A BALL QUOTIENT PARAMETRIZING TRIGONAL GENUS 4 CURVES

EDUARD LOOIJENGA

Abstract. We consider the moduli space of genus 4 curves endowed with a g_3^1 (which maps with degree 2 onto the moduli space of genus 4 curves). We prove that it defines a degree $\frac{1}{2}(3^{10}-1)$ cover of the nine-dimensional Deligne–Mostow ball quotient such that the natural divisors that live on that moduli space become totally geodesic (their normalizations are eight-dimensional ball quotients). This isomorphism differs from the one considered by S. Kondō, and its construction is perhaps more elementary, as it does not involve K3 surfaces and their Torelli theorem: the Deligne–Mostow ball quotient parametrizes certain cyclic covers of degree 6 of a projective line and we show how a level structure on such a cover produces a degree 3 cover of that line with the same discriminant, yielding a genus 4 curve endowed with a g_3^1.

§1. Introduction

Kondō constructed in [5] an isomorphism from a Zariski open subset of the moduli space of nonsingular complex genus 4 curves $\mathcal{M}_4$ onto a Zariski open subset of a ball quotient, where the latter covers the nine-dimensional ball quotient that appears in the Deligne–Mostow list. He also noted that this cannot be extended to the full $\mathcal{M}_4$, but that this is possible if we blow up the hyperelliptic locus $\mathcal{H}_4$ (which is of codimension 2 in $\mathcal{M}_4$). The exceptional divisor then maps to a hyperball quotient, a property that also holds for the generic point of the Theta-null locus. His approach is based on the fact that the generic point of $\mathcal{M}_4$ is represented by a curve X of bidegree $(3,3)$ on $\mathbb{P}^1 \times \mathbb{P}^1$. The $\boldsymbol{\mu}_3$-cover of $\mathbb{P}^1 \times \mathbb{P}^1$ that totally ramifies along X is a K3 surface with $\boldsymbol{\mu}_3$-action, and the Torelli theorem for such surfaces implies that its $\boldsymbol{\mu}_3$-Hodge structure is a complete invariant of that $\boldsymbol{\mu}_3$-surface and hence also of X (as the fixed point set of the $\boldsymbol{\mu}_3$-action). Such $\boldsymbol{\mu}_3$-Hodge structures are parametrized by a ball quotient of the above type.

The goal of this note is twofold. First, to observe that there is a modular interpretation of the hyperelliptic blowup as the moduli space of pairs consisting of a nonsingular complex genus 4 curve and a g_3^1 on that curve, where we note that a generic genus 4 curve has *two* g_3^1's and that this moduli space comes with an involution that exchanges them.

And second, to give this moduli space a complex hyperbolic structure without resorting to K3 surfaces and their Torelli theorem. For this, we start from the fact that the generic point of the Deligne–Mostow ball quotient parametrizes $\boldsymbol{\mu}_6$-covers $C \to \mathbb{P}^1$ totally ramified over a 12-element subset $D \subset \mathbb{P}^1$ and nowhere else. We then show how a certain level structure on C gives rise to a genus 4 curve X lying with degree 3 over $\mathbb{P}^1$ with discriminant D (so X comes with a g_3^1) and a correspondence between C and X.

Our approach is in the same spirit as our paper with Heckman [4] on the moduli space of rational elliptic surfaces (which here appears in the guise of the Theta-null locus). Most of that paper was concerned with the construction and comparison of certain compactifications of that moduli space (which accounts for its length), and it is likely that a similar program could be pursued here, where, of course, a compactification that parametrizes curves of arithmetic genus 4 endowed with a g_3^1 would be best.

Let us finally point out that our period map is quite different from the one introduced by Kondō, as the two discriminants that we associate with the two g_3^1's on a generic genus 4 curve will not lie in the same $\mathrm{PGL}_2(\mathbb{C})$-orbit and hence have different image in the Deligne–Mostow ball quotient.

After we posted the first version, Hang Xue drew my attention to his thesis [8] and a related publication [7] in which the difference of the two g_3^1's of a nonhyperelliptic genus 4 curve (regarded as an element of its Jacobian) is investigated in a universal manner.

I thank Gert Heckman for comments on that same version.

§2. The nine-dimensional Deligne–Mostow ball quotient

This section is mostly a review of known results. Its main purpose is to recall them in a way that suits our purpose and to establish notation.

2.1 $\boldsymbol{\mu_6}$-covers of $\mathbb{P}^1$

Let C be a nonsingular complex-projective curve endowed with a $\boldsymbol{\mu}_6$-action that has 12 fixed points, is free elsewhere, and is such that its orbit space P is a copy of $\mathbb{P}^1$. The discriminant of $C \to P$ is then of the form $5D_C$, with D_C a 12-element subset of P, considered as a reduced divisor and Riemann–Hurwitz shows that C will have genus 25. Note that the pair (P, D_C) determines C up to a covering transformation (in $\boldsymbol{\mu}_6$). We recall how this gives rise to a point in a nine-dimensional ball quotient, which happens to be the one of highest dimension that appears in the Deligne–Mostow list.

