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ON THE ZEROS OF POWER SERIES WITH EXPONENTIAL LOGARITHMIC COEFFICIENTS

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0. In this paper we investigate the zeros of power series

$$f(z)=\sum_0^\infty A(n)z^n \tag{1}$$

for some functions of coefficients A. In particular, we derive upper and lower bounds for the number of zeros of f in its domain of analyticity. For various choices of A such as $A(t)=(t+1)^\kappa$, $e^{\sqrt{t}}$, $R(t^\alpha)$ (R being a rational and real function), $(t+1)^\kappa \log^\lambda(t+1)$ ($\kappa, \lambda \in \mathbb{R}, \alpha > 0$) detailed investigations of the behaviour of the zeros are given in [2, 3, 4, 5, 6, 7, 9, 12]. The basic methods in obtaining upper and lower bounds for the number of zeros of f for an extended class of functions (1) are due to A. Peyerimhoff [9, 7, 3] and they have been extended by the authors [4, 5, 6, 12].

Throughout suppose that Q_s is a real polynomial with (exact) degree s and distinct zeros $\alpha_1, \ldots, \alpha_p$ that is

$$Q_s(z)=\prod_1^p (z-\alpha_\nu)^{k_\nu} \tag{2}$$

for some $k_\nu \in \mathbb{N}$, $\sum_1^p k_\nu = s$. Furthermore, we denote by P_k a polynomial of degree at most k, when $k \in \mathbb{N}_0$ and $P_k(z) \equiv 0$, when $-k \in \mathbb{N}$. P_k may be different at each occurrence.

We deal with power series (1) for the following choices of A where in some cases it is convenient to characterize A or a closely related function as a solution of some differential equation [cf. 7].

(i)

$$A(x)=\sum_{\nu=1}^p e^{\alpha_\nu x}P_{k_\nu-1}(x)+A_0(x) \tag{3}$$

where $A \in C_s[0,\infty)$ is a real solution of the differential equation

$$Q_s\left(\frac{d}{dx}\right)A(x)=\varphi(x), \qquad x>0, \tag{4}$$

φ being completely monotone for $x>0$.

(ii)

$$A(x)=R(\log(x+1)), \tag{5}$$

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where R denotes a real and rational function the poles of which are different from $\log n$, $n \in \mathbb{N}$.

(iii)

$$A(\log(x+1)) = \sum_{\nu=1}^{p} (x+1)^{\alpha_\nu} P_{k_\nu - 1}(\log(x+1)) \tag{6}$$

where $A \in C_s[0, \infty)$ is a real solution of the differential equation

$$Q_s\left(\frac{d}{dx}\right) A(x) = 0, \qquad x \in \mathbb{R}. \tag{7}$$

In the cases (ii) and (iii) the associated power series (1) admits unique analytic extension onto $\mathbb{C}^* = \{z \in \mathbb{C} \mid \text{if } \operatorname{Re} z \geq 1, \text{ then } \operatorname{Im} z \neq 0\}$, whereas for (3) f can be extended onto

$$\mathbb{C}_p^* = \mathbb{C}^* - \{e^{-\alpha_1}, \ldots, e^{-\alpha_p}\} \tag{8}$$

(see the proofs of Theorems 1, 2 and 3).

(i) If A satisfies (4) with $\alpha_\nu \leq 0$, $\nu = 1, \ldots, p$, and

$$A(0) = A'(0) = \cdots = A^{(q)}(0) = 0 \tag{9}$$

for some $q \in \{0, \ldots, s-1\}$, $s \geq 1$, then it was shown in [7, Theorem 4, p. 219] that

$$f(z) = \sum_{0}^{\infty} A(n+\tau) z^n, \qquad \tau \in [0, 1) \tag{10}$$

has at most s zeros in $\mathbb{C}^*$ (unless $f \equiv 0$) and at least $q+1$ different zeros which are ≤ 0. We shall show that the upper estimate remains true, if the restriction on α_ν is dropped completely and $\mathbb{C}^*$ is replaced by $\mathbb{C}_p^*$. Furthermore, we shall prove that the lower estimate for the number of negative zeros remains true, provided $\operatorname{Im} \alpha_\nu \not\equiv \pi \bmod 2\pi$, $\nu = 1, \ldots, p$ (see Theorem 1).

(ii) For (5) the representation of

$$R(z) = \frac{P_r(z)}{Q_s(z)}, \qquad r \geq 0, \qquad P_r(\alpha_\nu) \neq 0$$

will be written

$$R(z) = C_{r-s}(z) + \sum_{\nu=1}^{p} \sum_{\mu=1}^{k} \frac{a_{\nu,\mu}}{(z-\alpha_\nu)^\mu}, \qquad C_{r-s}(z) = \sum_{j=0}^{r-s} c_j z^j. \tag{11}$$

If $\alpha_\nu \in \mathbb{R} - \mathbb{N}_0$ and $m := \min\{k \in \mathbb{N}_0 \mid k \geq \max(0, \alpha_1, \ldots, \alpha_p)\}$, then it was shown in [3, theorem 3, p. 179] that the number of zeros of $f(z) = \sum_0^\infty R(n) z^n$ in $\mathbb{C}^*$ does not exceed $m + r$. Defining

$$l = \min\{k \in \mathbb{N}_0 \mid k \geq \max(0, e^{\alpha_1} - 1, \ldots, e^{\alpha_p} - 1)\}, \tag{12}$$

we prove that

$$f(z)=\sum_0^\infty R(\log(n+1))z^n \tag{13}$$

has at most $l+r$ zeros in $\mathbb{C}^*$, thereby showing again that an upper bound depends on the degree of P_r and the location of the poles of R only (see Theorem 2).

