Probability in the Engineering and Informational Sciences (2024), **38:**1 1–20
doi:10.1017/S0269964822000444

CAMBRIDGE
UNIVERSITY PRESS

RESEARCH ARTICLE

Tsallis value-at-risk: generalized entropic value-at-risk

Zhenfeng Zou[1], Zichao Xia[1] and Taizhong Hu[2]

[1]Department of Statistics and Finance, School of Management, University of Science and Technology of China, Hefei, Anhui 230026, China. E-mails: newzzf@mail.ustc.edu.cn, zichaox@mail.ustc.edu.cn
[2]IIF, School of Management, University of Science and Technology of China, Hefei, Anhui 230026, China.
E-mail: thu@ustc.edu.cn.

Keywords: Duality, Generalized q-exponential function, Generalized q-logarithm function, Generalized relative entropy, Luxemburg norm, Orlicz hearts, Orlicz norm, Orlicz spaces

MSC 2000: 91B30, 91G99

Abstract
Motivated by Ahmadi-Javid (*Journal of Optimization Theory Applications*, 155(3), 2012, 1105–1123) and Ahmadi-Javid and Pichler (*Mathematics and Financial Economics*, 11, 2017, 527–550), the concept of *Tsallis Value-at-Risk* (TsVaR) based on Tsallis entropy is introduced in this paper. TsVaR corresponds to the tightest possible upper bound obtained from the Chernoff inequality for the Value-at-Risk. The main properties and analogous dual representation of TsVaR are investigated. These results partially generalize the Entropic Value-at-Risk by involving Tsallis entropies. Three spaces, called the primal, dual, and bidual Tsallis spaces, corresponding to TsVaR are fully studied. It is shown that these spaces equipped with the norm induced by TsVaR are Banach spaces. The Tsallis spaces are related to the L^p spaces, as well as specific Orlicz hearts and Orlicz spaces. Finally, we derive explicit formula for the dual TsVaR norm.

1. Introduction

One of the important issues in insurance is insurance pricing. Over the past two decades, researchers have made great efforts to implement appropriate insurance pricing methods. One of the most popular methods is an axiomatic approach to characterize insurance prices, see Wang *et al.* [26]. The insurance pricing can be described as a functional from the set of nonnegative insurance risks to the extended nonnegative real numbers. How to measure the uncertainty is a key problem. “Entropy,” date back to 1865, is one of the best ways to measure uncertainty in probability theory. Shannon [22] introduced the *information entropy* of a discrete random variable Z with probability mass function $\{p_k\}$ by

$$H(Z) := \sum_k p_k \log p_k,$$

which is extended to the case of Z being a continuous random variable. Closely related to Shannon entropy is the quantity

$$H(\mathbb{Q} \,|\, \mathbb{P}) := \mathbb{E}_{\mathbb{P}}\left[\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} \ln\left(\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right)\right], \tag{1.1}$$

which is called *relative entropy* (also called *Kullback–Leibler divergence*), where $\mathbb{Q} \ll \mathbb{P}$, and $\mathbb{Q}$ and $\mathbb{P}$ are two probability measures. Relative entropy has been proved to have close connections with insurance, mathematical finance, risk measures, and others. For example, based on the variational representation of the relative entropy, Föllmer and Schied [11] defined entropic risk measure and studied its properties systemically, Delbaen *et al.* [8] solved the problem of hedging a contingent claim by maximizing the expected exponential utility of terminal net wealth, and Ahmadi-Javid [1,2] defined a new coherent risk measure called the *Entropic Value-at-risk* (EVaR). There are many generalizations of relative entropy in the literature, such as Rényi divergence [19,20] and generalized relative entropy (also called Tsallis relative entropy [24,25]).

Recently, Ma and Tian [16] established a variational representation for the generalized relative entropy, and Tian [23] provided new ideas and insights for pricing the non-attainable contingent claim in incomplete market under the generalized relative entropy. Inspired by these two papers, we propose *Tsallis Value-at-Risk* (TsVaR) based on Tsallis entropy by following the framework of Ahmadi-Javid [2] and Ahmadi-Javid and Pichler [4]. As shown by Tsallis [25], generalized relative entropy is defined by the generalized q-logarithm and generalized q-exponential functions. In order to introduce TsVaR (see Definition 3.1), we restrict ourself to consider nonnegative random variables and $q \in (0, 1]$. Under these restrictions, TsVaR corresponds to the tightest possible upper bound obtained from the Chernoff inequality for the Value-at-Risk (VaR). We show that TsVaR is not a coherent premium principle, even not a convex premium principle (see Definition 2.1 for formal definitions). This is caused by a lack of cash invariance in general. We also show that in the class of TsVaR with $q \in (0, 1]$, only EVaR is a coherent premium principle. Although TsVaR fails to be a convex premium principle, TsVaR's dual representation is an analogy with the generalized relative entropy.

The motivation of this paper is as follows. The first one is to generalize the concept of EVaR, defined by Ahmadi-Javid [2], to TsVaR, using the generalized q-logarithm and generalized q-exponential functions. The second one is to deepen our understanding of the application of entropy in risk measure and premium principle, and to investigate its theoretical properties.

As shown by Pichler [17,18], there is an intimate link between norms and coherent risk measures defined on the same model space. Specifically, a coherent risk measure ρ can be used to define an order-preserving semi-norm on the model space $\mathcal{X}$ by $\|\cdot\|_\rho := \rho(|\cdot|)$, whenever ρ is finite over $\mathcal{X}$. Under suitable assumptions on norms, coherent risk measures can be recovered from norms (see [18] Theorem 3.1). These considerations are also extended to vector-valued random variables (see [15]).

Fortunately, when consider the confidence level $\alpha \in (0, 1)$, the norm induced by TsVaR, called the TsVaR norm, is indeed a norm. TsVaR norms are proven to be equivalent to each other for different confidence levels, but they do not generate the L^p norms $\|\cdot\|_p$. For every $1 < p < (2-q)/(1-q)$, the L^p norms are bounded by the TsVaR norm, while the converse does not hold true. Thus, the largest model space contained in the domain of TsVaR is strictly larger than L^∞ but smaller than every L^p space. We also relate the TsVaR norm to the Orlicz (or Luxemburg) norm on the associated Orlicz space. Orlicz space has been used intensively to study risk measures (see [5,6]). Finally, we also present closed-form expression for the dual TsVaR norm.

The remainder of the paper is organized as follows. Section 2 introduces Tsallis relative entropy and premium principle. Section 3 defines the main object TsVaR and Tsallis spaces, and provides some basic properties about TsVaR. In Section 4, we first compare Tsallis spaces with other spaces, particularly with the L^p and Orlicz spaces, and then elaborate duality relations. In this section, we also present the closed formula of the dual TsVaR norm. Section 5 concludes the paper. Proofs of all lemmas and some propositions are postponed to Appendix A.

2. Preliminaries

2.1. *Premium principles*

Throughout, we consider an atomless probability space $(\Omega, \mathscr{F}, \mathbb{P})$. Let $\mathcal{X}$ be a model space, which is used to represent a set of nonnegative random losses. For $X \in \mathcal{X}$, a positive value of X represents the insurable loss of the insured. Let L_+^0 be the set of all nonnegative random variables, and L_+^k be the set of all random variables in L_+^0 with finite kth moment, where $k > 0$. For $X, Y \in \mathcal{X}$, we write $X \overset{\mathrm{d}}{=} Y$ whenever X and Y have the same distribution.

In this paper, we consider premium principles defined on the model space $\mathcal{X}$, which are understood as prices of the insurable loss of the insured. A premium principle ρ is a functional $\rho : \mathcal{X} \to \mathbb{R}_+ \cup \{+\infty\}$, where $\mathbb{R}_+ = [0, \infty)$.

Definition 2.1. *A mapping $\rho : \mathcal{X} \to \mathbb{R}_+ \cup \{+\infty\}$ is called a convex premium principle if it satisfies the following three properties: for all $X, Y \in \mathcal{X}$,*

(A1) *Cash invariance: $\rho(X + m) = \rho(X) + m$ for all $m \in \mathbb{R}_+$;*
(A2) *Monotonicity: $X \le Y \Longrightarrow \rho(X) \le \rho(Y)$;*
(A3) *Convexity: $\rho(\alpha X + (1-\alpha)Y) \le \alpha\rho(X) + (1-\alpha)\rho(Y)$ for all $\alpha \in [0, 1]$.*

A convex premium principle ρ is called a coherent premium principle if it satisfies

(A4) *Positive homogeneity: $\rho(\lambda X) = \lambda\rho(X)$ for all $X \in \mathcal{X}$ and $\lambda \in \mathbb{R}_+$.*

In addition, ρ is said to satisfy *law invariance* if $\rho(X) = \rho(Y)$ whenever $X \overset{\mathrm{d}}{=} Y$. In practice, it is often desirable that we are able to estimate or identify the values of risk premiums statistically, in which case law invariance is a natural requirement. The concepts of convex premium principle and coherent premium principle are analogies with convex risk measure and coherent risk measure [12]. In view of the simplicity and ease of calculation, *Value-at-Risk* (VaR) is one popular choice of premium principles, defined as

$$\mathrm{VaR}_\alpha(X) := \inf_{t \in \mathbb{R}_+} \{t : \mathbb{P}(X \le t) \ge \alpha\} \quad \text{for } X \in L_+^0,\ \alpha \in (0, 1].$$

2.2. *Tsallis relative entropy*

We recall from Tsallis [24] the q-generalization of the relative entropy, which is also called *Tsallis relative entropy*. For any two probability measures $\mathbb{Q}$ and $\mathbb{P}$ on $(\Omega, \mathscr{F})$, the *q-generalization of the relative entropy* is defined by

$$H_q(\mathbb{Q}|\mathbb{P}) := \begin{cases} \displaystyle\int \left(\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right)^q \ln_q\left(\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right) \mathrm{d}\mathbb{P}, & \text{if } \mathbb{Q} \ll \mathbb{P}, \\ +\infty, & \text{otherwise,} \end{cases}$$

where $q > 0$, and $\ln_q(x)$ is the *generalized q-logarithm function* defined on $(0, \infty)$, given by

$$\ln_q(x) := \begin{cases} \dfrac{x^{1-q} - 1}{1 - q}, & \text{for } q \ne 1, \\ \ln x, & \text{for } q = 1. \end{cases}$$

The inverse of $\ln_q(x)$ is called the *generalized q-exponential function*, given by

$$\exp_q(x) := \begin{cases} [1 + (1-q)x]^{1/(1-q)}, & \text{for } x > -\dfrac{1}{1-q} \text{ and } 0 < q < 1, \\ [1 + (1-q)x]^{1/(1-q)}, & \text{for } x < \dfrac{1}{q-1} \text{ and } q > 1, \\ \exp(x), & \text{for } x \in \mathbb{R} \text{ and } q = 1. \end{cases}$$

Note that $\ln_q(\cdot)$ and $\exp_q(\cdot)$ are well defined when $q > 0$. Clearly, $\ln_1(x) = \ln(x)$, $\exp_1(x) = \exp(x)$, and $H_1(\mathbb{Q}\,|\,\mathbb{P}) = H(\mathbb{Q}\,|\,\mathbb{P})$, where $H(\mathbb{Q}\,|\,\mathbb{P})$ is defined by (1.1).

