# DECREASING BAER SEMIGROUPS

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(Received 27 December, 1967)

**1. Introduction.** In [**8**] and [**9**] we initiated a study of lattice theory by means of Baer semigroups. Basically, a Baer semigroup is a multiplicative semigroup with 0 in which the left annihilator $L(x)$ of each element $x$ is a principal left ideal generated by an idempotent, while its right annihilator $R(x)$ is a principal right ideal generated by an idempotent. By [**8**, Lemma 2, p. 86], $L(0)$ has a unique idempotent generator 1 which is effective as a two-sided multiplicative identity for $S$. For any Baer semigroup $S$, if we use set inclusion to partially order both $\mathscr{L} = \mathscr{L}(S) = \{L(x) \mid x \in S\}$ and $\mathscr{R} = \mathscr{R}(S) = \{R(x) \mid x \in S\}$, we have by [**8**, Theorem 5, p. 86], that $\mathscr{L}$ and $\mathscr{R}$ form dual isomorphic lattices with 0 and 1. The Baer semigroup $S$ is said to *coordinatize* the lattice $L$ in case $\mathscr{L}(S)$ is isomorphic to $L$. In connection with this, it is important to note that by [**9**, Theorem 2.3, p. 1214], a poset $P$ with 0 and 1 is a lattice if and only if it can be coordinatized by a Baer semigroup.

Our goal in this paper is three-fold: (i) to show that a complete lattice is infinitely distributive if and only if it can be coordinatized by a decreasing Baer semigroup (see Definition 4.1); (ii) to begin an investigation of the algebraic properties of this class of semigroups; (iii) to use the Stone representation theorem to obtain a representation for complete infinitely distributive lattices.

**2. Basic terminology.** Let $P$ be a partially ordered set with 0 and 1. A mapping $\phi$: $P \to P$ is called *isotone* if $e \leqq f$ implies that $e\phi \leqq f\phi$ for all $e, f \in P$; it is called *residuated* if it is isotone and there is a necessarily unique isotone mapping $\phi^+ : P \to P$ (called the *residual* map associated with $\phi$) such that, for each $e \in P$,

$$e\phi^+\phi \leqq e \leqq e\phi\phi^+. \tag{1}$$

It follows from this that the pair $(\phi, \phi^+)$ sets up a Galois connection ([**2**], p. 124) between $P$ and its dual. It is now immediate that if $e = \bigvee_\alpha e_\alpha$ exists in $P$, then $e\phi$ is the join of the family $\{e_\alpha \phi\}$; dually, if $e_0 = \bigwedge_\alpha e_\alpha$ exists in $P$, then $e_0\phi^+$ is the meet of $\{e_\alpha \phi^+\}$. The set $S(P)$ of residuated maps on $P$ forms a semigroup with respect to function composition, and by [**9**, Theorem 2.3, p. 1214], this semigroup is a Baer semigroup if and only if $P$ is a lattice. For a more complete description of residuated maps we refer the reader to [**4**].

The residuated map $\phi$ on $P$ is called *range-closed* if its range is a principal ideal of $P$; it is called *dual range-closed* if the range of its associated residual map $\phi^+$ is a dual principal ideal of $P$.

Suppose now that $S$ is a Baer semigroup and $P = \mathscr{L}(S)$. Then each $x$ in $S$ induces a residuated map $\phi_x : P \to P$ by the rule

$$(Se)\phi_x = LR(ex), \tag{2}$$

where $e$ is an idempotent generator of $Se$; furthermore, by [**8**, Theorem 25, p. 94], the map

† Research supported in part by NSF Grant GP-5923.

$x \to \phi_x$ is a semigroup homomorphism of $S$ into $S(P)$ and $\{\phi_x \mid x \in S\}$ is a Baer semigroup which coordinatizes $P$. An element $x$ of $S$ is called *range-closed* or *dual range-closed* according to whether $\phi_x$ has the indicated property in $S(P)$.

While we are at the business of introducing terminology, let us point out that a lattice $L$ is called *infinitely distributive* if, for any family $\{e_\alpha\}$ of elements of $L$, the following infinite distributive law holds both in $L$ and in its dual:

$$\text{If } \bigvee_\alpha e_\alpha \text{ exists, then, for each } x \in L,\ \bigvee_\alpha (e_\alpha \wedge x) \text{ exists and equals } (\bigvee_\alpha e_\alpha) \wedge x. \tag{3}$$

## 3. Decreasing residuated maps.

DEFINITION 3.1. Let $P$ be a partially ordered set. A mapping $\phi : P \to P$ is called *decreasing* if $e \geqq e\phi$ for all $e \in P$, and *increasing* if $e \leqq e\phi$ for all $e \in P$.