(iii) It is known [12] that

$$f_{\kappa,m}(z)=\sum_0^\infty (n+1)^\kappa \log^m(n+1)z^n, \qquad \kappa\in\mathbb{R}, \qquad m\in\mathbb{N}_0 \tag{14}$$

has exactly m or $k+m$ zeros in $\mathbb{C}^*$, when $\kappa\le 0$ or $k<\kappa\le k+1$, $k\in\mathbb{N}_0$, respectively and these zeros are all ≤ 0 and simple.

If A satisfies (7) and (9) and if $\alpha_1<\cdots<\alpha_p$,

$$k=\max(-1,[\alpha_p]), \tag{15}$$

($[x]$ denotes the largest integer not exceeding x as customary)

$$n_p \text{ is the number of } \alpha_\nu \text{ being in } \mathbb{N}_0, \tag{16}$$

$j\in\mathbb{N}_0$ is determined by

$$\begin{array}{ll} j=0, & \text{if } \alpha_1\le 0, \\ j<\alpha_1\le j+1, & \text{if } \alpha_1>0, \end{array} \tag{17}$$

then we prove that

$$f(z)=\sum_0^\infty A(\log(n+1))z^n \tag{18}$$

has at most $k+s-n_p$ or $k+s-1-n_p$ zeros in $\mathbb{C}^*$, when $\alpha_1\le 0$ or $\alpha_1>0$ respectively. A lower bound for the number of different zeros being ≤ 0 is given by $j+q+1$ (see Theorems 3 and 4).

The situation becomes completely different for (10) and (18), if we admit that Q_s has non-real zeros, that is some α_ν's are "complex". For instance, if we consider

$$f(z)=\sum_0^\infty \sin\log(n+1)z^n \tag{19}$$

as a special case of (18) ($s=2$, $Q_2(z)=z^2+1$), then we have (see(14))

$$f(z)=\frac{1}{2i}(f_{i,0}(z)-f_{-i,0}(z)).$$

Observing that

$$f_{\kappa,0}(z)\sim\frac{-1}{\Gamma(1-\kappa)}\frac{1}{z(\log(-z))^\kappa},$$

as $z \to \infty$, $z \in \mathbb{C}^*$, $\kappa \notin \mathbb{N}$ [1, p. 226] we obtain for $z=-x$, $x \to \infty$

$$f(-x) \sim \frac{1}{x} \operatorname{Im} \frac{1}{\Gamma(1-i)(\log x)^i} = \frac{\alpha}{x} \sin(\log \log x + \beta),$$

α, $\beta \in \mathbb{R}$, $\alpha \neq 0$. Thus (19) has an infinite number of zeros on the negative real axis. Furthermore, it will be shown that (19) has infinitely many zeros on (0, 1) (see section 4). In general, when Q_s has "complex" zeros, we point out how to treat (10) and (18) by asymptotic methods. To this end we give generalizations of Watson's lemmata for asymptotic representations of Laplace integrals and loop integrals (see section 1).

1. In this section we collect some auxiliary results which will be used for obtaining upper bounds for the number of zeros and for deriving the asymptotic distribution of the zeros of some power series. The first lemma essentially gives a basic technique for handling our power series. Its proof is given in [3, p. 175] but since it is applied several times we restate it.

Lemma 1. *Suppose that* $g \in V[0, 1]$. *Then for* $z \in \mathbb{C}^*$

$$\prod_{j=1}^{n} (1-zt_j) \int_0^1 \frac{dg(t)}{1-zt} = P_{n-1}(z) + z^n \int_0^1 \prod_{j=1}^{n} (t-t_j) \frac{dg(t)}{1-zt}$$

with some polynomial P_{n-1}.

The following lemmata 2 and 3 are generalizations of Watson's lemmata for Laplace integrals and loop integrals [cf. 8, theorems 3.2, p. 113, and 5.1, p. 120]. Since the proofs are direct analogues of those of the theorems cited above, we omit them. Throughout for complex valued functions $a(t)$ and $b(t)$ defined in some angular neighbourhood of t_0, $a(t) \sim b(t)$, $t \to t_0$, means that $\lim_{t \to t_0} a(t)/b(t) = 1$.

Lemma 2. *Suppose that* $q(t)$ *is a complex valued function for* $t>0$ *with a finite number of discontinuities. Further assume that*

$$L(z) := \int_0^\infty e^{-zt} q(t)\, dt$$

and that ρ, $\lambda \in \mathbb{C}$, $\operatorname{Re} \rho > 0$.

(i) *If* L *has a finite abscissa of convergence and if*

$$q(t) \sim t^{\rho-1}\left(\log \frac{1}{t}\right)^{\lambda}, \qquad t \to +0,$$

then, as $z \to \infty$, $|\arg z| \leq (\pi/2) - \theta$, $\theta > 0$

$$L(z) \sim \frac{\Gamma(\rho)}{z^\rho} (\log z)^\lambda.$$

(ii) *If*

$$q(t) \sim t^{\rho-1}(\log t)^{\lambda}, \qquad t \to +\infty,$$

then, as $z \to 0$, $|\arg z| \le (\pi/2) - \theta$,

$$L(z) \sim \frac{\Gamma(\rho)}{z^{\rho}}\left(\log \frac{1}{z}\right)^{\lambda}.$$

(*Throughout* $\log z$, *its power, and the fractional powers of* z *are defined by their principal values.*)

LEMMA 3. *For given* $\varepsilon \in (0, 1)$ *let*

$$C_{\varepsilon} = \{t = re^{-i\pi} \mid \infty > r \ge \varepsilon\} \cup \{t = \varepsilon e^{i\varphi} \mid -\pi \le \varphi \le \pi\}$$
$$\cup \{t = re^{i\pi} \mid \varepsilon \le r < \infty\}$$

be Hankel's loop and $q(t)$ *a function being continuous on* C_{ε} *and holomorphic, but not necessarily single valued, in the annulus* $\{t \mid 0 < |t| < 2\varepsilon\}$. *Further we suppose that*

$$I(z) := \frac{1}{2\pi i} \oint_{C_{\varepsilon}} e^{zt} q(t)\, dt$$

has an abscissa of absolute convergence being different from $+\infty$, *and that*

$$q(t) \sim t^{\rho-1}\left(\log \frac{1}{t}\right)^{\lambda}$$

as $t \to 0$, $|\arg t| \le \pi$, *where* ρ, $\lambda \in \mathbb{C}$.