In view of (3.1) and the domains of the function $\exp_q(x)$ for different q, throughout the paper, we always consider the case of $0 < q \le 1$ unless stated otherwise. In order to accommodate to the domain

https://doi.org/10.1017/S0269964822000444 Published online by Cambridge University Press

of TsVaR (see Definition 3.1), we consider the model space $\mathcal{X} = L^0_+$, the space of nonnegative random variables. For recent discussion on risk measures and pricing principles based on Tsallis entropy, we refer to Ma and Tian [16] and Tian [23].

The next proposition gives some properties of the generalized q-logarithm and q-exponential functions, which will be helpful for our discussion. For more details, we refer to Tsallis [25].

Proposition 2.1. *The generalized q-logarithm and q-exponential functions have the following properties:*

(1) $\ln_q(\cdot)$ *is strictly increasing and concave, and* $\exp_q(\cdot)$ *is strictly increasing and convex.*
(2) $\ln_q(xy) = x^{1-q}\ln_q(y) + \ln_q(x)$ *for all* $x, y > 0$.
(3) $\ln_q(1/x) = -x^{q-1}\ln_q(x)$ *for all* $x > 0$.
(4) $f(\boldsymbol{x}) := \ln_q(\sum_{i=1}^n \exp_q(x_i))$ *is convex in* $\boldsymbol{x} = (x_1, \ldots, x_n)$.

3. Tsallis value-at-risk

In this section, we propose a new premium principle that corresponds to the tightest possible upper bound obtained from the Chernoff inequality for VaR. The basic idea follows from Chernoff inequality in Chernoff [7]: for any constant $t \in \mathbb{R}_+$ and $X \in L^0_+$,

$$\mathbb{P}(X \geq t) \leq \frac{\mathbb{E}[\exp_q(\lambda X)]}{\exp_q(\lambda t)}, \quad \forall \lambda > 0.$$

By solving the equation

$$\frac{\mathbb{E}[\exp_q(\lambda X)]}{\exp_q(\lambda t)} = \alpha$$

with respect to t for $\alpha \in (0, 1]$ and $\lambda > 0$, we obtain

$$t_X(\alpha, \lambda) := \frac{1}{\lambda}\ln_q\left(\frac{1}{\alpha}\mathbb{E}[\exp_q(\lambda X)]\right),$$

which satisfies that $\mathbb{P}(X \geq t_X(\alpha, \lambda)) \leq \alpha$. In fact, for each $\lambda > 0$, $t_X(\alpha, \lambda)$ is an upper bound for $\mathrm{VaR}_{1-\alpha}(X)$. We now consider the best upper bound of this type as a new premium principle that bounds $\mathrm{VaR}_{1-\alpha}(\cdot)$ by using the generalized q-exponential moments. Recall that we always assume $0 < q \leq 1$ and all random variables in L^0_+.

Definition 3.1. *Let $X \in L^0_+$ satisfying that $\mathbb{E}[\exp_q(\lambda_0 X)] < \infty$ for some $\lambda_0 > 0$. Then Tsallis Value-at-Risk* (TsVaR) *of X with confidence level $1 - \alpha$ for $\alpha \in (0, 1]$ is defined by*

$$\mathrm{TsVaR}_{1-\alpha}(X) := \inf_{\lambda>0} t_X(\alpha, \lambda) = \inf_{\lambda>0}\left\{\frac{1}{\lambda}\ln_q\left(\frac{1}{\alpha}\mathbb{E}[\exp_q(\lambda X)]\right)\right\}. \tag{3.1}$$

Furthermore, define three spaces of random variables

$$E := \{X \in L^0_+ : \mathbb{E}[\exp_q(\lambda X)] < \infty \text{ for all } \lambda > 0\},$$
$$E' := \left\{X \in L^0_+ : \frac{q}{1-q}\mathbb{E}[X^{1/q} - X] < \infty\right\},$$
$$E'' := \{X \in L^0_+ : \mathbb{E}[\exp_q(\lambda X)] < \infty \text{ for some } \lambda > 0\},$$

https://doi.org/10.1017/S0269964822000444 Published online by Cambridge University Press

which are called the primal, dual, and bidual Tsallis spaces, respectively. For $q = 1$, E' is understood as a limiting case, that is, $E' = \{X \in L^0_+ : \mathbb{E}[X \ln X] < \infty\}$, and E, E', E'' are called the primal, dual, and bidual entropic space, respectively. For $q \in (0,1)$, $E = E'' = L_+^{1/(1-q)}$.

Remark 3.1 (Lower and upper bounds)**.** For $\alpha \in (0,1]$, we have the following bounds for TsVaR:

$$\alpha^{q-1}\mathbb{E}(X) \leq \mathrm{TsVaR}_{1-\alpha}(X) \leq \alpha^{q-1}\,\text{ess-sup}(X), \quad X \in E''. \tag{3.2}$$

To see it, note that $\exp_q(\cdot)$ is a convex function by Proposition 2.1(1). Applying Jensen's inequality, the above lower bound follows since

$$\begin{aligned}\frac{1}{\lambda}\ln_q\left(\frac{1}{\alpha}\mathbb{E}[\exp_q(\lambda X)]\right) &= \frac{\alpha^{q-1}}{\lambda}\ln_q(\mathbb{E}[\exp_q(\lambda X)]) + \frac{1}{\lambda}\ln_q\left(\frac{1}{\alpha}\right)\\ &\geq \frac{\alpha^{q-1}}{\lambda}\ln_q([\exp_q(\lambda\mathbb{E}X)]) + \frac{1}{\lambda}\ln_q\left(\frac{1}{\alpha}\right) \geq \alpha^{q-1}\mathbb{E}X.\end{aligned}$$

The upper bound follows from

$$\begin{aligned}\mathrm{TsVaR}_{1-\alpha}(X) &= \inf_{\lambda>0}\left\{\frac{\alpha^{q-1}}{\lambda}\ln_q(\mathbb{E}[\exp_q(\lambda X)]) + \frac{1}{\lambda}\ln_q\left(\frac{1}{\alpha}\right)\right\}\\ &\leq \inf_{\lambda>0}\left\{\frac{\alpha^{q-1}}{\lambda}\ln_q(\mathbb{E}[\exp_q(\lambda\,\text{ess-sup}(X))]) + \frac{1}{\lambda}\ln_q\left(\frac{1}{\alpha}\right)\right\}\\ &= \inf_{\lambda>0}\left\{\alpha^{q-1}\,\text{ess-sup}(X) + \frac{1}{\lambda}\ln_q\left(\frac{1}{\alpha}\right)\right\} = \alpha^{q-1}\,\text{ess-sup}(X).\end{aligned}$$

From (3.2), it is known that $\mathrm{TsVaR}_\beta(X) \to +\infty$ as $\beta \nearrow 1$ when $\mathbb{E}[X] > 0$ and $0 < q < 1$. So, in Definition 3.1, we can not assume $\mathrm{TsVaR}_1(X) = \text{ess-sup}(X)$ for $X \in E''$ except $\mathrm{EVaR}_1(X) = \text{ess-sup}(X)$.

Below, we will see that TsVaR can not be a coherent premium principle, even for convex premium principle unless $q = 1$, since TsVaR is not cash invariance in general. To prove the convexity of $\mathrm{TsVaR}_{1-\alpha}(\cdot)$, we need the following two lemmas.

Lemma 3.2. *For $0 < q \leq 1$ and $\alpha \in (0,1]$, the function $g_\alpha(X,\lambda) := t_X(\alpha, 1/\lambda)$ is convex in (X,λ), where $\lambda > 0$ and $X \in E''$.*

Lemma 3.3. *For $0 < q \leq 1$ and $\alpha \in (0,1]$, the function $\inf_{\lambda>0}\{g_\alpha(X,\lambda)\}$ is convex in $X \in E''$.*

Proposition 3.4. $\mathrm{TsVaR}_{1-\alpha}(\cdot)$ *defined on E'' is a law invariant, monotonicity, positive homogeneity, and convex functional for every $\alpha \in (0,1]$.*

Proof. Note that $\mathrm{TsVaR}_{1-\alpha}(X) = \inf_{\lambda>0}\{g_\alpha(X,\lambda)\}$. The convexity of $\mathrm{TsVaR}_{1-\alpha}(\cdot)$ follows from Lemma 3.3. The law invariant, monotonicity, and positive homogeneity are obvious. □

$\mathrm{TsVaR}_{1-\alpha}(\cdot)$ is also subadditive on E'', that is,

$$\mathrm{TsVaR}_{1-\alpha}(X+Y) \leq \mathrm{TsVaR}_{1-\alpha}(X) + \mathrm{TsVaR}_{1-\alpha}(Y), \quad X, Y \in E'',$$

because $\mathrm{TsVaR}_{1-\alpha}(0) = 0$, and convexity is equivalent to subadditivity under positive homogeneity. However, in general, $\mathrm{TsVaR}_{1-\alpha}(\cdot)$ may fail to satisfy cash invariance. In Proposition 3.9, we will show that $\mathrm{TsVaR}_{1-\alpha}(\cdot)$ possesses some certain cash-subadditivity.

https://doi.org/10.1017/S0269964822000444 Published online by Cambridge University Press

In the following theorem, we show that $\mathrm{TsVaR}_{1-\alpha}(\cdot)$ is a coherent premium principle if and only if $q = 1$. In other words, with the exception of EVaR, TsVaR is not a coherent premium principle. EVaR is defined by (see [2])

$$\mathrm{EVaR}_{1-\alpha}(X) = \inf_{\lambda>0}\left\{\frac{1}{\lambda}\ln\left(\frac{1}{\alpha}\mathbb{E}[\exp(\lambda X)]\right)\right\}. \tag{3.3}$$

Theorem 3.5. *For $q \in (0,1]$, $\mathrm{TsVaR}_{1-\alpha}(\cdot)$ is a coherent premium principle for each $\alpha \in (0,1]$ if and only if $q = 1$.*

Proof. Sufficiency. When $q = 1$, TsVaR reduces to EVaR, which has been proven to be coherent by Ahmadi-Javid [2].