Since we find it convenient to make a distinction between what depends on the complex structure and what only depends on the underlying topology, we also fix a closed oriented surface Σ for of genus 25 endowed with a $\boldsymbol{\mu}_6$-action of the type above, so with 12 fixed points and acting freely elsewhere. We denote by $\pi : \Sigma \to S$ the formation of its $\boldsymbol{\mu}_6$-orbit space (a smooth 2-sphere), and by $D \subset S$ its set of critical values. Then any complex structure on S (which will make it a copy of the Riemann sphere) determines a unique one on Σ for which the $\boldsymbol{\mu}_6$-action and π are holomorphic, yielding a $\boldsymbol{\mu}_6$-curve as above with discriminant $5D$. We also note that a diffeomorphism of S which preserves D lifts to S and does so almost uniquely: two such lifts will lie in the same $\boldsymbol{\mu}_6$-orbit.

2.2 Hodge structure of a $\boldsymbol{\mu_6}$-cover of $\mathbb{P}^1$

We first note that the standard embedding $\chi : \boldsymbol{\mu}_6 \subset \mathbb{C}^\times$ and its complex conjugate are the only two primitive characters of $\boldsymbol{\mu}_6$. The group $\boldsymbol{\mu}_6$ acts on $H^1(\Sigma; \mathbb{C})$ and decomposes the latter into its character spaces: $H^1(\Sigma; \mathbb{C}) = \oplus_{i \in \mathbb{Z}/6} H^1(\Sigma; \mathbb{C})_{\chi^i}$. The intersection pairing gives rise to a nondegenerate Hermitian form on $H^1(\Sigma; \mathbb{C})$ defined by $(\alpha, \beta) \mapsto \sqrt{-1} \int_\Sigma \alpha \wedge \overline{\beta}$. It has signature $(25, 25)$, and for this form, the above decomposition into character spaces is perpendicular. In the presence of a complex structure, it is also compatible with the Hodge structure. For example, the decomposition

$$H^1(C; \mathbb{C})_\chi = H^1(C, \Omega_C)_\chi \oplus \overline{H^1(C, \Omega_C)_{\overline{\chi}}} \tag{2.1}$$

is perpendicular with respect to this Hermitian form with the first summand being positive definite of dimension 1 and the second summand negative definite of dimension 9. The group of covering transformations $\boldsymbol{\mu}_6$ acts trivially on $\mathbb{P}(H^1(\Sigma;\mathbb{C})_\chi)$ and so this projective space only depends on the pair (S,D); we therefore denote it by $\mathbb{P}(S,D)$. Its group $\mathrm{PU}(H^1(\Sigma;\mathbb{C})_\chi)$ of projective unitary transformations, which we denote for a similar reason by $\mathrm{PU}(S,D)$, is isomorphic to $\mathrm{PU}(1,9)$. The positive definite complex lines in $H^1(\Sigma;\mathbb{C})_\chi$ make up an open complex ball $\mathbb{B}(S,D) \subset \mathbb{P}(S,D)$ on which $\mathrm{PU}(S,D)$ acts transitively with compact stabilizers; indeed, this ball is the symmetric domain associated with $\mathrm{PU}(S,D)$. If there is no risk of confusion, we simply write $\mathbb{B}$ for $\mathbb{B}(S,D)$.

As we observed above, a complex structure on S determines one on Σ, turning it into a compact Riemann surface C with $\boldsymbol{\mu}_6$-action. The Hodge decomposition (2.1) then defines an element, the *Hodge point*, of $\mathbb{B}$. The main result of Deligne and Mostow [3], [6] implies that this Hodge point is a complete invariant of the relative isotopy class of this complex structure.

2.3 Action of a mapping class group

To make this last assertion precise, let $\mathrm{Diff}^+(S,D)$ denote the group of orientation-preserving diffeomorphisms of S which preserve D. Since this group $\mathrm{Diff}^+(S,D)$ acts on Σ up to covering transformations, it gives rise to a well-defined action on $\mathbb{P}(S,D)$ via $\mathrm{PU}(S,D)$ (and so preserves $\mathbb{B}$). This action factors through its connected component group that we shall denote $\mathrm{Mod}(S,D)$ and which we readily recognize as the spherical braid group on 12 strands. That group has as a distinguished conjugacy class the set of simple braids. We recall that a *simple braid* is given by an (unoriented) arc δ in S which connects two points of D and has no points of D in its interior: the associated simple braid $T_\delta \in \mathrm{Mod}(S,D)$ has its support in a regular neighborhood of δ and lets δ make a half-turn in a counterclockwise direction. These simple braids generate $\mathrm{Mod}(S,D)$. In fact, if we enumerate the points of D by $\{p_i\}_{i \in \mathbb{Z}/12}$ and let δ_i connect p_i with p_{i+1} in such a way as that their juxtaposition produces an embedded circle in S, then $T_{\delta_1},\ldots,T_{\delta_{10}}$ already generate $\mathrm{Mod}(S,D)$. It follows from the work of Deligne–Mostow that the image of $\mathrm{Mod}(S,D)$ in $\mathrm{PU}(S,D)$ is an arithmetic group, which therefore acts properly discretely on $\mathbb{B}$ with finite covolume. We denote this image group by $\Gamma \subset \mathrm{PU}(S,D)$.