Then, as $z \to \infty$, $|\arg z| \le (\pi/2) - \theta$, $\theta > 0$,

$$I(z) \sim \begin{cases} \dfrac{1}{\Gamma(1-\rho)} \dfrac{(\log z)^{\lambda}}{z^{\rho}}, & \text{if} \quad \rho \notin \mathbb{N} \\ \gamma_{\rho} \dfrac{\lambda (\log z)^{\lambda-1}}{z^{\rho}}, & \text{if} \quad \rho \in \mathbb{N},\ \lambda \ne 0, \end{cases}$$

where

$$\gamma_{\rho} = -\frac{d}{dx} \frac{1}{\Gamma(1-x)}\bigg|_{x=\rho}.$$

Finally, we use lemmata 2 and 3 to derive the asymptotic behaviour of

$$(14) \qquad f_{\kappa,m}(z) = \sum_{0}^{\infty} (n+1)^{\kappa} \log^{m}(n+1) z^{n}, \qquad \kappa \in \mathbb{C}, \qquad m \in \mathbb{N}_0$$

for $z \to \infty$ and $z \to 1$, $z \in \mathbb{C}^*$ (cf. lemma 3 in [12] with a different proof for $z = -x$, $x \to \infty$).

LEMMA 4. (i) *As* $z \to \infty$, $z \in \mathbb{C}^*$, *we have*

$$f_{\kappa,m}(z) \sim \begin{cases} \dfrac{(-1)^{\kappa+1}}{z^2} & \text{if} \quad m=0, \quad \kappa \in \mathbb{N} \\[2ex] \dfrac{(-1)^{m+1}(\log\log(-z))^m}{z\Gamma(1-\kappa)(\log(-z))^\kappa} & \text{if} \quad m \in \mathbb{N}_0, \quad \kappa \notin \mathbb{N} \\[2ex] \dfrac{(-1)^{m+1} m\gamma_\kappa (\log\log(-z))^{m-1}}{z(\log(-z))^\kappa} & \text{if} \quad m \in \mathbb{N}, \quad \kappa \in \mathbb{N} \end{cases} \tag{20}$$

where γ_κ *is defined in Lemma* 3.

(ii) *As* $z \to 1$, $z \in \mathbb{C}^*$, *we have*

$$f_{\kappa,m}(z) \sim \begin{cases} \dfrac{\Gamma(k+1)(-1)^m}{z(\log 1/z)^{\kappa+1}} \left(\log\log\dfrac{1}{z}\right)^m + H_1(z), & \text{if} \quad -\kappa \notin \mathbb{N} \\[2ex] \dfrac{(-1)^{m-\kappa}(\log\log 1/z)^{m+1}}{z(m+1)(-\kappa-1)!\,(\log 1/z)^{\kappa+1}} + H_2(z), & \text{if} \quad -\kappa \in \mathbb{N} \end{cases} \tag{21}$$

where $H_\mu(z)$ *denote functions being holomorphic at* $z=1$.

Proof. (i) By residue calculus (observe that $m \in \mathbb{N}_0$, see also [4, 5]) we have for $z \in \mathbb{C}_0^* = \{z \in \mathbb{C} \mid \text{if } \operatorname{Re} z \geq 0, \text{ then } \operatorname{Im} z \neq 0\}$

$$f_{\kappa,m}(z) = \frac{1}{z}\sum_1^\infty n^\kappa \log^m n\, z^n = \frac{-1}{2iz}\int_{1/2-i\infty}^{1/2+i\infty} \frac{t^\kappa \log^m t}{\sin \pi t} e^{t\log(-z)}\, dt$$

where $\log(-z) = \log|z| + i(\arg z - \pi)$, $0 < \arg z < 2\pi$. Now we deform the contour of integration into $(0 < \varepsilon < 1)$

$$C'_\varepsilon = \{t = \xi - i\varepsilon \mid -\infty < \xi \leq 0\} \cup \left\{t = \varepsilon e^{i\varphi} \,\middle|\, -\frac{\pi}{2} \leq \varphi \leq \frac{\pi}{2}\right\} \cup \{t = \xi + i\varepsilon \mid 0 \geq \xi > -\infty\}$$

and obtain for $z \in \mathbb{C}^*$, $|z| > 1$

$$f_{\kappa,m}(z) = \frac{-1}{2iz}\oint_{C'_\varepsilon} \frac{t^\kappa \log^m t}{\sin \pi t} e^{t\log(-z)}\, dt.$$

Now an application of Lemma 3 (see also [9, p. 205] for $z=-x$, $x \to \infty$ and $m=0$) leads to (20) provided $m \in \mathbb{N}_0$, $\kappa \notin \mathbb{N}$ or $m \in \mathbb{N}$, $\kappa \in \mathbb{N}$, since a deformation of C_ε into C'_ε in Lemma 3 does not affect the statement. If $m=0$ and $\kappa \in \mathbb{N}$, then see [10, Vol. I, p. 7, prob. 46].