LEMMA 3.2. *A residuated map $\phi$ on a partially ordered set $P$ is decreasing if and only if $\phi^+$ is increasing.*

*Proof.* If $e \geqq e\phi$, then $e\phi^+ \geqq e\phi\phi^+ \geqq e$; conversely, $e \leqq e\phi^+$ implies that $e\phi \leqq e\phi^+\phi \leqq e$.

LEMMA 3.3. *For a residuated map $\phi$ on the partially ordered set $P$, the following conditions are equivalent*:

(i) *$\phi$ is a decreasing idempotent*;
(ii) $\phi = \phi^+\phi$;
(iii) $\phi^+ = \phi\phi^+$;
(iv) *$\phi^+$ is an increasing idempotent.*

*Proof.* (i)$\Leftrightarrow$(iv). This follows from Lemma 3.2 and [**8**, Lemma 11, p. 89].
(iv)$\Rightarrow$(ii). If $x \leqq x\phi^+$, then $x\phi \leqq x\phi^+\phi \leqq x$. Hence

$$x\phi \leqq x\phi^+\phi = x\phi^+\phi\phi = (x\phi^+\phi)\phi \leqq x\phi \quad \text{or} \quad x\phi = x\phi^+\phi.$$

(ii)$\Rightarrow$(iii). If $x\phi = x\phi^+\phi$, then $x\phi^+ = x\phi^+\phi\phi^+ = x\phi\phi^+$.
(iii)$\Rightarrow$(iv). If $x\phi^+ = x\phi\phi^+$, then by (1), $x \leqq x\phi\phi^+ = x\phi^+$, and clearly

$$x\phi^+\phi^+ = x\phi^+\phi\phi^+ = x\phi^+.$$

Combining the above lemma with [**9**, Lemma 3.2, p. 1215], we now have

COROLLARY 3.4. *Let $\phi$ be an idempotent decreasing residuated map on the lattice $L$. Then*

(i) *$\phi$ is range-closed if and only if $x\phi = x \wedge 1\phi$ for all $x \in L$,*
(ii) *$\phi$ is dual range-closed if and only if $x\phi^+ = x \vee 0\phi^+$ for all $x \in L$.*

DEFINITION 3.5. A congruence relation $\Theta$ on a complete lattice $L$ is called *complete* if the condition $e_\alpha \equiv x\ (\Theta)$ for each index $\alpha$ implies that $\bigvee_\alpha e_\alpha \equiv x\ (\Theta)$ and $\bigwedge_\alpha e_\alpha \equiv x\ (\Theta)$.

We now establish a one-one correspondence between decreasing idempotent residuated mappings and complete congruences, thus generalizing a theorem of G. Bergmann [**1**].

THEOREM 3.6. *Let $\Theta$ be a complete congruence on the complete lattice $L$. For each $e \in L$, set $e\phi = \bigwedge\{x \in L \mid x \equiv e\ (\Theta)\}$. Then $\phi$ is a decreasing idempotent residuated map on $L$ whose associated residual map is given by $e\phi^+ = \bigvee\{x \in L \mid x \equiv e\ (\Theta)\}$. Indeed, there is a one-one correspondence between complete congruences and decreasing idempotent residuated maps.*

https://doi.org/10.1017/S0017089500000513 Published online by Cambridge University Press

*Proof.* Let $\Theta$ be a complete congruence on $L$, and define $\phi, \phi^+$ as above. Since $e \equiv e(\Theta)$, we clearly have $e\phi \leqq e \leqq e\phi^+$, and, since $\Theta$ is complete, we also have $e \equiv e\phi \equiv e\phi^+(\Theta)$. Thus $y$ is congruent to any one of $e$, $e\phi$ or $e\phi^+$ if and only if $e\phi \leqq y \leqq e\phi^+$. It is immediate that $e\phi = e\phi\phi = e\phi^+\phi$ and $e\phi^+ = e\phi^+\phi^+ = e\phi\phi^+$, so $\phi$ is a decreasing idempotent element of $S(L)$ with $\phi^+$ its associated residuated map.