2.4 A lattice over the Eisenstein ring

In order to make the action of a simple braid on $\mathbb{B}$ somewhat more explicit, we make some general observations first. Write τ for the primitive sixth root of unity with a positive imaginary part, $e^{\pi\sqrt{-1}/3}$, and regard it as a generator of $\boldsymbol{\mu}_6$. As a complex number, it satisfies $\tau^2 = \tau - 1$, but this identity is of course not valid in the group ring $\mathbb{Z}[\boldsymbol{\mu}_6]$. Given a $\mathbb{Z}[\boldsymbol{\mu}_6]$-module V, let us denote by V° (resp. $V_\circ$) the maximal submodule (resp. quotient module) on which τ satisfies the identity $\tau^2 = \tau - 1$. For example, $V_\circ$ is the quotient of V by the subgroup of $v \in V$ with nontrivial $\boldsymbol{\mu}_6$-stabilizer. Then V° and $V_\circ$ become modules over the ring of *Eisenstein integers* $\mathcal{E} := \mathbb{Z} + \mathbb{Z}\tau \subset \mathbb{C}$, the ring of integers the cyclotomic field $\mathbb{Q}(\sqrt[3]{1})$. Note that $\mathbb{R} \otimes V^\circ \to \mathbb{R} \otimes V_\circ$ is an isomorphism and the latter can be identified with the quotient of $\mathbb{R} \otimes V$ by the sum of the fixed point subspaces of τ^2 and τ^3.

Since $\mathcal{E}$ is a principal ideal domain, any finitely generated torsion-free $\mathcal{E}$-module is in fact free. So, if V is a finitely generated free $\mathbb{Z}$-module, then both $V_\circ$ and V° are finitely

generated free $\mathcal{E}$-modules. If we are in addition given a $\boldsymbol{\mu}_6$-invariant antisymmetric pairing $a : V \times V \to \mathbb{Z}$, then this gives rise to the $\frac{1}{2}\mathcal{E}$-valued Hermitian form on V° defined by

$$h_a(x,y) := \tfrac{1}{2}\tau a(x,y) - \tfrac{1}{2}a(x,\tau y)$$

(the justification for the perhaps somewhat unexpected normalization factor $\frac{1}{2}$ will be given below). Note that $h_a(x,x) = -\frac{1}{2}a(x,\tau x)$ and $h_a(x,y) - \overline{h_a(x,y)} = \frac{1}{2}(\tau - \overline{\tau})a(x,y) = \frac{1}{2}\theta a(x,y)$, where

$$\theta := \tau - \overline{\tau} = -1 + 2\tau = \sqrt{-3}$$

is the *first* purely imaginary number in $\mathcal{E}$. We apply this to the $\mathbb{Z}[\boldsymbol{\mu}_6]$-module $H_1(\Sigma)$. We find that $H_1(\Sigma)^\circ$ and $H_1(\Sigma)_\circ$ are free $\mathcal{E}$-modules of rank 10. The intersection pairing on $H_1(\Sigma)$ is nondegenerate on $H_1(\Sigma)^\circ$ and defines via the above prescription on this lattice the $\mathcal{E}$-valued Hermitian form

$$h(x,y) := \tfrac{1}{2}\tau(x \cdot y) - \tfrac{1}{2}(x \cdot \tau y).$$

We will write L for $H_1(\Sigma)^\circ$ endowed with this form.

Let us see what this induces on the preimage $\pi^{-1}\delta$ of an arc δ as above. We now assume δ oriented and let $\tilde{\delta}$ be a lift of δ over π. Then $c_\delta := (1 - \tau^3)\tilde{\delta}$ is an embedded oriented circle (hence a 1-cycle) and meets its τ-translate in two points, each with multiplicity 1. In other words, $c_\delta \cdot \tau c_\delta = 2$. Now, $H_1(\pi^{-1}\delta)^\circ$ has the $\mathcal{E}$-generator $a_\delta := (1+\tau)(1-\tau^3)\tilde{\delta} = (1+\tau)c_\delta$ (which is unique up multiplication by an element of $\boldsymbol{\mu}_6$), and the above formula shows that $h(a_\delta, a_\delta) = \frac{1}{2}. - 6 = -3$. A similar computation shows that $h(a_{\delta_i}, a_{\delta_{i+1}}) \in \boldsymbol{\mu}_6\theta$. So $h|L \times L$ takes values in $\theta\mathcal{E}$, so that $\theta^{-1}h$ defines a skew-Hermitian form

$$h' := \theta^{-1}h : L \times L \to \mathcal{E}.$$

Since we have the freedom of multiplying each a_δ with an element of $\boldsymbol{\mu}_6$, we can choose $a_{\delta_1}, \ldots, a_{\delta_{10}}$ in such a manner that it is an $\mathcal{E}$-basis of L with

$$h(a_{\delta_i}, a_{\delta_j}) = \begin{cases} -3, & \text{when } j = i, \\ \pm\theta, & \text{when } j = i \pm 1, \\ 0, & \text{otherwise.} \end{cases}$$