(ii) In this case an application of Plana's sum formula [11, p. 440] gives

$$\begin{aligned} f_{\kappa,m}(z) &= \frac{1}{z}\sum_1^\infty n^\kappa \log^m n z^n = \tfrac{1}{2}\log^m 1 + \frac{1}{z}\int_1^\infty t^\kappa \log^m t\, e^{-t\log(1/z)}\, dt \\ &\quad + i\int_0^\infty \frac{(1+iy)^\kappa \log^m(1+iy)z^{iy} - (1-iy)^\kappa \log^m(1-iy)z^{-iy}}{e^{2\pi y}-1}\, dy \\ &= \frac{1}{z}\int_1^\infty t^\kappa \log^m t e^{-t\log(1/z)}\, dt + H(z) \end{aligned}$$

where H is holomorphic at $z=1$. If $\operatorname{Re}\kappa>-1$, then Lemma 2, (ii) implies (21) directly. If $\operatorname{Re}\kappa\le -1$, then we put $\nu:=[-\operatorname{Re}\kappa]$ and apply Lemma 2, (ii) to

$$\left(\frac{d}{dz}z\right)^{\nu} f_{\kappa,m}(z)=\sum_{0}^{\infty}(n+1)^{\kappa+\nu}\log^{m}(n+1)z^{n}$$

which gives (note that $\operatorname{Re}\kappa+\nu>-1$)

$$\left(\frac{d}{dz}z\right)^{\nu} f_{\kappa,m}(z)\sim\frac{1}{z}\frac{\Gamma(\kappa+\nu+1)(-1)^{m}}{(\log 1/z)^{\kappa+\nu+1}}\left(\log\log\frac{1}{z}\right)^{m}$$

as $z\to 1$, $z\in\mathbb{C}^*$. Now ν times integration leads to (21).

2. In this section we deal with the power series (10) thereby extending the results in [7].

Theorem 1. *Suppose that* $Q_s\in\mathbb{R}[z]$, $s\ge 0$, *and that* $A\in C_s[0,\infty)$ *is a real solution of the differential equation*

$$Q_s\left(\frac{d}{dx}\right)A(x)=\varphi(x),\qquad x>0 \tag{4}$$

φ *being completely monotone for* $x>0$. *If* $\{\alpha_1,\ldots,\alpha_p\}$ *is the set of different zeros of* Q_s, *then*

$$f(z)=\sum_{0}^{\infty}A(n+\tau)z^{n},\qquad \tau\ge 0 \tag{10}$$

defines on $\mathbb{C}_p^*$ (*see* (8)) *uniquely a holomorphic function possessing at most* s *zeros unless* $f\equiv 0$. *Moreover, if*

$$\operatorname{Im}\alpha_\nu\not\equiv\pi \bmod 2\pi,\qquad \nu=1,\ldots,p, \tag{22}$$

$$A(0)=A'(0)=\cdots=A^{(q)}(0)=0 \tag{9}$$

for some $q\in\{0,\ldots,s-1\}$, $s\ge 1$, *and* $\tau\in[0,1)$, *then* f *has at least* $q+1$ *different zeros which are* ≤ 0. *The zeros of those being on the negative real axis have odd multiplicities unless* $f\equiv 0$.

Proof. Actually the proof for the upper estimate is hidden in that of theorem 3 in [7]. Therefore we only give a short outline for this part. The general solution of (4) is given by (3) with

$$A_0(x)=\frac{1}{2\pi i}\int_{+0}^{1}w\,dg(w)\int_{C_w}\frac{e^{(x-1)t}}{Q_s(t)(t-\log w)}\,dt,\qquad x>0, \tag{23}$$

where $\varphi(x)=\int_{+0}^{1}w^{x}\,dg(w)$, $x>0$, for some monotonically increasing g and C_w is a positively oriented and simply closed curve in the half plane $\operatorname{Re}t\le\max_{1\le\nu\le p}\operatorname{Re}\alpha_\nu+1$ containing $\alpha_1,\ldots,\alpha_p$, and $\log w$ in its interior. Since A_0

satisfies (4), it follows from lemma 1 in [7, p. 212] that

$$\int_{+0}^{1} \frac{dg(w)}{(1+\log 1/w)^{s-\nu}} < \infty, \qquad \nu = 0, \ldots, s. \tag{24}$$

This ensures the existence of the w-integral in (23) after having evaluated the contour integral along C_w by calculus of residues. To establish the analytic extension of f we observe that for sufficiently small $|z|$ we have, by (23) and (24),

$$\begin{aligned} \sum_{n=0}^{\infty} A_0(n+\tau)z^n &= \frac{1}{2\pi i}\sum_{0}^{\infty} z^n \int_{+0}^{1} w\,dg(w)\int_{C_w} \frac{e^{(n+\tau-1)t}}{Q_s(t)(t-\log w)}\,dt \\ &= \frac{1}{2\pi i}\int_{+0}^{1} w\,dg(w)\int_{C_w} \frac{e^{(\tau-1)t}}{(1-e^t z)Q_s(t)(t-\log w)}\,dt \end{aligned} \tag{25}$$

being holomorphic throughout $\mathbb{C}^*$. The part of f generated by the homogeneous solution of (4) obviously has the form

$$P_{s-1}(z)\Big/\prod_{\nu=1}^{p}(1-e^{\alpha_\nu}z)^{k_\nu} \tag{26}$$

for some $P_{s-1} \in \mathbb{R}[z]$. Now $f(z)$ is the sum of (25) and (26). Multiplying $f(z)$ by $\prod_1^p (1-e^{\alpha_\nu}z)^{k_\nu}$ the use of the technique of Lemma 1 immediately leads to (note that $\log(1/z)$ is outside of C_w for all $w \le 1$ if $|z|$ is sufficiently small)

$$\begin{aligned} &\prod_{\nu=1}^{p}(1-e^{\alpha_\nu}z)^{k_\nu} f(v) \\ &\quad = P_{s-1}(z) + z^s\int_{+0}^{1} w\,dg(w)\frac{1}{2\pi i}\int_{C_w}\prod_{\nu=1}^{p}\left(\frac{e^t-e^{\alpha_\nu}}{t-\alpha_\nu}\right)^{k_\nu}\frac{e^{(\tau-1)t}\,dt}{(1-e^t z)(t-\log w)} \\ &\quad = P_{s-1}(z) + z^s\int_{+0}^{1}\frac{w^\tau}{1-zw}\prod_{1}^{p}\left(\frac{w-e^{\alpha_\nu}}{\log w-\alpha_\nu}\right)^{k_\nu} dg(w) \end{aligned} \tag{27}$$

giving the analytic extension of f onto $\mathbb{C}_p^*$.