Necessary. Assume that $\mathrm{TsVaR}_{1-\alpha}(\cdot)$ is coherent for each $\alpha \in (0,1]$. Then, for any $m \in \mathbb{R}_+$ and $\alpha \in (0,1]$, we have $\mathrm{TsVaR}_{1-\alpha}(X+m) = \mathrm{TsVaR}_{1-\alpha}(X)+m$. Choose $X \equiv c \in \mathbb{R}_+$. From (3.1), it follows that

$$\begin{aligned}\mathrm{TsVaR}_{1-\alpha}(c+m) &= \inf_{\lambda>0}\left\{\frac{1}{\lambda}\ln_q\left(\frac{1}{\alpha}\exp_q(\lambda(c+m))\right)\right\}\\ &= \inf_{\lambda>0}\left\{\alpha^{q-1}(c+m)+\frac{1}{\lambda}\ln_q\left(\frac{1}{\alpha}\right)\right\} = \alpha^{q-1}(c+m),\end{aligned}$$

and

$$\begin{aligned}\mathrm{TsVaR}_{1-\alpha}(c)+m &= \inf_{\lambda>0}\left\{\frac{1}{\lambda}\ln_q\left(\frac{1}{\alpha}\exp_q(\lambda c)\right)\right\}+m\\ &= \inf_{\lambda>0}\left\{\alpha^{q-1}c+\frac{1}{\lambda}\ln_q\left(\frac{1}{\alpha}\right)\right\}+m = \alpha^{q-1}c+m,\end{aligned}$$

Thus, $\alpha^{q-1}(c+m) = \alpha^{q-1}c+m$ for any $m \in \mathbb{R}_+$ and $\alpha \in (0,1)$. This implies $q = 1$. □

Although TsVaR is not a coherent premium principle in general, we establish its dual representation, which reveals its relationship with the generalized relative entropy. To state and prove this result, we need two lemmas. The first one is the variational representation for the generalized relative entropy, and the second one is a special case of Lemma 1.3 in Ahmadi-Javid [1].

Lemma 3.6 [16] Theorem 4.1. *For $\lambda > 0$, $q > 0$, and $q \neq 1$, we have*

$$\ln_q \mathbb{E}[\exp_q(\lambda X)] = \sup_{\mathbb{Q}\ll\mathbb{P}}\left\{\lambda\mathbb{E}\left[\left(\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right)^q X\right]-H_q(\mathbb{Q}|\mathbb{P})\right\},\quad X\in E''. \tag{3.4}$$

Lemma 3.7. *For $0 < q \le 1$ and $\alpha \in (0,1]$,*

$$\begin{aligned}&\inf_{\lambda>0}\left\{\sup_{\mathbb{Q}\ll\mathbb{P}}\left\{\mathbb{E}\left[\left(\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right)^q X\right]+\lambda\left(\alpha^{1-q}\ln_q\left(\frac{1}{\alpha}\right)-H_q(\mathbb{Q}|\mathbb{P})\right)\right\}\right\}\\ &\quad= \sup_{\mathbb{Q}\ll\mathbb{P},H_q(\mathbb{Q}|\mathbb{P})\le-\ln_q(\alpha)}\mathbb{E}\left[\left(\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right)^q X\right],\quad X\in E''.\end{aligned} \tag{3.5}$$

Theorem 3.8. *For $0 < q \le 1$, the dual representation of $\mathrm{TsVaR}_{1-\alpha}$ has the form*

$$\mathrm{TsVaR}_{1-\alpha}(X) = \sup_{\mathbb{Q}\in\mathcal{Q}}\left\{\alpha^{q-1}\mathbb{E}\left[\left(\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right)^q X\right]\right\},\quad X\in E'',$$

where $\mathcal{Q} = \{\mathbb{Q}\ll\mathbb{P} : H_q(\mathbb{Q}|\mathbb{P}) \le -\ln_q(\alpha)\}$.

https://doi.org/10.1017/S0269964822000444 Published online by Cambridge University Press

Proof. For $q = 1$, TsVaR is EVaR, and the corresponding proof can be found in Ahmadi-Javid [2]. For $q \in (0, 1)$ and $X \in E''$, by Proposition 2.1(2) and Lemmas 3.6 and 3.7, we have

$$
\begin{aligned}
\mathrm{TsVaR}_{1-\alpha}(X) &= \inf_{\lambda>0}\left\{\frac{1}{\lambda}\ln_q\left(\frac{1}{\alpha}\mathbb{E}[\exp_q(\lambda X)]\right)\right\} \\
&= \inf_{\lambda>0}\left\{\frac{1}{\lambda}\left(\alpha^{q-1}\ln_q \mathbb{E}[\exp_q(\lambda X)] + \ln_q\frac{1}{\alpha}\right)\right\} \\
&= \inf_{\lambda>0}\left\{\frac{1}{\lambda}\left(\alpha^{q-1}\sup_{\mathbb{Q}\ll\mathbb{P}}\left\{\lambda\,\mathbb{E}\left[\left(\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right)^q X\right] - H_q(\mathbb{Q}|\mathbb{P})\right\} + \ln_q\left(\frac{1}{\alpha}\right)\right)\right\} \\
&= \inf_{\lambda>0}\left\{\sup_{\mathbb{Q}\ll\mathbb{P}}\left\{\alpha^{q-1}\mathbb{E}\left[\left(\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right)^q X\right] - \frac{1}{\lambda}\left[\alpha^{q-1}H_q(\mathbb{Q}|\mathbb{P}) - \ln_q\left(\frac{1}{\alpha}\right)\right]\right\}\right\} \\
&= \sup_{\mathbb{Q}\ll\mathbb{P},\, H_q(\mathbb{Q}|\mathbb{P})\le -\ln_q(\alpha)}\left\{\alpha^{q-1}\mathbb{E}\left[\left(\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right)^q X\right]\right\}.
\end{aligned}
$$

The desired result now follows. □

By Theorem 3.8, we will show in the next proposition that $\mathrm{TsVaR}_{1-\alpha}(\cdot)$ satisfies some restricted cash-subadditivity. For more discussion on cash-subadditivity, we refer to El Karoui and Ravanelli [10] and Han *et al.* [13].

Proposition 3.9. *For* $0 < q \le 1$ *and* $\alpha \in (0, 1]$, *we have*

$$
\mathrm{TsVaR}_{1-\alpha}(X + m) \le \mathrm{TsVaR}_{1-\alpha}(X) + \alpha^{q-1}m, \quad X \in E'',\ m \in \mathbb{R}_+.
$$

Proof. By Theorem 3.8, we have

$$
\mathrm{TsVaR}_{1-\alpha}(X) = \sup_{\mathbb{Q}\in\mathcal{Q}}\left\{\alpha^{q-1}\mathbb{E}\left[\left(\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right)^q X\right]\right\}.
$$

where $\mathcal{Q} = \{\mathbb{Q} \ll \mathbb{P} : H_q(\mathbb{Q}|\mathbb{P}) \le -\ln_q(\alpha)\}$. Then,

$$
\begin{aligned}
\mathrm{TsVaR}_{1-\alpha}(X + m) &= \sup_{\mathbb{Q}\in\mathcal{Q}}\left\{\alpha^{q-1}\mathbb{E}\left[\left(\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right)^q (X + m)\right]\right\} \\
&= \sup_{\mathbb{Q}\in\mathcal{Q}}\left\{\alpha^{q-1}\mathbb{E}\left[\left(\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right)^q X\right] + m\alpha^{q-1}\mathbb{E}\left[\left(\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right)^q\right]\right\} \\
&\le \sup_{\mathbb{Q}\in\mathcal{Q}}\left\{\alpha^{q-1}\mathbb{E}\left[\left(\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right)^q X\right] + m\alpha^{q-1}\left[\mathbb{E}\left(\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right)\right]^q\right\} \\
&= \mathrm{TsVaR}_{1-\alpha}(X) + \alpha^{q-1}m,
\end{aligned}
$$

where the inequality follows from Jensen's inequality. □

From Definition 3.1, it follows that

$$
\begin{aligned}
\mathrm{TsVaR}_{1-\alpha}(X) &= \inf_{\lambda>0}\left\{\frac{1}{\lambda}\ln_q\left(\frac{1}{\alpha}\mathbb{E}[\exp_q(\lambda X)]\right)\right\} \\
&= \inf_{\lambda>0}\left\{\frac{1}{\lambda}\alpha^{q-1}\ln_q(\mathbb{E}[\exp_q(\lambda X)]) + \frac{1}{\lambda}\ln_q\left(\frac{1}{\alpha}\right)\right\} \\
&= \inf_{\lambda>0}\left\{\frac{1}{\lambda}\alpha^{q-1}\ln_q(\mathbb{E}[\exp_q(\lambda X)]) - \frac{1}{\lambda}\alpha^{q-1}\ln_q(\alpha)\right\},
\end{aligned}
$$

https://doi.org/10.1017/S0269964822000444 Published online by Cambridge University Press

which shows that $\mathrm{TsVaR}_{1-\alpha}(X)$ only depends on $\{\ln_q \mathbb{E}[\exp_q(\lambda X)], \lambda > 0\}$. The next proposition demonstrates how these functions can be represented by means of TsVaR.

Proposition 3.10. *For $X \in E$ and $\lambda > 0$,*

$$\begin{aligned} \ln_q \mathbb{E}[\exp_q(\lambda X)] &= \sup_{0<\alpha\le 1} \{\lambda \alpha^{1-q} \mathrm{TsVaR}_{1-\alpha}(X) + \ln_q(\alpha)\}, \\ \mathbb{E}[\exp_q(\lambda X)] &= \sup_{0<\alpha\le 1} \exp_q(\lambda \alpha^{1-q} \mathrm{TsVaR}_{1-\alpha}(X) + \ln_q(\alpha)), \\ t_X(1,\lambda) &= \sup_{0<\alpha\le 1} \{\alpha^{1-q} \mathrm{TsVaR}_{1-\alpha}(X) + \lambda^{-1} \ln_q(\alpha)\}. \end{aligned} \tag{3.6}$$

Proof. We only prove (3.6). By Proposition 2.1(2), we have

$$\begin{aligned} -\mathrm{TsVaR}_{1-\alpha}(X) &= \sup_{x>0} \left\{ -x \ln_q \left(\frac{1}{\alpha} \mathbb{E} \left[\exp_q \left(\frac{X}{x} \right) \right] \right) \right\} \\ &= \sup_{x>0} \left\{ -x \ln_q \left(\frac{1}{\alpha} \right) - x \alpha^{q-1} \ln_q \mathbb{E} \left[\exp_q \left(\frac{X}{x} \right) \right] \right\}. \end{aligned}$$

Thus,

$$\begin{aligned} -\alpha^{1-q} \mathrm{TsVaR}_{1-\alpha}(X) &= \sup_{x>0} \left\{ -x \alpha^{1-q} \ln_q \left(\frac{1}{\alpha} \right) - x \ln_q \mathbb{E} \left[\exp_q \left(\frac{X}{x} \right) \right] \right\} \\ &= \sup_{x>0} \left\{ x \ln_q(\alpha) - x \ln_q \mathbb{E} \left[\exp_q \left(\frac{X}{x} \right) \right] \right\} \\ &= \sup_{x\ge 0} \{ x \ln_q(\alpha) - g(x) \}, \end{aligned}$$

where

$$g(x) = \begin{cases} x \ln_q(\mathbb{E}[\exp_q(X/x)]), & \text{if } x > 0, \\ \text{ess-sup}(X), & \text{if } x = 0. \end{cases}$$

One can observe that the function $-[\exp_q(y)]^{1-q} \mathrm{TsVaR}_{1-\exp_q(y)}(X)$ with domain $(-1/(1-q), 0]$ for $0 < q < 1$ and $(-\infty, 0]$ for $q = 1$ is the conjugate of function $g(x)$ with domain $[0, +\infty)$. Since $g(x)$ is convex and closed by Lemma 3.2, $g(x)$ is the conjugate of its own conjugate. This completes the proof. □

The next proposition compares the values of TsVaR for different confidence levels α.