This also shows that L is isomorphic to the lattice thus denoted in [2] (this explains the coefficient $\frac{1}{2}$ in our definition of h) and if we replace h by $-h$, it becomes isomorphic to the lattice denoted Λ in [4]. In [2], L is in fact constructed from the even $\mathbb{Z}$-lattice $\mathbf{E}_8 \perp \mathbf{E}_8 \perp \mathbf{U} \perp \mathbf{U}$. Going in the opposite direction, this amounts to the statement that we recover this lattice if we take the underlying abelian group and the even symmetric pairing on it defined by the real part of h multiplied with $-\frac{2}{3}$.

2.5 Mirror hyperballs
The simple braid T_δ acts in L as the unitary transformation s_δ defined by

$$s_\delta(x) = x + \tfrac{1}{3}(1 - \tau^2)h(x, a_\delta)a_\delta = x + \tau h'(x, a_\delta)a_\delta.$$

It is what is called in [2] a *triflection* in $\mathcal{E}a_\delta$: it multiplies a_δ with the third root of unity τ^2 and is the identity on the orthogonal complement of a_δ. It indeed preserves L, as follows from the fact that h' takes its values in $\mathcal{E}$.

Poincaré duality identifies $H^1(\Sigma)^\circ$ with $H_1(\Sigma)^\circ = L$, and we use this to identify $\mathbb{P}(S, D)$ with $\mathbb{P}(\mathbb{C} \otimes_{\mathcal{E}} L)$. In particular, T_δ acts on $\mathbb{B}$ as a triflection. So its fixed point set $\mathbb{B}^{T_\delta}$ is a hyperball (to which we shall also refer as a *mirror* in $\mathbb{B}$). Since the T_δ make up a single conjugacy class in $\mathrm{Mod}(S, D)$, the group $\mathrm{Mod}(S, D)$ permutes the mirror hyperballs transitively.

The group of unitary transformations of the Hermitian $\mathcal{E}$-module L acts properly discretely on $\mathbb{B}$ with kernel its center $\boldsymbol{\mu}_6$. The group Γ is a priori contained in the quotient of this group by its center (which is an arithmetic subgroup of $\mathrm{PU}(S, D)$), but, as Allcock has shown (see [2, Th. 5.1]), is in fact equal to it. So Γ acts properly discontinuously on $\mathbb{B}$ and the Baily–Borel theory asserts that $\mathbb{B}_\Gamma := \Gamma \backslash \mathbb{B}$ has the structure of a quasi-projective orbifold that can be completed to a normal projective variety by adding a finite set of *cusps* (in this case, just one, see below). The image of the Hodge point in this ball quotient $\mathbb{B}_\Gamma$ is a complete invariant of the isomorphism type of the $\boldsymbol{\mu}_6$-curve C, or equivalently, of the pair (S, D) with the given complex structure.

The $\mathrm{Mod}(S, D)$-centralizer of T_δ acts on the mirror $\mathbb{B}^{T_\delta}$ with image an arithmetic subgroup and the resulting ball quotient of $\mathbb{B}^{T_\delta}$ appears in the Deligne–Mostow list as the one defined by the sequence $(\frac{2}{12}, \frac{1}{12}, \frac{1}{12}, \ldots, \frac{1}{12})$.

The group Γ is as a quotient of $\mathrm{Mod}(S, D)$ obtained by imposing the relation $T_\delta^3 = 1$. The mirrors are locally finite on $\mathbb{B}$, and since they are transitively permuted by Γ, they define an irreducible totally geodesic hypersurface $\mathcal{D}$ in $\mathbb{B}_\Gamma$ that we shall call the *confluence locus*. We write $\mathbb{B}^\circ$ for the complement of the union of the mirrors (an open subset of $\mathbb{B}$). So this is the preimage of $\mathbb{B}_\Gamma^\circ \setminus \mathcal{D}$.

2.6 The Deligne–Mostow isomorphism

Let $\mathcal{M}(12)$ denote the $\mathrm{PGL}(2, \mathbb{C})$-orbit space of the configuration space of 12-element subsets of $\mathbb{P}^1$ (this is also equal to the $\mathfrak{S}_{12}$-orbit space of $\mathcal{M}_{0,12}$). The above construction defines a homomorphic map

$$\mathcal{M}(12) \to \mathbb{B}_\Gamma.$$

If we think of a 12-element subset of $\mathbb{P}^1$ as a reduced degree 12-divisor, then the above map extends to the locus $\mathcal{M}(12)^{\mathrm{st}}$ of Hilbert–Mumford stable positive degree 12-divisors, that is, divisors for which the multiplicities are at most 5. The Deligne–Mostow theory asserts that this extension is in fact an isomorphism

$$\mathcal{M}(12)^{\mathrm{st}} \xrightarrow{\cong} \mathbb{B}_\Gamma$$

that takes $\mathcal{M}(12)$ onto $\mathbb{B}_\Gamma^\circ$. It even extends to an isomorphism of their natural compactifications: the GIT compactification on the left and the Baily–Borel compactification on the right. Both are one-point compactifications, with the added point on the left being represented by a 6-divisible divisor on $\mathbb{P}^1$ whose support consists of two points. In particular, there is just one cusp.