Since $Q_s \in \mathbb{R}[z]$, it follows that $P_{s-1} \in \mathbb{R}[z]$ and that

$$\prod_{1}^{p}\left(\frac{w-e^{\alpha_\nu}}{\log w-\alpha_\nu}\right)^{k_\nu} \ge 0.$$

Now the upper bound follows from theorem 1 in [9, p. 196].

To prove the lower bound we put

$$a_\nu := \left[\frac{|\operatorname{Im}\alpha_\nu|}{\pi}\right] \quad\text{and}\quad \beta_\nu := \begin{cases} \alpha_\nu - \pi i a_\nu \operatorname{sign}(\operatorname{Im}\alpha_\nu), & \text{if } a_\nu \text{ is even} \\ \alpha_\nu - \pi i(a_\nu+1)\operatorname{sign}(\operatorname{Im}\alpha_\nu), & \text{if } a_\nu \text{ is odd.} \end{cases}$$

Clearly $\alpha_\nu - \beta_\nu \equiv 0 \bmod 2\pi i$ and $|\operatorname{Im}\beta_\nu| < \pi$, by (22).

From (23) we have

$$A_0(x)=\int_{+0}^{1} w\,dg(w)\left\{\sum_{\nu=1}^{p}\operatorname*{res}_{t=\alpha_\nu}\frac{e^{(x-1)t}}{Q_s(t)(t-\log w)}+\frac{w^{x-1}}{Q_s(\log w)}\right\}. \tag{28}$$

By the periodicity of the exponential we obtain that $A_0(n)$ and $A(n)$ do not change, when the α_ν's in the exponents are replaced by the β_ν's. Further, we may assume that $\operatorname{Re}\beta_\nu\le 0$, for otherwise we consider

$$\tilde f(\xi)=f(e^k z)=\sum_0^\infty \tilde A(n)\xi^n$$

where $\xi:=e^k z$, $\tilde A(x)=A(x)e^{-kx}$ for some $k\in\mathbb{N}_0$. Since all these manipulations do not affect the assumptions of Theorem 1 we may suppose w.l.o.g. that $|\operatorname{Im}\alpha_\nu|<\pi$ and $\operatorname{Re}\alpha_\nu\le 0$, $\nu=1,\ldots,p$.

Next, we put $\theta:=\max_{1\le\nu\le p}\operatorname{Im}\alpha_\nu$. Hence it follows from (3) and (28) that for every positive ε and γ

$$|A(\gamma+\rho e^{i\varphi})|<e^{(\theta+\varepsilon)\rho}\quad |\varphi|\le\frac{\pi}{2}, \tag{29}$$

when ρ is sufficiently large. By (24), (29) holds for $\gamma=0$ and $(q<s)$ $|A^{*(q)}(x)|<e^{(\theta+\varepsilon)|x|}$ for sufficiently large $|x|$, where $A^*(x):=A(ix)$, $x\in\mathbb{R}$. Clearly (9) implies $A^*(0)=A^{*\prime}(0)=\cdots=A^{*(q)}(0)=0$. Now the lower bound follows from the theorem in [6].

Remarks. (i) Formula (27) shows that f has at most $s-1$ zeros in $\mathbb{C}-\{e^{-\alpha_1},\ldots,e^{-\alpha_p}\}$ if $\varphi\equiv 0$.

(ii) The function

$$f(z)=\sum_0^\infty n^2\cos\pi n\ z^n=\frac{-z(1-z)}{(1+z)^3}$$

shows that we cannot omit condition (22).

Applying the lower estimate given in Theorem 1 to $g(z)=f(-z)$ we obtain

Corollary. *If in Theorem 1 we assume in addition that $\varphi\equiv 0$ and $\operatorname{Im}\alpha_\nu\not\equiv 0 \bmod 2\pi$, then $f(z)=\sum_0^\infty A(n)z^n$ has at least $q+1$ different zeros which are ≥ 0. The zeros of those being on the positive real axis have odd multiplicities unless $f\equiv 0$.*

As an application we consider $f(z)=\sum_0^\infty n^k\sin\alpha n z^n$, $k\in\mathbb{N}_0$, $0<\alpha<\pi$ $(Q_s(z)=(z^2+\alpha^2)^{k+1}$, $s=2k+2$, $q=k$, $\alpha_1=i\alpha$, $\alpha_2=-i\alpha$, $\varphi\equiv 0)$. It follows that f has exactly $2k+1$ zeros in $\mathbb{C}-\{e^{i\alpha},e^{-i\alpha}\}$, all zeros are simple, and, besides $z=0$, there are k positive and k negative zeros.

3. In this section we deal with the power series (13). The main result is given by

Theorem 2. *Suppose that $R(z)=P_r(z)/Q_s(z)$, $P_r\in\mathbb{R}[z]$. Further assume that*

$r\geq 0$, Q_s is given by (2), *$P_r(\alpha_\nu)\neq 0$, $\alpha_\nu\neq \log n$, $n\in\mathbb{N}$, $\nu=1,\ldots,p$.*
Then

$$f(z)=\sum_{0}^{\infty} R(\log(n+1))z^n \tag{13}$$

defines on $\mathbb{C}^*$ uniquely a holomorphic function. Further, if $\alpha_\nu\in\mathbb{R}$, $\nu=1,\ldots,p$, then the number of zeros of f in $\mathbb{C}^*$ does not exceed $l+r$ (l is defined by (12)).