Proposition 3.11. *For $0 < \alpha_1 \le \alpha_2 < 1$, $q \in (0, 1]$, we have*

$$\left(\frac{\alpha_2}{\alpha_1} \right)^{1-q} \mathrm{TsVaR}_{1-\alpha_2}(X) \le \mathrm{TsVaR}_{1-\alpha_1}(X), \quad X \in E'', \tag{3.7}$$

and

$$\mathrm{TsVaR}_{1-\alpha_1}(X) \le \frac{\ln_q(\alpha_1^2) - \ln_q(\alpha_1)}{\ln_q(\alpha_2^2) - \ln_q(\alpha_2)} \cdot \mathrm{TsVaR}_{1-\alpha_2}(X), \quad X \in E''. \tag{3.8}$$

https://doi.org/10.1017/S0269964822000444 Published online by Cambridge University Press

Proof. The proof of the case $q = 1$ can be found in Proposition 2.12 of Ahmadi-Javid and Pichler [4]. Next, we assume $q \in (0, 1)$. Note that

$$\begin{aligned}\frac{1}{\lambda}\ln_q\left[\frac{1}{\alpha_1}\mathbb{E}[\exp_q(\lambda X)]\right] &= \frac{1}{\lambda}\ln_q\left(\frac{\alpha_2}{\alpha_1}\frac{1}{\alpha_2}\mathbb{E}[\exp_q(\lambda X)]\right)\\ &= \frac{1}{\lambda}\left(\frac{\alpha_2}{\alpha_1}\right)^{1-q}\ln_q\left(\frac{1}{\alpha_2}\mathbb{E}[\exp_q(\lambda X)]\right) + \frac{1}{\lambda}\ln_q\left(\frac{\alpha_2}{\alpha_1}\right)\\ &\geq \frac{1}{\lambda}\left(\frac{\alpha_2}{\alpha_1}\right)^{1-q}\ln_q\left(\frac{1}{\alpha_2}\mathbb{E}[\exp_q(\lambda X)]\right).\end{aligned}$$

Then, taking the infimum over $\lambda > 0$ in the both sides of the above inequality yields that

$$\mathrm{TsVaR}_{1-\alpha_1}(X) \geq \left(\frac{\alpha_2}{\alpha_1}\right)^{1-q}\mathrm{TsVaR}_{1-\alpha_2}(X).$$

This proves (3.7).

To prove (3.8), note that $\ln_q \mathbb{E}[\exp_q(\lambda X)] \geq 0$ since $X \geq 0$. Thus, for $q \in (0, 1)$, we have

$$\begin{aligned}\frac{1}{\lambda}\ln_q\left[\frac{1}{\alpha_1}\mathbb{E}[\exp_q(\lambda X)]\right] &= \frac{1}{\lambda}\alpha_1^{q-1}\ln_q\mathbb{E}[\exp_q(\lambda X)] + \frac{1}{\lambda}\ln_q\left(\frac{1}{\alpha_1}\right)\\ &= \frac{1}{\lambda}\frac{\alpha_1^{q-1}}{\alpha_2^{q-1}}\alpha_2^{q-1}\ln_q\mathbb{E}[\exp_q(\lambda X)] + \frac{1}{\lambda}\frac{\ln_q(\alpha_1^{-1})}{\ln_q(\alpha_2^{-1})}\ln_q\left(\frac{1}{\alpha_2}\right)\\ &\leq \frac{1}{\lambda}\frac{\alpha_1^{q-1}}{\alpha_2^{q-1}}\cdot\frac{\ln_q(\alpha_1^{-1})}{\ln_q(\alpha_2^{-1})}\alpha_2^{q-1}\ln_q\mathbb{E}[\exp_q(\lambda X)] + \frac{1}{\lambda}\frac{\alpha_1^{q-1}}{\alpha_2^{q-1}}\cdot\frac{\ln_q(\alpha_1^{-1})}{\ln_q(\alpha_2^{-1})}\ln_q\left(\frac{1}{\alpha_2}\right)\\ &= \frac{\alpha_1^{q-1}}{\alpha_2^{q-1}}\frac{\ln_q(\alpha_1^{-1})}{\ln_q(\alpha_2^{-1})}\cdot\frac{1}{\lambda}\ln_q\left(\frac{1}{\alpha_2}\mathbb{E}[\exp_q(\lambda X)]\right)\\ &= \frac{\ln_q(\alpha_1^2) - \ln_q(\alpha_1)}{\ln_q(\alpha_2^2) - \ln_q(\alpha_2)}\cdot\frac{1}{\lambda}\ln_q\left(\frac{1}{\alpha_2}\mathbb{E}[\exp_q(\lambda X)]\right),\end{aligned}\tag{3.9}$$

where the last equality follows since

$$\begin{aligned}\frac{\alpha_1^{q-1}}{\alpha_2^{q-1}}\frac{\ln_q(\alpha_1^{-1})}{\ln_q(\alpha_2^{-1})} &= \frac{\alpha_1^{q-1}(\alpha_1^{q-1}-1)/(1-q)}{\alpha_2^{q-1}(\alpha_2^{q-1}-1)/(1-q)}\\ &= \frac{(\alpha_1^{2(q-1)}-1)-(\alpha_1^{q-1}-1)}{(\alpha_2^{2(q-1)}-1)-(\alpha_2^{q-1}-1)} = \frac{\ln_q(\alpha_1^2)-\ln_q(\alpha_1)}{\ln_q(\alpha_2^2)-\ln_q(\alpha_2)}.\end{aligned}$$

By taking the infimum over all $\lambda > 0$ in the both sides of (3.9), we conclude that

$$\mathrm{TsVaR}_{1-\alpha_1}(X) \leq \frac{\ln_q(\alpha_1^2)-\ln_q(\alpha_1)}{\ln_q(\alpha_2^2)-\ln_q(\alpha_2)}\cdot\mathrm{TsVaR}_{1-\alpha_2}(X).$$

This completes the proof of the proposition. □

Next, we establish the *strong monotonicity* of TsVaR. Ahmadi-Javid and Fallah-Tafti [3] showed that EVaR also possesses this property, which does not hold for other popular (coherent or non-coherent) monotone risk measures such as the VaR or Expected Shortfall. Recall the definition of strongly monotonicity.

Definition 3.2. *A risk measure ρ is called strongly monotone if it holds that $\rho(X) > \rho(Y)$ for any pair of random variables X and Y in the domain of ρ that satisfy the conditions*

(C1) $X \geq Y$ *and* $\mathbb{P}(X > Y) > 0$.
(C2) *ess-sup*$(X) >$ *ess-sup*(Y) *or ess-sup*$(X) =$ *ess-sup*$(Y) = +\infty$.

Theorem 3.12 (Strong monotonicity of TsVaR)**.** *Let X and Y be random variables in the space E'' that satisfy Conditions* (C1) *and* (C2). *Then,*

$$\mathrm{TsVaR}_{1-\alpha}(X) > \mathrm{TsVaR}_{1-\alpha}(Y) \quad \textit{for any } \alpha \in (0,1].$$

Proof. From (3.1) and Proposition 2.1, we have

$$\mathrm{TsVaR}_{1-\alpha}(X) = \inf_{\lambda>0}\{t_X(\alpha,\lambda)\} = \inf_{\lambda>0}\left\{\frac{1}{\lambda}\alpha^{q-1}\ln_q(\mathbb{E}[\exp_q(\lambda X)]) + \frac{1}{\lambda}\ln_q\left(\frac{1}{\alpha}\right)\right\}.$$

Since $\exp_q(\cdot)$ is strictly increasing, Condition (C1) ensures that $\mathbb{E}[\exp_q(\lambda X)] > \mathbb{E}[\exp_q(\lambda X)]$ for $\lambda > 0$. Thus,

$$\begin{aligned} t_X(\alpha,\lambda) &= \frac{1}{\lambda}\alpha^{q-1}\ln_q(\mathbb{E}[\exp_q(\lambda X)]) + \frac{1}{\lambda}\ln_q\left(\frac{1}{\alpha}\right) \\ &> \frac{1}{\lambda}\alpha^{q-1}\ln_q(\mathbb{E}[\exp_q(\lambda Y)]) + \frac{1}{\lambda}\ln_q\left(\frac{1}{\alpha}\right) = t_Y(\alpha,\lambda) \end{aligned}$$

since $\ln_q(\cdot)$ is strictly increasing. Moreover, by Condition (C2), we have

$$\lim_{\lambda\to\infty} t_X(\alpha,\lambda) = \alpha^{q-1}\,\text{ess-sup}(X) > \alpha^{q-1}\,\text{ess-sup}(Y) = \lim_{\lambda\to\infty} t_Y(\alpha,\lambda)$$

or

$$\lim_{\lambda\to 0} t_X(\alpha,\lambda) = \lim_{\lambda\to 0} t_Y(\alpha,\lambda) = +\infty.$$

Note that $t_X(\alpha,\lambda)$ is continuous in $\lambda > 0$. Therefore,

$$\mathrm{TsVaR}_{1-\alpha}(X) = \inf_{\lambda>0}\{t_X(\alpha,\lambda)\} > \inf_{\lambda>0}\{t_Y(\alpha,\lambda)\} = \mathrm{TsVaR}_{1-\alpha}(Y).$$

This ends the proof. □

In the end of this section, we give an example to compare the EVaR and TsVaR for a random variable with $U(0,1)$ distribution.