§3. Trigonal structures on a genus 4 curve

3.1 g_3^1's on a genus 4 curve

Let X be a smooth connected complex-projective curve of genus 4, and let P be a pencil of degree 3 on X (in other words, a g_3^1). If P has no base point, then the pencil defines a morphism $X \to P$ of degree 3 and by Riemann–Hurwitz, the discriminant divisor $D(X, P)$

of this morphism is of degree 12. Note that its multiplicities will be ≤ 2. In case P has a base point, then the residual base-point-free pencil P' will have degree 2 and this makes X hyperelliptic with P' being the hyperelliptic pencil. We will see that it is then natural to define the discriminant divisor of P to be the degree 12-divisor $D(X,P)$, that is (via the identification of P with P'), the sum of the discriminant divisor of $X \to P'$ (the image of the Weierstraß points, so of degree 10) plus twice the image in P' of the fixed point P. So its multiplicities are then ≤ 3.

Recall that if X is nonhyperelliptic, then a canonical image of X in $\mathbb{P}^3$ is the transversal intersection of a cubic surface with a quadric Q surface, the latter being unique and either smooth (so isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$) or a quadric cone. In the first case, the projections on the two factors give us two pencils of degree 3. In the second case, projection away from the vertex of the cone also yields a pencil of degree 3; the line bundle in question is then an effective (even) theta characteristic. It is well known that this exhausts all the g_3^1's in genus 4. For future reference, we state this as a lemma (and only sketch the proof).

LEMMA 3.1. *Let X be a smooth connected complex-projective curve of genus 4. If X is nonhyperelliptic, then any g_3^1 on X is as above: it comes with a residual g_3^1 (in the sense that its members supplement the members of the given g_3^1 to a canonical divisor) and there are no others (so the two being equal if and only if the g_3^1 is a theta characteristic). If X is hyperelliptic, then any g_3^1 on X is the sum of an $x \in X$ and its hyperelliptic g_2^1; in that case, the pencil is a theta characteristic if and only if x is a Weierstraß point.*

Proof. This is well known, as it is in a sense a geometric formulation of Clifford's theorem in genus 4. The proof runs as follows. If P is a g_3^1 on X, then Clifford's theorem implies that P is a complete linear system. By Riemann–Roch, we have a supplementary g_3^1 on X, denoted P', whose members supplement the members of P to a canonical divisor. So each member of P (resp. P') defines a line in $\check{\mathbb{P}}(H^0(X, \Omega_X)) \cong \mathbb{P}^3$ which meets the canonical image of X in at most three points and P' (resp. P) is realized by the pencil of planes through that line. We have $P = P'$ if and only if P is a theta characteristic. The rest of the argument is left to the reader. □

3.2 A moduli stack of genus 4 curves with a g_3^1

We interpret this in terms of moduli spaces. Pairs (X,P) as above define a Deligne–Mumford stack that we shall denote by $\mathcal{M}_4(g_3^1)$. It comes with an involution which assigns to (X,P) the residual pair (X,P') that is characterized by the property that $P + P'$ lies in the canonical system. The forgetful morphism $\mathcal{M}_4(g_3^1) \to \mathcal{M}_4$ evidently factors through the orbit space of this involution. Recall that the moduli space $\mathcal{M}_4$ contains the Theta-null locus $\mathcal{M}_4^\Theta$ (the locus for which the curve admits an even effective theta characteristic) as a substack of codimension one and that this substack $\mathcal{M}_4^\Theta$ in turn contains the hyperelliptic locus $\mathcal{H}_4$ as a substack of codimension one.

It follows from Lemma 3.1 that $\mathcal{M}_4(g_3^1) \to \mathcal{M}_4$ is over $\mathcal{M}_4 \setminus \mathcal{H}_4$ a double covering with the covering transformation being induced by the residual involution (and so ramifies over $\mathcal{M}_4^\Theta \setminus \mathcal{H}_4$). This lemma also shows that the preimage $\mathcal{H}_4(g_3^1)$ of $\mathcal{H}_4$ in $\mathcal{M}_4(g_3^1)$ is the universal hyperelliptic curve of genus 4 (so that will be a divisor). The strict transform of $\mathcal{M}_4^\Theta$ in $\mathcal{M}_4(g_3^1)$ is another divisor $\mathcal{M}_4(g_3^1)^\Theta$ on $\mathcal{M}_4(g_3^1)$ which meets $\mathcal{H}_4(g_3^1)$ in the locus parametrizing the Weierstraß points (this locus is geometrically connected, since the monodromy is transitive on the Weierstrass points).