Proof. Throughout the proof we denote the number of zeros of f in $\mathbb{C}^*$ by N. Using representation (11) we have for $|z|<1$

$$f(z)=\sum_{0}^{\infty} C_{r-s}(\log(n+1))z^n+\sum_{\nu=1}^{p}\sum_{\mu=1}^{k_\nu} a_{\nu,\mu}\sum_{n=0}^{\infty}\frac{z^n}{(\log(n+1)-\alpha_\nu)^\mu}.$$

From formula (1.7) and the remarks immediately following in [12] we get ($m=1,\ldots,r-s$)

$$f_{0,m}(z)=\sum_{0}^{\infty}\log^m(n+1)z^n=\frac{z}{1-z}\int_0^1\frac{1-t}{\log 1/t}\frac{P_{m-1}(\log_2 1/t)}{1-zt}\,dt,$$

where $\log_2 1/t=\log\log 1/t$. Hence it follows that

$$\begin{aligned}\sum_{0}^{\infty} C_{r-s}(\log(n+1))z^n&=\sum_{j=0}^{r-s}c_j\sum_{n=0}^{\infty}\log^j(n+1)z^n\\&=\frac{c_0}{1-z}+\frac{z}{1-z}\int_0^1\frac{1-t}{\log 1/t}\frac{P_{r-s-1}(\log_2 1/t)}{1-zt}\,dt.\end{aligned}$$

Furthermore, we have for $n\geq l$, $\mu\geq 1$,

$$\begin{aligned}\frac{1}{(\log(n+1)-\alpha_\nu)^\mu}&=\frac{1}{(\mu-1)!}\int_0^\infty e^{-\tau(\log(n+1)-\alpha_\nu)}\tau^{\mu-1}\,d\tau\\&=\frac{1}{(\mu-1)!}\int_0^\infty\frac{e^{\alpha_\nu\tau}}{(n+1)^\tau}\tau^{\mu-1}\,d\tau\\&=\frac{1}{(\mu-1)!}\int_0^1 dt\;t^n\int_0^\infty d\tau\frac{e^{\alpha_\nu\tau}\tau^{\mu-1}}{\Gamma(\tau)}e^{(\tau-1)\log_2(1/t)}\end{aligned}$$

and hence

$$\sum_{0}^{\infty}\frac{z^n}{(\log(n+1)-\alpha_\nu)^\mu}=P_{l-1}(z)+\frac{z^l}{(\mu-1)!}\int_0^1 dt\frac{t^l}{1-zt}\int_0^\infty d\tau\frac{e^{\alpha_\nu\tau}\tau^{\mu-1}}{\Gamma(\tau)}e^{(\tau-1)\log_2(1/t)}.$$

Now, using Lemma 1, we finally obtain for $z\in\mathbb{C}^*$ (observe that $c_0=0$, when $r<s$)

$$(1-z)f(z)=P_l(z)+z^{l+1}\int_0^1\frac{(1-t)t^l}{\log 1/t}\frac{V(t)}{1-zt}\,dt,\quad\text{if}\quad r\geq s \tag{30}$$

$$f(z)=P_{l-1}(z)+z^l\int_0^1\frac{t^l}{\log 1/t}\frac{V(t)}{1-zt}\,dt,\quad\text{if}\quad r<s \tag{31}$$

where

$$V(t)=P_{r-s-1}\left(\log_2\frac{1}{t}\right)-\int_0^\infty e^{\tau\log_2(1/t)}\sum_{\nu=1}^{p}e^{\alpha_\nu\tau}\sum_{\mu=1}^{k_\nu}\frac{a_{\nu,\mu}}{(\mu-1)!}\,\tau^{\mu-1}\frac{d\tau}{\Gamma(\tau)}$$

thereby giving an explicit representation of the analytic extension of f onto $\mathbb{C}^*$. Since P_l and P_{l-1} are real polynomials we may apply theorem 1 in [3], that is we need an upper estimate of the number of real zeros of

$$H(\xi):=V(e^{-e^{-\xi t}})=P_{r-s-1}(\xi)-\int_0^\infty e^{-\tau\xi}\frac{E(\tau)}{\Gamma(\tau)}\,d\tau,$$

where

$$E(\tau):=\sum_{\nu=1}^{p}e^{\alpha_\nu\tau}\sum_{\mu=1}^{k_\nu}\frac{a_{\nu,\mu}}{(\mu-1)!}\,\tau^{\mu-1}.$$

(i) Suppose that $r=s$. Then, since E has at most $s-1$ real zeros [10, vol. II, p. 48, prob. 75], H has at most $s-1$ real zeros [10, vol. II, p. 50, prob. 80]. Thus, by theorem 1 in [3] and (30), $N\le l+1+s-1=l+r$.

(ii) Suppose that $r<s$. Observing that E has at most r positive zeros in this case (cf. the proof of theorem 3 in [3]), by (31) as above, we obtain $N\le l+r$.

(iii) Finally we assume that $r>s$. By Rolle's theorem, the number of real zeros of H does not exceed that of

$$H^{(r-s)}(\xi)=(-1)^{r-s+1}\int_0^\infty e^{-\tau\xi}\frac{\tau^{r-s}}{\Gamma(\tau)}E(\tau)\,d\tau$$

by more than $r-s$. Thus, as above we obtain $N\le l+1+s-1+r-s=l+r$, by (30). This completes the proof.

In case, when R has non-real poles, the asymptotic methods developed in [3, 5, in particular theorem 1 in 5] show how to find sufficient conditions for f to have an infinity of zeros on the negative real axis. Since the results and arguments are very similar to those in [3, 5] we omit them. We only show that $z=\infty$ is the only possible limit point of zeros.