Example 3.13. For $X \sim U(0,1)$ and $q \in (0,1)$, it follows from (3.1) and (3.3) that

$$\begin{aligned} \mathrm{TsVaR}_{1-\alpha}(X) &= \inf_{\lambda>0}\left\{\frac{1}{\lambda}\ln_q\left(\frac{(1+(1-q)\lambda)^{(2-q)/(1-q)}-1}{\alpha\lambda(2-q)}\right)\right\}, \\ \mathrm{EVaR}_{1-\alpha}(X) &= \inf_{\lambda>0}\left\{\lambda\ln\left(\lambda\exp\left(\frac{1}{\lambda}\right)-\lambda\right)-\lambda\ln\alpha\right\}, \quad \alpha \in (0,1). \end{aligned}$$

Figure 1 plots $\mathrm{EVaR}_{1-\alpha}(X)$ and $\mathrm{TsVaR}_{1-\alpha}(X)$ with respect to different α with $q = 0.4, 0.8$ and 1. We observe that the premium principle calculating via TsVaR is conservative when considering small α, and that as $q \nearrow 1$, TsVaR converges to EVaR, which is consistent with our conclusion. The advantage of TsVaR is that the size of q can be determined according to the current state, which coincided with the idea of Tsallis [25], that is, the parameter q in Tsallis relative entropy can be viewed as a bias of the original probability measure. If the insurer believes that the insurable loss X has potential huge loss or uncertainty about the distribution of X, the insurer can choose smaller q and α to ensure its own safety.

https://doi.org/10.1017/S0269964822000444 Published online by Cambridge University Press

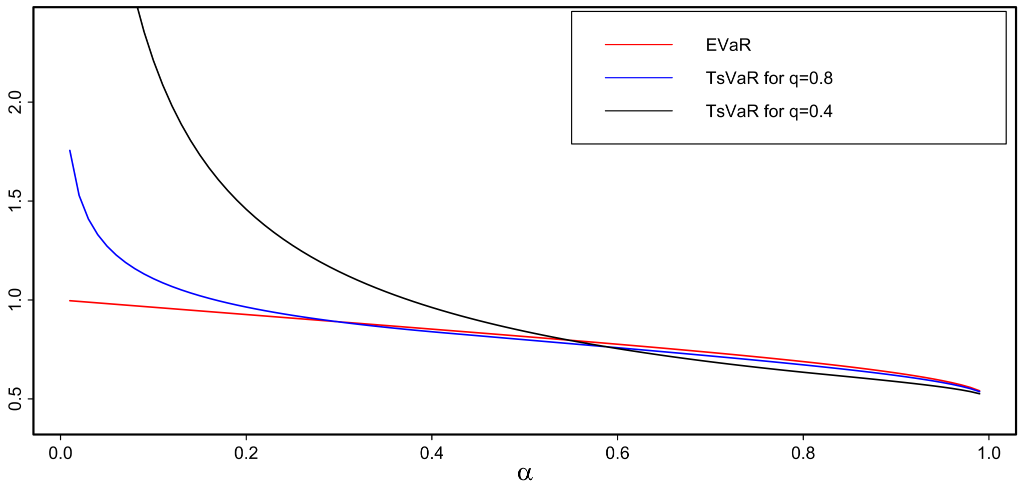

Figure 1. $\mathrm{EVaR}_{1-\alpha}$ *and* $\mathrm{TsVaR}_{1-\alpha}$ *for a random variable* $X \sim U(0,1)$ *with* $q = 0.4, 0.8,$ *and* 1.

4. Banach spaces

Given a coherent risk measure ρ, one may define the semi-norm $\|\cdot\|_\rho := \rho(|\cdot|)$ on the same model space (see [17]). It is well known that risk functional ρ is Lipschitz continuous with respect to the associated norm $\|\cdot\|_\rho$. In this section, we consider the TsVaR *norm*, which is generated by TsVaR as follows, $\|\cdot\| = \mathrm{TsVaR}_\alpha(|\cdot|)$. Recall that we consider the nonnegative random variables. Then, the norm can simplify to $\|\cdot\| = \mathrm{TsVaR}_\alpha(\cdot)$. We first show both E and E'' equipped with TsVaR norm $\|\cdot\|$ are Banach spaces.

Theorem 4.1. *For* $0 < q \le 1$ *and* $0 < \alpha < 1$, *denote* $\|\cdot\| := \mathrm{TsVaR}_\alpha(\cdot)$. *Then, the pairs*

$$(E, \|\cdot\|) \quad \text{and} \quad (E'', \|\cdot\|)$$

are (different) Banach spaces.

Proof. First, we prove that $\|\cdot\|$ induced by TsVaR_α is a norm. Since $\|\cdot\|$ is a semi-norm, it suffices to prove that $\|X\| = 0$ for $X \in E''$ implies $X = 0$, a.s. Assume $\mathrm{TsVaR}_\alpha(X) = 0$. From (3.2), it follows that $\mathbb{E}[X] = 0$, implying $X = 0$ almost sure. Thus, $\|\cdot\|$ is a normal.

Next, we show that E and E'' are complete under the normal $\|\cdot\|$. Assume that $0 < q < 1$ since the case of $q = 1$ reduces to Theorem 2.14 in Ahmadi-Javid and Pichler [4]. We only consider the space E'' since $E = E''$ when $0 < q < 1$. Let $\{X_n\}$ be a Cauchy sequence in E''. For $\epsilon > 0$, there exists $n_0 > 0$ such that $\|X_n - X_m\| < \epsilon$ whenever $m, n > n_0$, and thus $|\|X_m\| - \|X_n\|| \le \|X_n - X_m\| < \epsilon$. Thus, $\lim_{n\to\infty} \|X_n\|$ exists and is finite, and $\|X_n\| < C$ for all $n \ge 1$.

Now recall that $\|X_n\| = \mathrm{TsVaR}_\alpha(X_n)$, so there exists $\lambda_n > 0$ in (3.1) such that

$$\frac{1}{\lambda_n} \ln_q \left[\frac{1}{1-\alpha} \mathbb{E}[\exp_q(\lambda_n X_n)] \right] < C. \tag{4.1}$$

Since $\mathbb{E}[\exp_q(\lambda_n X_n)] \ge 1$, we have

$$\frac{1}{\lambda_n} \ln_q \frac{1}{1-\alpha} \le \frac{1}{\lambda_n} \ln_q \left(\frac{1}{1-\alpha} \mathbb{E}[\exp_q(\lambda_n X_n)] \right) < C.$$

https://doi.org/10.1017/S0269964822000444 Published online by Cambridge University Press

It follows that

$$\lambda_n > \lambda^* := C^{-1} \ln_q \left(\frac{1}{1-\alpha} \right) > 0,$$

and $\mathbb{E}[\exp_q(\lambda X_n)]$ is well-defined by (4.1) for every $\lambda < \lambda^*$. Recall from Remark 3.1 that $(1-\alpha)^{q-1}\mathbb{E}(X) \leq \mathrm{TsVaR}_\alpha(X)$. Thus, $\{X_n\}$ is a Cauchy sequence for L^1 as well, which implies that there exists $X \in L^1$ such that $X_n \xrightarrow{L_1} X$. It remains to show that $X \in E''$ and that $\mathrm{TsVaR}_\alpha(|X - X_n|) \to 0$ as $n \to \infty$.

- We prove $X \in E''$. If there exists a subsequence $\{n_k\}$ such that $\lambda_{n_k} \to \lambda^*$, from (4.1), it follows that

$$\liminf_{k\to\infty} \mathbb{E}[\exp_q(\lambda_{n_k} X_{n_k})] \leq (1-\alpha) \exp_q(\lambda^* C).$$

  Thus, by Fatou's lemma, we have

$$\mathbb{E}[\exp_q(\lambda^* X)] \leq \liminf_{k\to\infty} \mathbb{E}[\exp_q(\lambda^* X_{n_k})] \leq \liminf_{k\to\infty} \mathbb{E}[\exp_q(\lambda_{n_k} X_{n_k})] \leq (1-\alpha)\exp_q(\lambda^* C) < \infty,$$

  implying $X \in E''$. If $\liminf_{n\to\infty} \lambda_n > \lambda^*$, then there exists a subsequence $\{n_k\}$ such that $\lambda_{n_k} > \lambda^*$ for any $k \geq 1$. Thus, by Lemma 3.2 and (4.1),

$$\begin{aligned}
\frac{1}{\lambda^*} \ln_q(\mathbb{E}[\exp_q(\lambda^* X_{n_k})]) &= \frac{1}{\lambda^*} \ln_q \left(\mathbb{E}\left[\exp_q \left(\frac{\lambda^*}{\lambda_{n_k}} \cdot \lambda_{n_k} X_{n_k} + \left(1 - \frac{\lambda^*}{\lambda_{n_k}}\right) \cdot 0 \right)\right]\right) \\
&\leq \frac{1}{\lambda_{n_k}} \ln_q(\mathbb{E}[\exp_q(\lambda_{n_k} X_{n_k})]) \\
&\leq \frac{1}{\lambda_{n_k}} \ln_q \left(\frac{1}{1-\alpha} \mathbb{E}[\exp_q(\lambda_{n_k} X_{n_k})] \right) < C,
\end{aligned}$$

  implying that $\mathbb{E}[\exp_q(\lambda^* X_{n_k})] < \exp_q(\lambda^* C)$. Again, applying Fatou's lemma, we have

$$\mathbb{E}[\exp_q(\lambda^* X)] \leq \liminf_{k\to\infty} \mathbb{E}[\exp_q(\lambda^* X_{n_k})] \leq \exp_q(\lambda^* C) < \infty,$$

  implying $X \in E''$.
- We prove that $\mathrm{TsVaR}_\alpha(|X - X_n|) \to 0$. For $\epsilon > 0$, choose $n_0 > 0$ such that $\mathrm{TsVaR}_\alpha(|X_n - X_m|) < \epsilon$ for $m, n > n_0$. Therefore, there exists $\lambda_{n,m} > 0$ such that

$$\frac{1}{\lambda_{n,m}} \ln_q \left(\frac{1}{1-\alpha} \right) \leq \frac{1}{\lambda_{n,m}} \ln_q \left(\frac{1}{1-\alpha} \mathbb{E}[\exp_q(\lambda_{n,m}|X_n - X_m|)] \right) < \epsilon,$$

  implying that $\lambda_{n,m} > \lambda_1^* := \epsilon^{-1} \ln_q(1/(1-\alpha))$. Again, by Lemma 3.2, we have

$$\begin{aligned}
\frac{1}{\lambda_1^*} \ln_q(\mathbb{E}[\exp_q(\lambda_1^*|X_n - X_m|)]) &= \frac{1}{\lambda_1^*} \ln_q \left(\mathbb{E}\left[\exp_q \left(\frac{\lambda_1^*}{\lambda_{n,m}} \cdot \lambda_{n,m}|X_n - X_m| \right)\right]\right) \\
&\leq \frac{1}{\lambda_{n,m}} \ln_q(\mathbb{E}[\exp_q(\lambda_{n,m}|X_n - X_m|)]) \\
&\leq \frac{1}{\lambda_{n,m}} \ln_q \left[\frac{1}{1-\alpha} \mathbb{E}[\exp_q(\lambda_{n,m}|X_n - X_m|)] \right] < \epsilon.
\end{aligned}$$

  Applying Fatou's lemma, we have

$$\mathbb{E}[\exp_q(\lambda_1^*|X - X_m|)] \leq \liminf_{n\to\infty} \mathbb{E}[\exp_q(\lambda_1^*|X_n - X_m|)] < \exp_q(\lambda_1^* \epsilon).$$

https://doi.org/10.1017/S0269964822000444 Published online by Cambridge University Press

Hence,

$$\begin{aligned}
&\frac{1}{\lambda_1^*}\ln_q\left(\frac{1}{1-\alpha}\mathbb{E}[\exp_q(\lambda_1^*|X-X_m|)]\right)\\
&\quad=\frac{1}{\lambda_1^*}(1-\alpha)^{q-1}\ln_q\mathbb{E}[\exp_q(\lambda_1^*|X-X_m|)]+\frac{1}{\lambda_1^*}\ln_q\left(\frac{1}{1-\alpha}\right)\\
&\quad\le(1-\alpha)^{q-1}\epsilon+\epsilon.
\end{aligned}$$

Since ϵ is arbitrary, it follows that $\text{TsVaR}_\alpha(|X_n-X|)\to 0$. Thus, E'' is complete.