Suppose conversely that $\phi$ is a decreasing idempotent element of $S(L)$. Notice that, by Lemma 3.3, $e\phi = f\phi$ if and only if $e\phi^+ = f\phi^+$. Thus, if we define $\Theta$ by the rule

$$e \equiv f\ (\Theta) \quad \Leftrightarrow \quad e\phi = f\phi,$$

it will follow that, if $e_\alpha \equiv x\ (\Theta)$ for each index $\alpha$, then

$$e_\alpha \phi = x\phi \Rightarrow (\bigvee_\alpha e_\alpha)\phi = \bigvee_\alpha (e_\alpha \phi) = x\phi,$$
$$e_\alpha \phi^+ = x\phi^+ \Rightarrow (\bigwedge_\alpha e_\alpha)\phi^+ = \bigwedge_\alpha (e_\alpha \phi^+) = x\phi^+,$$

so that we may conclude that $\bigvee_\alpha e_\alpha \equiv x\ (\Theta)$ and $\bigwedge_\alpha e_\alpha \equiv x\ (\Theta)$. Therefore $\Theta$ is a complete congruence. The proof is completed by noting that $x \equiv y\ (\Theta)$ if and only if $x\phi = y\phi$, so that $\Theta$ is determined by $\phi$.

Corollary 3.7. *If $\Theta$ is a complete congruence on $L$ and if $e_\alpha \equiv f_\alpha\ (\Theta)$ for all $\alpha$, then $\bigvee_\alpha e_\alpha \equiv \bigvee_\alpha f_\alpha\ (\Theta)$ and $\bigwedge_\alpha e_\alpha \equiv \bigwedge_\alpha f_\alpha\ (\Theta)$.*

*Proof.* With $\phi$ and $\phi^+$ defined as in the theorem, we have

$$(\bigvee_\alpha e_\alpha)\phi = \bigvee_\alpha (e_\alpha \phi) = \bigvee_\alpha (f_\alpha \phi) = (\bigvee_\alpha f_\alpha)\phi$$
$$(\bigwedge_\alpha e_\alpha)\phi^+ = \bigwedge_\alpha (e_\alpha \phi^+) = \bigwedge_\alpha (f_\alpha \phi^+) = (\bigwedge_\alpha f_\alpha)\phi^+.$$

## 4. Decreasing Baer semigroups.

Definition 4.1. A Baer semigroup $S$ will be called *decreasing* if for each $x \in S$ the induced residuated map $\phi_x$ on $\mathscr{L}(S)$ is decreasing. Then $S$ is decreasing if and only if $Se \in \mathscr{L}$ together with $x \in S$ implies that $(Se)\phi_x = LR(ex) \leqq Se$.

We shall show in §5 that, if $S$ is decreasing, then $\mathscr{L}(S)$ is infinitely distributive, and that every complete infinitely distributive lattice arises in this way. Meanwhile, we begin an investigation of the semigroup properties of decreasing Baer semigroups.

Theorem 4.2. *For a Baer semigroup $S$ the following conditions are equivalent:*

(i) *$S$ is decreasing.*
(ii) *For each $x \in S$, $L(x)$ is a two-sided ideal.*
(iii) *For each $x \in S$, $R(x)$ is a two-sided ideal.*
(iv) *$xy = 0$ implies that $xSy = (0)$.*
(v) *$Se \in \mathscr{L}(S)$ with $e = e^2$ implies that $ex = exe$ for all $x$ in $S$.*
(vi) *$eS \in \mathscr{R}(S)$ with $e = e^2$ implies that $xe = exe$ for all $x$ in $S$.*

*Proof.* (i) $\Rightarrow$ (ii). Let $Se = L(x)$. Then for each $y \in S$, $LR(ey) \leqq Se$ implies that $ey \in L(x)$. It follows that $Se$ is a two-sided ideal of $S$.

(ii) $\Rightarrow$ (iv). If $L(y)$ is a two-sided ideal, then $xy = 0$ implies that $x \in L(y)$, so, for each $a \in S$, $xa \in L(y)$ implies that $xay = 0$. It follows that $xSy = (0)$.

(iv) $\Rightarrow$ (v). Suppose (iv) holds. Let $Se = L(y)$ with $e = e^2$. Then $ey = 0$ implies that $exy = 0$ for all $x \in S$, so $ex \in L(y)$ and hence $ex = exe$.

(v) $\Rightarrow$ (i). If $ex = exe$, then $ey = 0$ implies that $exy = exey = 0$, so $R(e) \leqq R(ex)$. It follows that $(Se)\phi_x = LR(ex) \leqq LR(e) = Se$, and hence $S$ is decreasing.