First we observe that representations (30) and (31) remains true possibly with some $l'>l$, if R has non-real poles. Since V is holomorphic on $(0, 1)$, $z=1$, ∞ are the only limit points of zeros of f. Using Plana's sum formula [11, p. 440] we obtain

$$f(z)=P_{l'-1}(z)+\tfrac{1}{2}R(\log(l'+1))z^{l'}+\int_{l'}^\infty R(\log(t+1))e^{-t\log(1/z)}\,dt$$

$$+iz^{l'}\int_0^\infty\frac{R(\log(1+l'+iy))z^{iy}-R(\log(1+l'-iy))z^{-iy}}{e^{2\pi y}-1}\,dy$$

where the latter integral is bounded for $z\to 1$, $z\in\mathbb{C}^*$. Further we get from

Lemma 2, (ii), that

$$\int_{l'}^{\infty} R(\log(t+1))e^{-t\log(1/z)}\,dt \sim \frac{K(-1)^{r-s}}{\log 1/z}\left(\log\log\frac{1}{z}\right)^{r-s}$$

as $z \to 1$, $z \in \mathbb{C}^*$, for some real $K \neq 0$ and thus

$$f(z) \sim \frac{K(-1)^{r-s}}{\log 1/z}(\log\log 1/z)^{r-s} \qquad z \to 1, \qquad z \in \mathbb{C}^*.$$

This shows that $z=1$ is never a limit point of zeros of f.

4. In this section we investigate the zeros of the power series (18). The main results are concerned with the case of real α_ν's only.

THEOREM 3. *Suppose that* $Q_s \in \mathbb{R}[z]$, $s \geq 1$, *is given by* (2) *with* $\alpha_1 < \cdots < \alpha_p$. *Further assume that* $A \in C_s[0, \infty)$ *is a real solution of the differential equation*

$$(7) \qquad Q_s\left(\frac{d}{dx}\right)A(x) = 0, \qquad x \in \mathbb{R}.$$

Then (*unless* $f \equiv 0$)

$$(18) \qquad f(z) = \sum_0^\infty A(\log(n+1))z^n$$

defines (*uniquely*) *on* $\mathbb{C}^*$ *a holomorphic function possessing at most* $k+s-n_p$ *or* $k+s-1-n_p$ *zeros, if* $\alpha_1 \leq 0$ *or* $\alpha_1 > 0$ *respectively* (*k and* n_p *are defined by* (15) *and* (16) *respectively*).

Proof. The general solution of (7) is given by

$$(32) \qquad A(x) = \sum_1^p e^{\alpha_\nu x} P_{k_\nu - 1}(x).$$

From formulae (1.6), (1.7), (1.8), and the remarks immediately following (1.7) in [12] we get, $\kappa \in \mathbb{R}$, $m \in \mathbb{N}_0$, $z \in \mathbb{C}^*$

$$(33) \qquad f_{\kappa,m}(z) = \int_0^1 \frac{P_m(\log_2 1/t)}{1-zt}\frac{dt}{(\log 1/t)^{\kappa+1}}, \quad \text{if} \quad \kappa < 0,$$

$$(34) \quad f_{\kappa,m}(z) = \frac{1}{(1-z)^{[\kappa]+1}}\left\{P_{[\kappa]}(z) + z^{[\kappa]+1}\int_0^1 \frac{P_m(\log_2 1/t)}{1-zt}\frac{(1-t)^{[\kappa]+1}}{(\log 1/t)^{\kappa+1}}\,dt\right\},$$

$$\text{if} \quad \kappa \geq 0,$$

and in particular

$$(35) \quad f_{\kappa,m}(z) = \frac{1}{(1-z)^{[\kappa]+1}}\left\{P_{[\kappa]-1}(z) + z^{[\kappa]}\int_0^1 \frac{P_m(\log_2 1/t)}{1-zt}\frac{(1-t)^{[\kappa]+1}}{t(\log 1/t)^{\kappa+1}}\,dt\right\},$$

$$\text{if} \quad \kappa > 0,$$

where P_m in (34) and (35) reduce to some polynomials P_{m-1}, if $\kappa \in \mathbb{N}_0$. Suppose that $\alpha_1 > 0$. Then, by (32), f is a sum of terms of type (35). Put $\kappa = \alpha_\nu$, $m = \kappa_\nu - 1$. Applying the technique of Lemma 1 to (35) we get

$$(1-z)^{k+1} f_{\kappa,m}(z) = P_{k-1}(z) + z^k \int_0^1 \frac{P_m(\log_2 1/t)}{1-zt} \frac{(1-t)^{k+1}}{t(\log 1/t)^{\kappa+1}}\, dt$$

and hence

$$(36) \qquad (1-z)^{k+1} f(z) = P_{k-1}(z) + z^k \int_0^1 \frac{V(t)}{1-zt} \frac{(1-t)^{k+1}}{t \log 1/t}\, dt, \qquad \alpha_1 > 0$$

where

$$(37) \qquad V(t) = \sum_{\nu=1}^{p} (\log 1/t)^{-\alpha_\nu} P_{k_\nu - 1}(\log_2 1/t)$$

and $P_{k_\nu-1}$ becomes $P_{k_\nu-2}$ if the corresponding α_ν is a positive integer according the remark following (35).

If $\alpha_1 \leqq 0$, then some of the terms have forms (33) or (34). Similarly as above we obtain the slightly weaker result

$$(38) \qquad (1-z)^{k+1} f(z) = P_k(z) + z^{k+1} \int_0^1 \frac{V(t)}{1-zt} \frac{(1-t)^{k+1}}{\log 1/t}\, dt, \qquad \alpha_1 \leqq 0$$

with the same $V(t)$ as in (37).

By the remarks following (35) and (37) V has at most $s-1-n_p$ zeros on $(0, 1)$ [10, vol. II, p. 48, prob. 75]. Now, using (36) and (38), theorem 1 in [3] completes the proof.

Lower bounds for the number of negative zeros are given by

THEOREM 4. *Suppose that the assumptions of Theorem* 3 *are satisfied. Moreover, assume that*

$$(9) \qquad A(0) = A'(0) = \cdots = A^{(q)}(0) = 0$$

for some $q \in \{0, \ldots, s-1\}$. *Then*

$$(18) \qquad f(z) = \sum_0^\infty A(\log(n+1)) z^n$$

has at least $j+q+1$ *different zeros which are* ≤ 0 (j *is defined by* (17)).