This completes the proof of the theorem. □

In the next theorem, we proceed with a comparison of the TsVaR norm with the L^p-norms $\|\cdot\|_p$.

Theorem 4.2. *For* $0<q\le 1$ *and* $0<\alpha<1$, *denote* $\|\cdot\|=\text{TsVaR}_\alpha(\cdot)$. *Then,*

$$(1-\alpha)^{q-1}\|X\|_1\le\|X\|,\quad X\in E'', \tag{4.2}$$

and

$$\|X\|\le(1-\alpha)^{q-1}\|X\|_\infty,\quad X\in L_+^\infty. \tag{4.3}$$

Furthermore, for every $1<p<\kappa_q$, *there exists a finite constant* $c_{p,q}$ *such that*

$$\|X\|_p\le c_{p,q}\cdot\text{TsVaR}_\alpha(X),\quad X\in E'', \tag{4.4}$$

where

$$\kappa_q=\begin{cases}(2-q)/(1-q), & 0<q<1,\\ +\infty, & q=1,\end{cases}$$

and

$$c_{p,q}=\max\left\{(1-\alpha)^{1-q},\ \frac{\ln_q(1-\beta)^2-\ln_q(1-\beta)}{\ln_q(1-\alpha)^2-\ln_q(1-\alpha)}\cdot(1-\beta)^{1-q}\right\}$$

with $\beta=1-\exp_q(1-p)$.

Proof. Note that (4.2) and (4.3) follow from (3.2), and that Theorem 3.1 in Ahmadi-Javid and Pichler [4] gives the proof for the case $q=1$.

To prove (4.4), we assume $0<q<1$. For any fixed p such that $1<p<\kappa_q$, it follows that $1-p>1-\kappa_q=-1/(1-q)$. Define $\beta=1-\exp_q(1-p)$ so that $\beta\in(0,1)$, and denote $\phi(x)=(\ln_q(x))^p$. It can be checked that

$$\phi''(x)=\frac{p}{x^{2q}}(\ln_q(x))^{p-2}\left(p-1+q\ln_q\frac{1}{x}\right),$$

https://doi.org/10.1017/S0269964822000444 Published online by Cambridge University Press

which is negative and, hence, $\phi(x)$ is concave, when $x \geq 1$. Since $\exp_q(\lambda X)/\exp_q(1-p) \geq 1$, by Jensen's inequality, we have

$$\begin{aligned}\frac{1}{\lambda}\ln_q\left(\frac{1}{1-\beta}\mathbb{E}[\exp_q(\lambda X)]\right) &= \frac{1}{\lambda}\left(\phi\left[\frac{1}{1-\beta}\mathbb{E}[\exp_q(\lambda X)]\right]\right)^{1/p} \\ &\geq \frac{1}{\lambda}\left[\mathbb{E}\circ\phi\left(\frac{\exp_q(\lambda X)}{1-\beta}\right)\right]^{1/p} \\ &= \frac{1}{\lambda}\left(\mathbb{E}\left[(1-\beta)^{q-1}\lambda X + \ln_q\left(\frac{1}{1-\beta}\right)\right]^p\right)^{1/p} \\ &\geq \frac{1}{\lambda}(\mathbb{E}[(1-\beta)^{q-1}\lambda X]^p)^{1/p} = (1-\beta)^{q-1}\|X\|_p. \qquad (4.5)\end{aligned}$$

Taking the infimum in (4.5) among all $\lambda > 0$ yields that

$$\|X\|_p \leq (1-\beta)^{1-q}\cdot \mathrm{TsVaR}_\beta(X).$$

Therefore, the desired result (4.4) follows from Proposition 3.11 immediately. □

4.1. Relation to Orlicz spaces

In this subsection, we discuss the relationship between the Tsallis spaces and their equivalent Orlicz hearts and Orlicz spaces. Let $\Phi : [0,+\infty) \to [0,+\infty)$ be convex with $\Phi(0) = 0$, $\Phi(1) = 1$ and $\lim_{x\to\infty}\Phi(x) = \infty$. The convex conjugate of Φ is defined as

$$\Psi(z) := \sup_{x\in\mathbb{R}_+}\{xz - \Phi(x)\}, \quad z \in \mathbb{R}_+.$$

These functions are called a pair of complementary Young functions in the context of Orlicz spaces with

$$xz \leq \Phi(x) + \Psi(z), \quad x, z \in \mathbb{R}_+.$$

Definition 4.1. *For a pair Φ and Ψ of complementary Young functions, define the spaces*

$$\begin{aligned}L^\Phi &= \{X \in L^0_+ : \mathbb{E}[\Phi(cX)] < \infty \text{ for some } c > 0\},\\ M^\Phi &= \{X \in L^0_+ : \mathbb{E}[\Phi(cX)] < \infty \text{ for all } c > 0\},\end{aligned}$$

and the norms

$$\begin{aligned}\|X\|_\Phi &= \inf\left\{\lambda > 0 : \mathbb{E}\left[\Phi\left(\left|\frac{X}{\lambda}\right|\right)\right] \leq 1\right\},\\ \|X\|^*_\Phi &= \sup_{\mathbb{E}[\Psi(|Y|)]\leq 1}\mathbb{E}[XY] = \inf_{t>0}\left\{\frac{1}{t}(1+\mathbb{E}[\Phi(t|X|)])\right\}.\end{aligned}$$

The norms $\|X\|_\Phi$ and $\|X\|^*_\Phi$ are called the *Luxemburg norm* and *Orlicz norm*, respectively. The spaces L^Φ and M^Φ are called the *Orlicz space* and *Orlicz heart*, respectively. Similarly, define L^Ψ and M^Ψ as well as the norms $\|X\|_\Psi$ and $\|X\|^*_\Psi$.

In the rest of this paper, we assume $0 < q \leq 1$ and consider the following pair of Young functions

$$\Phi(x) := \begin{cases} x, & \text{for } x \leq 1,\\ \exp_q(x-1), & \text{for } x > 1,\end{cases} \qquad (4.6)$$

https://doi.org/10.1017/S0269964822000444 Published online by Cambridge University Press

and

$$\Psi(z) := \begin{cases} 0, & \text{for } z \le 1, \\ qz \ln_q(z^{1/q}), & \text{for } z > 1. \end{cases} \tag{4.7}$$

Proposition 4.3. *For $0 < q \le 1$, let Φ and Ψ be a pair of Young functions defined by (4.6) and (4.7), respectively. Then $E = M^{\Phi}$, $E' = M^{\Psi} \subset L^{\Psi}$, and $E'' = L^{\Phi}$. Indeed, for $0 < \alpha < 1$, the norms*

$$\| \cdot \| = \mathrm{TsVaR}_{\alpha}(\cdot) \text{ and } \| \cdot \|_{\Phi}^{*}$$

are equivalent on E''. Particularly, we have

$$\|X\| \le c_q(1-\alpha)^{q-1}\|X\|_{\Phi}^{*}, \quad Y \in E'',$$

where $c_q = \max\{\exp_q(1), -\ln_q(1-\alpha)\}$.

Proof. It is easy to show that, for $x \ge 0$,

$$\ln_q(x) \le x - 1, \quad \exp_q(x) - 1 - \ln_q(1-\alpha) \le c_q\dot{(}1 + \Phi(x)),$$

where $c_q = \max\{\exp_q(1), -\ln_q(1-\alpha)\}$. Then, for $X \in L_{+}^{0}$,

$$\begin{aligned} \ln_q\left(\frac{1}{1-\alpha}\mathbb{E}[\exp_q(\lambda X)]\right) &= (1-\alpha)^{q-1}(\ln_q \mathbb{E}[\exp_q(\lambda X)] - \ln_q(1-\alpha)) \\ &\le (1-\alpha)^{q-1}(\mathbb{E}[\exp_q(\lambda X)] - 1 - \ln_q(1-\alpha)) \\ &\le c_q(1-\alpha)^{q-1}(1 + \mathbb{E}[\Phi(\lambda X)]). \end{aligned}$$

Thus, for $X \in L^{\Phi}$, we have

$$\mathrm{TsVaR}_{\alpha}(X) \le c_q(1-\alpha)^{q-1} \inf_{\lambda>0}\left\{\frac{1}{\lambda}(1 + \mathbb{E}\Phi[\lambda X])\right\} = c_q(1-\alpha)^{q-1}\|X\|_{\Phi}^{*}.$$

This proves that $L^{\Phi} \subset E''$. To prove the converse inequality, let $X \in E''$, that is, there exists $\lambda > 0$ such that $\mathbb{E}[\exp_q(\lambda X)] < \infty$. Then $X \in L^{\Phi}$ since $\Phi(x) \le \exp_q(x)$. So we get $L^{\Phi} = E''$. Similarly, $E = M^{\Phi}$.

Now consider the identity map $A : (L^{\Phi}, \|\cdot\|_{\Phi}^{*}) \to (E'', \mathrm{TsVaR}_{\alpha}(\cdot))$, which is bounded. By the above reasoning, A is bijective. Since $(E'', \mathrm{TsVaR}_{\alpha}(\cdot))$ is a Banach space by Theorem 4.1, it follows from the bounded inverse theorem (open mapping theorem, see [21] Corollary 2.12) that the inverse A^{-1} is continuous as well, that is, there is a constant $c' < \infty$ such that

$$\|X\|_{\Phi}^{*} \le c' \,\mathrm{TsVaR}_{\alpha}(X), \quad X \in E''.$$

Therefore, $\| \cdot \| = \mathrm{TsVaR}_{\alpha}(\cdot)$ and $\| \cdot \|_{\Phi}^{*}$ are equivalent on E''.