Our goal is to realize $\mathcal{M}_4(g_3^1)$ as an open subset of a quotient of $\mathbb{B}$ relative some finite index subgroup of Γ. We determine this subgroup in two ways: first in a purely topological manner and subsequently more in the spirit of algebraic geometry as a level structure on the $\boldsymbol{\mu}_6$-curve Σ.

3.3 The approach via a classification of coverings

This begins with addressing the question of how many topological types of smooth connected degree three coverings of the 2-sphere S exist that have the 12-element subset $D \subset S$ as reduced discriminant. Such a covering is given by its restriction to $S \setminus D$ (which is then unramified). So if we choose a base point $o \in S \setminus D$, then it can be given by a group homomorphism $\rho : \pi_1(S \setminus D, o) \to \mathfrak{S}_3$ which assigns to a simple loop around a point of D a transposition. The image of ρ must be a transitive subgroup of $\mathfrak{S}_3$ and hence must be all of $\mathfrak{S}_3$. Two such epimorphisms define isomorphic coverings if and only if they differ by an inner automorphism of $\mathfrak{S}_3$. So if we let $\mathrm{Epi}'(\pi_1(S \setminus D, o), \mathfrak{S}_3)$ stand for the set of group of surjective homomorphisms $\pi_1(S \setminus D, o) \to \mathfrak{S}_3$ that take simple loops to transpositions, then the set of topological types is naturally identified with

$$R(S, D) := \mathfrak{S}_3 \backslash \mathrm{Epi}'(\pi_1(S \setminus D, o), \mathfrak{S}_3).$$

We determine its number of elements. We choose smooth arcs $\{\gamma_i\}_{i \in \mathbb{Z}/12}$ from o to the distinct points of D that only meet at o and depart from there along rays in $T_o S$ in a counterclockwise order. This means that the associated simple loop associated with γ_i defines a $c_i \in \pi_1(S \setminus D, o)$ so that $c_{11} \cdots c_0 = 1$ (we read composition of loops from right to left) and $c_1, \ldots, c_{11}$ are free generators. Then a ρ as above is given by its values on $c_1, \ldots, c_{11}$. Note that $\rho(c_1), \ldots, \rho(c_{11})$ is a sequence in the 3-element set of transpositions $\{(12), (23), (31)\}$ of $\mathfrak{S}_3$ that cannot be nonconstant, but can otherwise be arbitrary. So there are $3^{11} - 3$ such ρ. The action of $\mathfrak{S}_3$ on this set by conjugation is free and so

$$\#R(S, D) = (3^{11} - 3)/6 = (3^{10} - 1)/2 = 3^9 + 3^8 + \cdots + 1.$$

This number is also equal to $\#\mathbb{P}^9(\mathbb{F}_3)$. We shall see that this is not a coincidence.

Evidently, the group of automorphisms of $\pi_1(S \setminus D, o)$ which preserves the simple loops acts on $R(S, D)$. The group $\mathrm{Mod}(S, D)$ maps to the outer automorphism group of $\pi_1(S \setminus D, o)$. Since we have divided out by $\mathfrak{S}_3$-conjugation, this gives us a genuine action of $\mathrm{Mod}(S, D)$ on $R(S, D)$.

LEMMA 3.2. *The action of $\mathrm{Mod}(S, D)$ on $R(S, D)$ is through Γ and is transitive.*

Proof. Recall that an (unoriented) arc δ in S connecting two distinct points of D whose relative interior does not meet any point of D represents a simple braid in $\mathrm{Mod}(S, D)$. If we choose an interior point of δ as our base point o, then this yields two simple loops and hence two transpositions of the fiber over o. Then T_δ acts trivially on ρ if these two transpositions are equal and, as a straightforward verification shows, is of order 3 otherwise. It follows that this action of $\mathrm{Mod}(S, D)$ on $R(S, D)$ factors through Γ.

For the transitivity property, it is enough to show that for a given degree 3 covering $\Sigma' \to S$ as above, there exists a standard generating set $(c_0, \ldots, c_{11})$ of $\pi_1(S \setminus D, o)$ as above and a numbering of the fiber over o such that $\rho_{\Sigma'}(c_i)$ equals (12) for i even and (23) for i odd. Since $\mathrm{Mod}(S, D)$ acts transitively on the collection of standard generating sets, this will indeed suffice.

Suppose that for $i \leq 10$, we have constructed arcs $\gamma_1, \ldots, \gamma_i$ that depart at o along tangent rays in a counterclockwise order, are otherwise disjoint and are such that $\rho(c_1), \ldots \rho(c_i)$ are as desired, and with the property that the preimage in Σ' over $\Delta_i := S \setminus \{\gamma_1 \cup \cdots \cup \gamma_i\}$ (a copy of an open disk) is connected. This means that the monodromy over $\Delta_i \setminus D$ is still the full $\mathfrak{S}_3$. The reader will then have no trouble finding an arc γ_{i+1} which departs along a ray in the sector spanned by the rays of departure of γ_i and γ_1 and which stays in $\{0\} \cup \Delta$ and whose associated monodromy is the prescribed transposition. If $i \leq 9$, extra care is needed to ensure that the preimage in Σ' over $\Delta_{i+1} := S \setminus \{\gamma_1 \cup \cdots \cup \gamma_{i+1}\}$ is still connected. That this is indeed possible follows from the fact that the preimage over Δ_i is connected and has at least three points of simple ramification.