Proof. If $\alpha_1 \leq 1$, that is $j=0$, then the statement follows readily from [6]. If $\alpha_1 > 1$, then we have, by (17), that

$$(39) \qquad 0 < \alpha_1 - j \leq 1, \qquad j \geq 1.$$

Now we get from [6] again that

$$h(z) := \sum_0^\infty (n+1)^{-j} A(\log(n+1)) z^n$$

has at least $q+1$ different zeros being ≤ 0. Obviously we have

$$f(z)=\left(\frac{d}{dz}z\right)^j h(z)=\frac{d}{dz}\left(z\frac{d}{dz}\right)^{j-1} zh(z).$$

Now (39) and (20) show that

$$\left(z\frac{d}{dz}\right)^{\nu} zh(z)\to 0 \quad \text{as} \quad z\to-\infty, \qquad \nu=0,\ldots,j-1.$$

Hence, by a Rolle type argument [see e.g. 10, vol. II, p. 39, prob. 16] to the interval $(-\infty, \varepsilon)$ we obtain inductively that f has at least $j+q+1$ different zeros which are ≤ 0. This completes the proof.

As an application we consider the function (see (14) and [12])

$$(14) \qquad f(z)=f_{\kappa,m}(z)=\sum_0^{\infty}(n+1)^{\kappa}\log^m(n+1)z^n, \qquad \kappa\in\mathbb{R}, \qquad m\in\mathbb{N}_0.$$

We have $p=1$, $s=m+1$, $\alpha_1=\kappa$, $k=\max(-1,[\kappa])$; $j=0$, if $\kappa\leq 0$, and $j<\kappa\leq j+1$, if $\kappa>0$; $q=m-1$. Now it is easily verified, by Theorems 3 and 4 that $m+j$ is the exact number of zeros of $f_{\kappa,m}$ in $\mathbb{C}^*$; all of them are ≤ 0 and simple.

If we drop the assumption on the reality of α_ν, then it was pointed out by Example (19) that we encounter functions having infinitely many zeros in $\mathbb{C}^*$. In general, that is α_ν may be non real, Lemma 4 yields asymptotic expansions for f being of the type

$$f(z)\sim\frac{1}{z}\sum_{\nu=1}^{p}(\log(-z))^{-\alpha_\nu}P_{r_\nu}(\log\log(-z))+\frac{K}{z^2} \qquad z\to\infty, \qquad z\in\mathbb{C}^*,$$

and

$$f(z)\sim\frac{1}{z}\sum_{\nu=1}^{p}\left(\log\frac{1}{z}\right)^{-\alpha_\nu-1}P_{r_\nu}\left(\log\log\frac{1}{z}\right)+K \qquad z\to 1, \qquad z\in\mathbb{C}^*,$$

$(K\in\mathbb{R})$. From these asymptotic expansions sufficient conditions (depending on the location of the α_ν's mainly) for the existence of an infinity of zeros accumulating at $z=\infty$ and $z=1$ can be derived as in [3, 5]. Moreover, $z=\infty$ and $z=1$ are the only possible limit points of zeros; for representations (36) and (38) remain true possibly with some $k'>k$ and it follows again from the analyticity of V on $(0, 1)$ that the zeros cannot accumulate at some $x_0\in(1,\infty)$.

For

$$(19) \qquad f(z)=\sum_0^{\infty}\sin\log(n+1)z^n$$

we get from Lemma 4 (21) that $(s=2, Q_s(z)=z^2+1, \alpha_1=i, \alpha_2=-i)$ as

$z = x \to 1-0$

$$f(x) \sim \frac{1}{2ix \log 1/x}\left(\frac{\Gamma(1+i)}{(\log 1/x)^i} - \frac{\Gamma(1-i)}{(\log 1/x)^{-i}}\right) = \frac{\operatorname{Im}\Gamma(1+i)/(\log 1/x)^i}{x \log 1/x}$$

$$= \frac{a}{x \log 1/x} \sin\left(\log\log\frac{1}{x} + b\right), \qquad a, b \in \mathbb{R}, \qquad a \neq 0.$$

This shows that f has infinitely many zeros on (0, 1).

REFERENCES

1. W. Ford, *Studies on divergent series and summability and the asymptotic developments of functions defined by MacLaurin series*, Chelsea Publ. Comp., New York, (1960).
2. W. Gawronski, *Asymptotic distribution of the zeros of power series*, Serd. Bulg. math. publ. vol. 5, 13–23 (1979).
3. W. Gawronski and A. Peyerimhoff, *On the zeros of power series with rational coefficients*, Arch. Math., **29,** 173–186 (1977).
4. W. Gawronski, *On the zeros of power series with rational coefficients II*, Arch. Math., **31,** 346–355 (1978).
5. W. Gawronski, *On the zeros of power series with rational coefficients III*, Arch. Math., **32,** 368–376 (1979).
6. W. Gawronski, *A lower bound for the number of negative zeros of power series*, Canad. Math. Bull., vol. 22 (1), 47–52 (1979).
7. W. B. Jurkat and A. Peyerimhoff, *On power series with negative zeros*, Tôhoku Math. J., vol. 24, no. 2, 207–221 (1972).
8. F. Olver, *Introduction to asymptotics and special functions*, Academic Press, Inc., New York, London, (1974).
9. A. Peyerimhoff, *On the zeros of power series*, Mich. Math. J., vol. 13, 193–214 (1966).
10. G. Pólya and G. Szegö, *Aufgaben und Lehrsätze aus der Analysis I, II*, Springer-Verlag, Berlin, Heidelberg, New York, (1970).
11. G. Sansone and J. Gerretsen, *Lectures on the theory of functions of a complex variable*, vol. I, P. Noordhoff, Groningen, (1960).
12. U. Stadtmüller, *On the zeros of power series with logarithmic coefficients*, Canad. Math. Bull., vol. 22 (2), 221–233 (1979).

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