Finally, note that Ψ does not satisfy (Δ_2) condition, that is, $\Psi(2x) \le k\psi(x)$ for every $k > 2$ whenever x is large enough. By Theorems 2.1.11 and 2.1.17 of Edgar *et al.* [9], we know $M^{\Psi} \subset L^{\Psi}$. This completes the proof of the proposition. □

4.2. Characterization of the dual norm

In this subsection, we consider the associated dual norm of TsVaR_{α}:

$$\|Z\|^{*} := \sup_{X \in E'', \mathrm{TsVaR}_{\alpha}(X) \le 1} \mathbb{E}[XZ] = \sup_{X \in E'', X \ne 0} \frac{\mathbb{E}[XZ]}{\mathrm{TsVaR}_{\alpha}(X)}, \quad Z \in E'.$$

https://doi.org/10.1017/S0269964822000444 Published online by Cambridge University Press

Theorem 4.4. *For* $0 < q \leq 1$ *and* $0 < \alpha < 1$, *we have*

$$\|Z\|^* = \sup_{c>0} \frac{\mathbb{E}[Z \ln_q((Z/c)^{1/q} \vee 1)]}{\ln_q\left(\frac{1}{1-\alpha}\mathbb{E}[(Z/c)^{1/q} \vee 1]\right)}, \quad Z \in E', \tag{4.8}$$

where $x \vee y = \max\{x, y\}$.

Proof. The proof is similar to that of Theorem 4.4 in Ahmadi-Javid and Pichler [4]. For completeness, we give the details. Note that, for $Z \in E'$,

$$\begin{aligned}
\|Z\|^* &= \sup_{\mathrm{TsVaR}_\alpha(X)\leq 1} \mathbb{E}[XZ] = \sup_{X\neq 0} \frac{\mathbb{E}[XZ]}{\mathrm{TsVaR}_\alpha(X)} \\
&= \sup_{X\neq 0} \frac{\mathbb{E}[XZ]}{\inf_{\lambda>0}\left\{\lambda^{-1}\ln_q\left(\frac{1}{1-\alpha}\mathbb{E}[\exp_q(\lambda X)]\right)\right\}} \\
&= \sup_{X\neq 0}\sup_{\lambda>0} \frac{\mathbb{E}[\lambda XZ]}{\ln_q\left(\frac{1}{1-\alpha}\mathbb{E}[\exp_q(\lambda X)]\right)} \\
&= \sup_{X\neq 0} \frac{\mathbb{E}[XZ]}{\ln_q\left(\frac{1}{1-\alpha}\mathbb{E}\left[\exp_q(X)\right]\right)}.
\end{aligned} \tag{4.9}$$

For any $c > 0$, define $X_c = \ln_q((Z/c)^{1/q} \vee 1)$. Then $X_c \geq 0$ and

$$\|Z\|^* \geq \sup_{c>0} \frac{\mathbb{E}[Z \ln_q((Z/c)^{1/q} \vee 1)]}{\ln_q\left(\frac{1}{1-\alpha}\mathbb{E}[(Z/c)^{1/q} \vee 1]\right)}.$$

To obtain the converse inequality, from (4.9), it follows that $d \geq \|Z\|^*$ is equivalent to

$$\mathbb{E}[XZ] - d \cdot \ln_q\left(\frac{1}{1-\alpha}\mathbb{E}[\exp_q(X)]\right) \leq 0, \quad \forall X \geq 0.$$

We maximize the left-hand side of the above expression with respect to $X \geq 0$. The Lagrangian of this maximization problem is

$$L(X, \mu) = \mathbb{E}[XZ] - d \cdot \ln_q\left(\frac{1}{1-\alpha}\mathbb{E}[\exp_q(X)]\right) - \mathbb{E}[X\mu],$$

where μ is the Lagrange multiplier associated with the constraint $X \geq 0$. The Lagrangian L is differentiable, and its directional derivative with respect to X in direction $H \in E''$ is

$$\frac{\partial}{\partial X} L(X, \mu)H = \mathbb{E}[(Z - \mu)H] - d\left(\frac{1}{1-\alpha}\right)^{1-q}[\mathbb{E}\exp_q(X)]^{-q}\mathbb{E}[(\exp_q(X))^q H].$$

The derivative vanishes in every direction H so that $Z - \mu = c[\exp_q(X)]^q$ with $c = d(1-\alpha)^{q-1}$ $[\mathbb{E}\exp_q(X)]^{-q} > 0$. By complimentary slackness for the optimal X and μ,

$$X > 0 \iff \mu = 0 \iff Z = c[\exp_q(X)]^q > c,$$

which is equivalent to

$$X = \begin{cases} \ln_q(Z/c)^{1/q}, & \text{if } X > 0 \\ 0, & \text{if } X = 0 \end{cases} = \ln_q((Z/c)^{1/q} \vee 1).$$

https://doi.org/10.1017/S0269964822000444 Published online by Cambridge University Press

This completes the proof of the theorem. □

In the next proposition, we reconsider the domain of the objective function in (4.8).

Proposition 4.5. *For* $0 < q \leq 1$ *and* $0 < \alpha < 1$, *the objective function*

$$h_\alpha(c) := \frac{\mathbb{E}[Z \ln_q((Z/c)^{1/q} \vee 1)]}{\ln_q\left(\frac{1}{1-\alpha}\mathbb{E}((Z/c)^{1/q} \vee 1)\right)}$$

in the expression of the dual norm (4.8) can be continuously extend to $[0, \infty)$, *and*

$$\lim_{c \downarrow 0} h_\alpha(c) = (1-\alpha)^{1-q} \|Z\|_{1/q}. \tag{4.10}$$

Furthermore, the supremum is attained at some $c \geq 0$. *If* $Z \neq 0$ *is bounded, then the optimal* c *satisfies* $0 \leq c < \|Z\|_\infty$.

Proof. First, note that $h_\alpha(c)$ is continuous in $c \in (0, \infty)$ and $h_\alpha(c) \geq 0$. Note that

$$h_\alpha(c) \longrightarrow \frac{(1-\alpha)^{1-q}\mathbb{E}[Z^{1/q}]}{(\mathbb{E}[Z^{1/q}])^{1-q}} = (1-\alpha)^{1-q}\|Z\|_{1/q}, \quad c \to 0,$$

and $h_\alpha(c) \longrightarrow 0$ as $c \to +\infty$. On the other hand, for $c \geq \|Z\|_\infty$, the numerator of h_α is 0 and, hence, $h_\alpha(c) = 0$. Thus, the desired result follows from the continuity of $h_\alpha(\cdot)$ on $[0, \|Z\|_\infty]$. □

From Theorem 4.4, for given $Z \in E'$, we can identify a random variable $X^* \in E''$, which maximizes

$$\|Z\|^* = \sup_{X \in E'', X \neq 0} \frac{\mathbb{E}[XZ]}{\mathrm{TsVaR}_\alpha(X)}. \tag{4.11}$$

Proposition 4.6. *For* $0 < q \leq 1$ *and* $0 < \alpha < 1$, *let* $Z \in E'$, *and suppose that* $c^* > 0$ *be optimal in (4.8). Then,*

$$X := \ln_q((Z/c^*)^{1/q} \vee 1) \tag{4.12}$$

satisfies the equality

$$\|Z\|^* = \frac{\mathbb{E}[XZ]}{\mathrm{TsVaR}_\alpha(X)}.$$

Proof. By (4.8) and (4.12), we have

$$\|Z\|^* = \frac{\mathbb{E}[ZX]}{\ln_q\left(\frac{1}{1-\alpha}\mathbb{E}[\exp_q(X)]\right)}.$$

Thus,

$$\mathbb{E}[ZX] \leq \|Z\|^* \cdot \mathrm{TsVaR}_\alpha(X) \leq \|Z\|^* \cdot \ln_q\left(\frac{1}{1-\alpha}\mathbb{E}[\exp_q(X)]\right) = \mathbb{E}[ZX],$$

where the inequalities follows from (4.11) and (3.1), respectively. This completes the proof. □

For $0 < q \leq 1$ and $0 < \alpha < 1$, let $X \in E''$ be fixed. How to identify a random variable $Z \in E'$, which maximizes

$$\mathrm{TsVaR}_\alpha(X) := \sup_{Z \in E', Z \neq 0} \frac{\mathbb{E}[XZ]}{\|Z\|^*}. \tag{4.13}$$

Ahmadi-Javid and Pichler [4] gave a positive answer for $q = 1$. It is still an open question for $0 < q < 1$.

https://doi.org/10.1017/S0269964822000444 Published online by Cambridge University Press

5. Conclusion

In this paper, we generalize the concept of EVaR, defined by Ahmadi-Javid [2], to TsVaR, using the generalized q-logarithm and generalized q-exponential functions. TsVaR is not a coherent premium principle, even not a convex premium principle. This is caused by a lack of cash invariance in general. We show that in the class of TsVaR with $q \in (0, 1]$, only EVaR corresponding to $q = 1$ is a coherent premium principle. Dual representation for TsVaR is established by using the variational representation for the generalized relative entropy, which is due to Ma and Tian [16]. We compare TsVaR's under different confidence levels α and obtain the strong monotonicity of TsVaR. Finally, we consider the norm and dual norm induced by TsVaR constrained on the related spaces E, E', and E'', which are called the primal, dual, and bidual Tsallis spaces, respectively. It is proven that $(E, \|\cdot\|)$ and $(E'', \|\cdot\|)$ are Banach spaces when the norm $\|\cdot\|$ is induced by TsVaR. We also give the explicit formula of the dual TsVaR norm.

Acknowledgments. The authors are grateful to the Editor and one anonymous referee for their comprehensive reviews of an earlier version of this paper.

Funding Statement. T. Hu was supported by the National Natural Science Foundation of China (No. 71871208) and by Anhui Center for Applied Mathematics.

Conflict of interest. The authors declare no conflict of interest.

