each of the ξ_j. The approximated Lagrangian then takes the form,

$$L = \frac{1}{2} M_{ij}(x) \dot{\vec{x^i}} \cdot \dot{\vec{x^j}} + \frac{1}{2} \tilde{g}_{ij}(x) [\dot{\xi}^i + w^i_k(x) \dot{\vec{x^k}}][\dot{\xi}^j + w^i_l(x) \dot{\vec{x^l}}]. \tag{21.76}$$

By computing the pairwise interactions between the separated dyons, one can determine the functions $M_{ij}(x), \tilde{g}_{ij}(x)$ and $w^i_j(x)$. We refer the reader to [214] for the derivation and we cite here the results,

$$M_{ij} = m_i - \textstyle\sum_{k \neq 1} \frac{4\pi \vec{\alpha}^*_i \cdot \vec{\alpha}^*_k}{e^2 r_{ik}}, \quad i = j \tag{21.77}$$

$$M_{ij} = \textstyle\sum_{k \neq 1} \frac{4\pi \vec{\alpha}^*_i \cdot \vec{\alpha}^*_j}{e^2 r_{ij}}, \quad i \neq j \tag{21.78}$$

$$\vec{W}_{ij} = -\textstyle\sum_{k \neq 1} \vec{\alpha}^*_i \cdot \vec{\alpha}^*_k \vec{w}_{ik}, \quad i = j \tag{21.79}$$

$$\vec{W}_{ij} = \vec{\alpha}^*_i \cdot \vec{\alpha}^*_i w_{ij}, \quad i \neq j \tag{21.80}$$

and $K = \frac{(4\pi)^2}{e^4} M^{-1}$. It can be further shown that the metric of the moduli space of a two monopole BPS solution can be determined exactly and it takes the form of a Taub–Nut metric or Atiya–Hitchin metric [19] depending whether the monopoles are distinct or the same. This is beyond the scope of this book and we refer the reader to for instance the review [214].

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