This establishes that (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iv) $\Leftrightarrow$ (v). The remaining equivalences follow from the symmetry of (iv).

COROLLARY 4.3. *The multiplicative semigroup of a Baer ring is decreasing if and only if every idempotent is central* (*in the sense that $ex = xe$ for all $x$*).

DEFINITION 4.4. A Baer semigroup $S$ is called *strongly regular* [9, p. 1218] if $Sx \in \mathscr{L}(S)$ and $xS \in \mathscr{R}(S)$ for every $x \in S$.

We may now state

COROLLARY 4.5. *Every decreasing strongly regular Baer semigroup $S$ is a union of groups.*

*Proof.* Such a semigroup is regular, and, by Theorem 4.2, every idempotent $e$ commutes with every $x$ in $S$. Hence by [3, Theorem 1.17, p. 28], $S$ is an inverse semigroup. It is now easily seen that for each idempotent $e$, $\{x \in S \mid Sx = Se\}$ is a group. Since $S$ is strongly regular, this completes the proof.

As a final item concerning the nature of decreasing Baer semigroups we have

THEOREM 4.6. *For a Baer semigroup $S$, the following conditions are equivalent*:

(i) *$S$ has no non-zero nilpotent elements.*

(ii) *$L(x) = R(x)$ for each $x \in S$.*

(iii) *$S$ is decreasing and has no non-zero nilpotent ideals.*

*Proof.* (i) $\Rightarrow$ (ii). If $xy = 0$, then $(yx)(yx) = y(xy)x = 0$ shows that $yx = 0$; hence $R(x) \subseteq L(x)$. Similarly, $L(x) \subseteq R(x)$.

(ii) $\Rightarrow$ (iii). If $L(x) = R(x)$ for each $x \in S$, then $L(x)$ is a two-sided ideal, so by Theorem 4.2, $S$ is decreasing. In order to show that there are no non-zero nilpotent ideals, it clearly suffices to show that if $A$ is an ideal of $S$ such that $A^2 = (0)$, then $A = (0)$. If $A$ is such an ideal and $x \in A$, then $x^2 \in A^2 = (0)$ implies that $x^2 = 0$. Now if $Se = L(x)$ with $e = e^2$, then since $L(x) = R(x)$, $xe = 0$. Hence $x \in L(x) = Se$ implies that $x = xe = 0$; consequently $A = (0)$.

(iii) $\Rightarrow$ (i). It suffices to show that $x^2 = 0$ implies that $x = 0$. Now, if $x^2 = 0$, then $SxS$ is an ideal and since $S$ is decreasing,

$$(SxS)(SxS) = S(xSSx)S = (0),$$

by Theorem 4.2(iv). By (iii), $SxS = (0)$; hence $x = 0$.

**5. Infinitely distributive lattices.** In this final section we tackle the problem of linking Baer semigroups with infinitely distributive lattices. The key item is provided by Corollary 4.3 which states that, if $\phi$ is a decreasing range-closed idempotent, then $x\phi = x \wedge 1\phi$ holds for all $x$. It follows that if $x = \bigvee_\alpha x_\alpha$ exists, then $x \wedge 1\phi$ is the join of the family $\{x_\alpha \wedge 1\phi\}$. Dually, if $\phi$ is a decreasing dual range-closed idempotent and if $y = \bigwedge_\alpha y_\alpha$ exists, then $y \vee 0\phi^+$ is the meet of the family $\{y_\alpha \vee 0\phi^+\}$. If we apply this to a decreasing Baer semigroup $S$, then by

[9, Lemma 3.1, p. 1214], $Se \in \mathscr{L}(S)$ implies that $e$ is range-closed, and so for each $Sx \in \mathscr{L}$, $(Sx)\phi_e = Sx \wedge Se$. It follows that if $\bigvee_\alpha Sx_\alpha$ exists in $\mathscr{L}(S)$, then $(\bigvee_\alpha Sx_\alpha) \wedge Se$ is the join of the family $\{Sx_\alpha \wedge Se\}$, and this shows that the infinite distributive law (3) of §2 holds. A dual argument shows that $\mathscr{R}(S)$ satisfies (3), and the dual isomorphism between $\mathscr{R}(S)$ and $\mathscr{L}(S)$ now shows $\mathscr{L}(S)$ to be infinitely distributive.