References

[1] Ahmadi-Javid, A. (2012). Addendum to: Entropic value-at-risk: A new coherent risk measure. *Journal of Optimization Theory Applications* 155(3): 1124–1128.
[2] Ahmadi-Javid, A. (2012). Entropic value-at-risk: A new coherent risk measure. *Journal of Optimization Theory Applications* 155(3): 1105–1123.
[3] Ahmadi-Javid, A. & Fallah-Tafti, M. (2019). Portfolio optimization with entropic value-at-risk. *European Journal of Operational Research* 279(1): 225–241.
[4] Ahmadi-Javid, A. & Pichler, A. (2017). An analytical study of norms and Banach spaces induced by the entropic value-at-risk. *Mathematics and Financial Economics* 11: 527–550.
[5] Cheridito, P. & Li, T. (2008). Dual characterization of properties of risk measures on Orlicz hearts. *Mathematics and Financial Economics* 2(1): 29–55.
[6] Cheridito, P. & Li, T. (2009). Risk measures on Orlicz hearts. *Mathematics Finance* 19(2): 189–214.
[7] Chernoff, H. (1952). A measure of asymptotic efficiency for tests of a hypothesis based on the sum of observations. *The Annals of Mathematical Statistics* 23(4): 493–507.
[8] Delbaen, F., Grandits, P., Rheinländer, T., Samperi, D., Schweizer, M., & Stricker, C. (2002). Exponential hedging and entropic penalties. *Mathematical Finance* 12(2): 99–123.
[9] Edgar, G., Sucheston, L., & Edgar, G. (1992). *Stopping times and directed processes*. Cambridge: Cambridge University Press.
[10] El Karoui, N. & Ravanelli, C. (2009). Cash subadditive risk measures and interest rate ambiguity. *Mathematical Finance* 19(4): 561–590.
[11] Föllmer, H. & Schied, A. (2002). Convex measures of risk and trading constraints. *Finance and Stochastics* 6(4): 429–447.
[12] Föllmer, H. & Schied, A. (2011). *Stochastic finance: An introduction in discrete time*, 3rd ed. Berlin: Walter de Gruyter.
[13] Han, X., Wang, Q., Wang, R., & Xia, J. (2021). Cash-subadditive risk measures without quasi-convexity. *arXiv preprint*, arXiv:2110.12198.
[14] Jeyakumar, V. & Wolkowicz, H. (1992). Generalizations of Slater's constraint qualification for infinite convex programs. *Mathematical Programming* 57(1): 85–101.
[15] Kalmes, T. & Pichler, A. (2018). On Banach spaces of vector-valued random variables and their duals motivated by risk measures. *Banach Journal of Mathematical Analysis* 12(4): 773–807.
[16] Ma, H. & Tian, D. (2021). Generalized entropic risk measures and related BSDEs. *Statistics & Probability Letters* 174: Article number 5109110.
[17] Pichler, A. (2013). The natural Banach space for version independent risk measures. *Insurance: Mathematics and Economics* 53(2): 405–415.
[18] Pichler, A. (2017). A quantitative comparison of risk measures. *Annals of Operations Research* 254(1): 251–275.
[19] Pichler, A. & Schlotter, R. (2020). Entropy based risk measures. *European Journal of Operational Research* 285: 223–236.
[20] Rényi, A. (1961). On measures of entropy and information. *Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability*, Vol. 1, pp. 547–561.
[21] Rudin, W. (1973). *Functional analysis*. New York: McGraw-Hill.

[22] Shannon, C. (1948). A mathematical theory of communication. *The Bell System Technical Journal* 27: 379–423.
[23] Tian, D. (2022). Pricing principle via Tsallis relative entropy in incomplete market. *arXiv preprint*, arXiv:2201.05316.
[24] Tsallis, C. (1988). Possible generalization of Boltzmann-Gibbs statistics. *Journal of Statistical Physics* 52(1): 479–487.
[25] Tsallis, C. (2009). *Introduction to nonextensive statistical mechanics: approaching a complex world*. New York: Springer.
[26] Wang, S., Young, V., & Panjer, H. (1997). Axiomatic characterization of insurance prices. *Insurance: Mathematics and Economics* 21(2): 173–183.

Appendix A

Proof. We only prove part (4) since the others are trivial. Denote $\boldsymbol{z} = (z_1, \ldots, z_n) = (\exp_q(x_1), \ldots, \exp_q(x_n))$, and let $\boldsymbol{z}^\top$ represent the transpose of $\boldsymbol{z}$ and $\mathbf{1} = (1, \ldots, 1)$. Then, the Hessian matrix of $f(\boldsymbol{x})$ is

$$\nabla^2 f(\boldsymbol{x}) = \frac{q}{(\mathbf{1}\boldsymbol{z}^\top)^{q+1}}[(\mathbf{1}\boldsymbol{z}^\top)\,\mathrm{diag}(\boldsymbol{z}^{2q-1}) - (\boldsymbol{z}^\top)^q \boldsymbol{z}^q].$$

To prove that $f(\boldsymbol{x})$ is convex, it suffices to show that for all $\boldsymbol{v} = (v_1, \ldots, v_n)$, we have

$$\boldsymbol{v}\nabla^2 f(\boldsymbol{x})\boldsymbol{v}^\top = \frac{q}{(\mathbf{1}\boldsymbol{z}^\top)^{q+1}}\left[\left(\sum_{i=1}^n z_i\right)\left(\sum_{i=1}^n v_i^2 z_i^{2q-1}\right) - \left(\sum_{i=1}^n v_i z_i^q\right)^2\right] \geq 0,$$

which follows from the Cauchy–Schwarz inequality. □

Proof. First note that the set $\{(X, \lambda) : \lambda > 0, \mathbb{E}[\exp_q(\lambda X)] < \infty\}$ is convex. Then, it suffices to show that for $\beta \in [0, 1]$,

$$g_\alpha(\beta X + (1-\beta)Y, \beta\lambda_1 + (1-\beta)\lambda_2) \leq \beta g_\alpha(X, \lambda_1) + (1-\beta) g_\alpha(Y, \lambda_2),$$

which can be rewrite as

$$\begin{aligned}&[\beta\lambda_1 + (1-\beta)\lambda_2]\ln_q\left[\frac{1}{\alpha}\mathbb{E}\left[\exp_q\left(\frac{\beta X + (1-\beta)Y}{\beta\lambda_1 + (1-\beta)\lambda_2}\right)\right]\right]\\ &\quad\leq \beta\lambda_1 \ln_q\left[\frac{1}{\alpha}\mathbb{E}\left[\exp_q\left(\frac{X}{\lambda_1}\right)\right]\right] + (1-\beta)\lambda_2\ln_q\left[\frac{1}{\alpha}\mathbb{E}\left[\exp_q\left(\frac{Y}{\lambda_2}\right)\right]\right].\end{aligned}$$

By Proposition 2.1(2), the above inequality reduces to

$$\begin{aligned}&[\beta\lambda_1 + (1-\beta)\lambda_2]\ln_q \mathbb{E}\left[\exp_q\left(\frac{\beta X + (1-\beta)Y}{\beta\lambda_1 + (1-\beta)\lambda_2}\right)\right]\\ &\quad\leq \beta\lambda_1 \ln_q \mathbb{E}\left[\exp_q\left(\frac{X}{\lambda_1}\right)\right] + (1-\beta)\lambda_2 \ln_q \mathbb{E}\left[\exp_q\left(\frac{Y}{\lambda_2}\right)\right].\end{aligned} \tag{A.1}$$

Setting $\lambda = \beta\lambda_1 + (1-\beta)\lambda_2$ and $w = \beta\lambda_1/\lambda$, (A.1) is equivalent to

$$\ln_q \mathbb{E}\left[\exp_q\left(w\frac{X}{\lambda_1} + (1-w)\frac{Y}{\lambda_2}\right)\right] \leq w\ln_q \mathbb{E}\left[\exp_q\left(\frac{X}{\lambda_1}\right)\right] + (1-w)\ln_q \mathbb{E}\left[\exp_q\left(\frac{Y}{\lambda_2}\right)\right],$$

which follows from Proposition 2.1(4). This completes the proof of the lemma. □

Proof. For any $\epsilon > 0$ and $X, Y \in E''$, there exist $\lambda_1, \lambda_2 > 0$ such that $\mathbb{E}[\exp_q(\lambda_1 X)] < \infty$, $\mathbb{E}[\exp_q(\lambda_2 Y)] < \infty$ and

$$g_\alpha(X, \lambda_1) \leq \inf_{\lambda>0}\{g_\alpha(X, \lambda)\} + \epsilon, \quad g_\alpha(Y, \lambda_2) \leq \inf_{\lambda>0}\{g_\alpha(X, \lambda)\} + \epsilon$$

https://doi.org/10.1017/S0269964822000444 Published online by Cambridge University Press

by using the continuity of $g_\alpha(X,\lambda)$ in $\lambda > 0$. By Lemma 3.2, $g_\alpha(X,\lambda)$ is convex in (X,λ). So, for any $\beta \in [0,1]$,

$$\begin{aligned}\inf_{\lambda>0}\{g_\alpha(\beta X+(1-\beta)Y,\lambda)\} &\le g_\alpha(\beta X+(1-\beta)Y,\beta\lambda_1+(1-\beta)\lambda_2)\\ &\le \beta g_\alpha(X,\lambda_1)+(1-\beta)g_\alpha(Y,\lambda_2)\\ &\le \beta\inf_{\lambda>0}\{g_\alpha(X,\lambda)\}+(1-\beta)\inf_{\lambda>0}\{g_\alpha(X,\lambda)\}+\epsilon.\end{aligned}$$

Therefore, the desired result follows by letting $\epsilon \downarrow 0$. □

Proof. We use the idea in the proof of Lemma 1.3 in Ahmadi-Javid [1]. Define $f(x) = x^q \ln_q(x)$, and denote $Y = \mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P}$. Then (3.5) can be rewritten as

$$\sup_{\lambda>0} L(\lambda) = \inf_{\mathbb{Q}\ll\mathbb{P},\mathbb{E}[f(Y)]\le -\ln_q(\alpha)}\{-\mathbb{E}[Y^q X]\},$$

where

$$\begin{aligned}L(\lambda) &= \inf_{Y\in S}\left\{-\mathbb{E}[XY^q]+\lambda\left(\mathbb{E}[f(Y)]-\alpha^{1-q}\ln_q\frac{1}{\alpha}\right)\right\}\\ &= \inf_{Y\in S}\{-\mathbb{E}[XY^q]+\lambda(\mathbb{E}[f(Y)]+\ln_q(\alpha))\}\end{aligned}$$

is the Lagrangian function associated with the optimization problem in the right-hand side, and $S = \{Y \in L^1_+ : \mathbb{E}(Y) = 1\}$. Hence, it suffices to show that the optimal duality gap for optimization problem in the right-hand side is zero. This is possible by showing that the generalized Slater's constraint qualification in Jeyakumar and Wolkowicz [14] holds for this problem, that is, there exists $\widehat{Y} \in L^1$ satisfying $\mathbb{E}[\widehat{Y}] = 1$, $\widehat{Y} \in L^1_+$, and $\mathbb{E}[f(\widehat{Y})] < \alpha^{1-q}\ln_q(1/\alpha) = -\ln_q(\alpha)$. Note that $\widehat{Y} = 1$ fulfills these conditions. Thus, we complete the proof of the lemma. □

Cite this article: Zou Z, Xia Z and Hu T (2024). Tsallis value-at-risk: generalized entropic value-at-risk. *Probability in the Engineering and Informational Sciences* **38**, 1–20. https://doi.org/10.1017/S0269964822000444