On the other hand, if $i = 10$, we are done, for then the value of ρ on $c_0 := c_1^{-1} \cdots c_{11}^{-1}$ will be $\left((23)(12)\right)^5 (23) = (12)$. This completes the proof. □

Fix a degree 3 covering $\Sigma' \to S$ as above and denote by $r = r(\Sigma'/S) \in R(S, D)$ the associated element and denote by $\Gamma' \subset \Gamma$ its Γ-stabilizer. It follows from Lemma 3.2 that Γ' is a subgroup Γ of index $(3^{10} - 1)/2$. By construction, the isomorphism type of a connected smooth degree 3 covering of a copy of $\mathbb{P}^1$ with reduced discriminant of degree 12 determines a Γ'-orbit in $\mathbb{B}^\circ$. So, if $\mathcal{M}_4(g_3^1)^\circ$ stands for the open subset of $\mathcal{M}_4(g_3^1)$ that parametrizes pairs (X, P) for which P has precisely 12 nonreduced members (in other words, for which X is nonhyperelliptic and for which $X \to P$ has reduced discrimininant), then we find the following corollary.

COROLLARY 3.3. *The open subset $\mathcal{M}_4(g_3^1)^\circ$ is naturally isomorphic to the ball quotient $\mathbb{B}_{\Gamma'}^\circ$.*

In order to get all of $\mathcal{M}_4(g_3^1)$, we must allow the discriminant divisor D to become nonreduced by letting two of its points coalesce. If we let this happen in the interior of a closed disk $\Delta \subset P$ which contains no other points of D, then the covering over its boundary $\partial\Delta$ is connected (the monodromy is of order 3) or is trivial. In the first case, this confluence creates a point of total ramification and we still have defined a genus 4 covering of S. In the last case, the confluence creates an ordinary double point of the covering. This we will therefore not allow, except for one particular situation that we should not throw out: if the covering has in addition a section over $P \setminus \Delta$, then the degeneration will be the union of a smooth double covering $X' \to P$ (whose discriminant is $D \cap (P \setminus \Delta)$) and a component $\tilde{P}$ which maps isomorphically onto P and meets X' transversally over z. So this amounts to specifying a hyperelliptic curve of genus 4 and a point on that curve; in other words, an element of $\mathcal{H}_4(g_3^1)$. This happens, for example, when $\rho(c_0) = \rho(c_1) \neq \rho(c_3) = \cdots = \rho(c_{11})$, and we take for Δ a thin regular neighborhood $\gamma_0 \cup \gamma_1$.

We can state this in terms of the Γ-action on $R(S, D)$. The confluence of two points along an arc δ as above represents a triflection in Γ. It may or may not act trivially on $R(S, D)$. If the action is nontrivial, then this confluence creates a point of total ramification and we still have defined a genus 4 covering of S. Otherwise, this creates an ordinary double point. These three cases translate into saying that the stabilizer Γ' has three orbits in the collection of mirrors. Equivalently, the preimage of the confluence divisor $\mathcal{D}$ in $\mathbb{B}_\Gamma$ under the natural map

$$\mathbb{B}_{\Gamma'} \to \mathbb{B}_\Gamma$$

has three irreducible components $\mathcal{D}_{\mathrm{rm}}, \mathcal{D}_{\mathrm{sg}}, \mathcal{D}_{\mathcal{H}}$ whose generic points are characterized topologically by the degree 3 covering acquiring, respectively, (rm) a point of total ramification, (sg) an ordinary double point whose complement is connected, and $(\mathcal{H})$ an ordinary double point which disconnects. We can now improve upon Corollary 3.3 as follows.

THEOREM 3.4. *The above construction gives an identification*

$$\mathcal{M}_4(g_3^1) \cong \mathbb{B}_{\Gamma'} \setminus \mathcal{D}_{\mathrm{sg}},$$

thus endowing the left-hand side with an (incomplete) complex-hyperbolic metric. This identifies the preimage of $\mathcal{D}_{\mathcal{H}}$ with the universal hyperelliptic curve of genus 4.

For a different approach to this theorem (that ultimately yields a more precise statement), we first need to discuss level 3 structures for the Eisenstein lattice L.

3.4 The approach via a level structure

We begin with the observation that $\mathcal{E}/\theta\mathcal{E}$ is an $\mathbb{F}_3$-vector space of dimension one on which $\tau \in \boldsymbol{\mu}_6$ acts as minus the identity (because $\tau + 1 = \tau^{-1}\theta \in \theta\mathcal{E}$). It follows that $\mathbb{F}_3 \otimes_{\mathcal{E}} L = L/\theta L$ is an $\mathbb{F}_3$-vector space of rank 10 on which the skew-Hermitian form h' induces a symplectic form $h'_{\mathbb{F}_3} := \mathbb{F}_3 \otimes h'$. The following lemma generalizes an argument of Allcock (appearing in the proof of Theorem 5.2 of [2]).