Suppose next that $L$ is a complete infinitely distributive lattice. Then, for each $e \in L$, the mapping $x\mu_e = x \wedge e$ is residuated with the associated residual map given by

$$x\mu_e^+ = \bigvee\{y \in L \mid y \wedge e \leqq x\}.$$

Dually, the map $x\nu_e = \bigwedge\{y \in L \mid y \vee e \geqq x\}$ is residuated with $x\nu_e^+ = x \vee e$. Let $S$ denote the semigroup formed by the decreasing residuated maps in $L$. By [8, Lemma 13, p. 89], for $\psi, \phi \in S$,

$$\psi\phi = 0 \quad \text{if and only if} \quad 1\psi \leqq 0\phi^+.$$

Now $1\psi \leqq 0\phi^+$ holds if and only if $\psi = \psi\mu_{0\phi^+}$, and so the left annihilator of $\phi$ is $S\mu_{0\phi^+}$. Similarly, $R(\phi) = \nu_{1\phi}S$, showing that $S$ is a Baer semigroup. Furthermore, as in the proof of [8, Theorem 15, p. 90], the mapping $S\mu_e \to e$ is an isomorphism of $\mathscr{L}(S)$ onto $L$. It follows that $S$ coordinatizes $L$ and that $S$ is decreasing. We summarize the situation in the next theorem.

THEOREM 5.1. *If $S$ is a decreasing Baer semigroup, then $\mathscr{L}(S)$ is an infinitely distributive lattice. Every complete infinitely distributive lattice may be coordinatized by a decreasing Baer semigroup.*

COROLLARY 5.2. *A finite lattice is distributive if and only if it can be coordinatized by a decreasing Baer semigroup.*

If $L$ is a complete infinitely distributive lattice, we now seek a representation for $L$ analogous to the Stone representation theorem for Boolean algebras. By [5, Corollary 2, p. 79], $L$ may be regarded as a complete sublattice of a complete Boolean algebra $A$ (with the same zero and unit elements). Hence by [7, Theorem 3.8, p. 9], there is an increasing idempotent residuated mapping $\theta$ on $A$ such that (i) range $\theta = L$, and (ii) the mapping $\alpha \to \theta\alpha$ is a semigroup isomorphism of $S(L)$ onto $S[A; L]$, where $S[A; L]$ denotes the set of all $\phi \in S(A)$ such that range $\phi \subseteq L$ and range $\phi^+ \subseteq L$. Notice now that range $\phi \subseteq L$ if and only if $\phi = \phi\theta$ and range $\phi^+ \subseteq L$ if and only if $\phi^+ = \phi^+\theta^+$, i.e. if and only if $\phi = \theta\phi$. It follows that $S[A; L] = \theta S(A)\theta$. We have yet to relate this to some sort of representation theorem for $L$.

The crucial item here is provided in [6]. By the Stone representation theorem [10], if $A$ is a Boolean algebra, then there is a Boolean space (i.e., a compact, totally disconnected Hausdorff space) $X$ such that $A$ is isomorphic to the algebra of all clopen subsets of $X$. Here the word *clopen* denotes a set which is both open and closed. A relation $R$ on $X$ is called *Boolean* if the inverse image under $R$ of a clopen subset of $X$ is clopen, while the direct image of a point in $X$ is closed. It follows quickly from [6, Theorem 8, p. 233], that $S(A)$ is isomorphic to the semigroup $B(X)$ formed by those relations $R$ on $X$ such that both $R$ and $R^{-1}$ are Boolean. By [11, Theorem 9, p. 116], every increasing residuated map on $A$ may be regarded as a reflexive transitive relation $R$ in $B(X)$. This proves

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THEOREM 5.3. *Let $L$ be a complete infinitely distributive lattice. Then there exists a Boolean space $X$ and a reflexive transitive relation $R$ in $B(X)$ such that $S(L)$ is isomorphic to $R[B(X)]R$.*

COROLLARY 5.4. *If $L$ is a finite distributive lattice, there exists a set $X$ and a reflexive transitive relation $R$ on $X$ such that $S(L)$ is isomorphic to $R[\mathscr{R}(X)]R$, where $\mathscr{R}(X)$ denotes the semigroup of relations on $X$.*

We close this paper by posing a question: *Is it possible to obtain a representation in the spirit of Theorems* 5.1 *and* 5.3, *for completely distributive lattices* (*see* [**2**], *p.* 119)?

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