LEMMA 3.5. *The symplectic form $h'_{\mathbb{F}_3}$ is nondegenerate on $\mathbb{F}_3 \otimes_{\mathcal{E}} L$ (so that it is isomorphic to $\mathbb{F}_3^{10}$ equipped with its standard symplectic form), and Γ acts on $\mathbb{F}_3 \otimes_{\mathcal{E}} L$ with image its full symplectic group (a copy of $\mathrm{Sp}_{10}(\mathbb{F}_3)$). The image of a triflection is a symplectic transvection.*

Proof. The basis $a_{\delta_1}, \ldots, a_{\delta_{10}}$ of L maps to a basis $\alpha_1, \ldots, \alpha_{10}$ of $\mathbb{F}_3 \otimes_{\mathcal{E}} L$ with the property that $h'_{\mathbb{F}_3}(\alpha_i, \alpha_j)$ is ± 1 if $j = i \pm 1$ and is zero otherwise. The triflection defined by a_{δ_i} becomes $x \mapsto x + \tau h'_{\mathbb{F}_3}(x, \alpha_i)\alpha_i = x - h'_{\mathbb{F}_3}(x, \alpha_i)\alpha_i$, which is indeed a symplectic transvection. Let V be the free abelian group with basis $\tilde{\alpha}_1, \ldots, \tilde{\alpha}_{10}$ endowed with the symplectic form that assigns to the pair $(\tilde{\alpha}_i, \tilde{\alpha}_j)$ the value ± 1 if $j = i \pm 1$ and is zero otherwise, so that its reduction mod 3 gives $h'_{\mathbb{F}_3}$. This lattice is well known in singularity theory (it is the one for the Milnor fiber of a plane curve singularity of type A_{10}, defined by $z_1^2 + z_2^{11} = 0$ as given on a standard basis of vanishing cycles). It was proved by A'Campo [1] that the subgroup of $\mathrm{Sp}(V)$ generated by these symplectic transvections contains the principal congruence level 2 subgroup. This implies that it maps onto the full symplectic group of its mod 3 reduction. □

The covering group $\boldsymbol{\mu}_6$ acts on the 10-dimensional $\mathbb{F}_3$-vector space $\mathbb{F}_3 \otimes_{\mathcal{E}} L$ by scalars and so the nine-dimensional projective space $\mathbb{P}(\mathbb{F}_3 \otimes_{\mathcal{E}} L)$ only depends on the pair (S, D).

We regard L as a primitive subgroup of $H^1(\Sigma)$, so that a one-dimensional subspace $\ell \subset \mathbb{F}_3 \otimes_{\mathcal{E}} L$ defines (by restriction) a surjection $H_1(\Sigma) \cong \mathrm{Hom}(H^1(\Sigma), \mathbb{Z}) \to \mathrm{Hom}(\ell, \mathbb{F}_3) = \ell^{\vee}$. This surjection is $\boldsymbol{\mu}_6$-equivariant if we let $\boldsymbol{\mu}_6$ act on $\ell^{\vee}$ via the obvious character $\boldsymbol{\mu}_6 \twoheadrightarrow \boldsymbol{\mu}_2 = \{\pm 1\}$. This also yields an unramified $\ell^{\vee}$-covering $\Sigma_\ell \to \Sigma$.

LEMMA 3.6. *The $\boldsymbol{\mu}_6$-action on Σ lifts to Σ_ℓ. If Σ_ℓ' denotes the orbit space of this lift, then Σ_ℓ' is an oriented surface of genus 4 such that the obvious map $\Sigma_\ell' \to S$ is orientation preserving of degree 3 with discriminant D (in other words, defines an element of $R(S, D)$). Moreover, this degree 3 cover is independent of the lift of the $\boldsymbol{\mu}_6$-action, in the sense that for any two lifts, the two degree 3 covers of S are isomorphic by a unique isomorphism.*

Proof. We first show that the $\boldsymbol{\mu}_6$-action on Σ lifts to Σ_ℓ. Recall that if we choose a base point $x \in \Sigma$, then Σ_ℓ can be obtained as a quotient of the space of paths in Σ that begin at x: two such paths α, β define the same point of Σ_ℓ if and only if α and β have the same endpoint and the class of the loop $\beta^{-1}\alpha$ in $\pi_1(\Sigma, x)$ has zero image in $\ell^\vee$. If we choose the base point x to be a fixed point of the $\boldsymbol{\mu}_6$-action, then $\boldsymbol{\mu}_6$ will act on the space of such of paths. The homomorphism $\pi_1(\Sigma, x) \to \ell^\vee$ is $\boldsymbol{\mu}_6$-equivariant, in particular, the $\boldsymbol{\mu}_6$-action on $\pi_1(\Sigma, x)$ preserves its kernel. This implies that we have lifted the $\boldsymbol{\mu}_6$-action on Σ to Σ_ℓ.

This lift is not canonical, but two such lifts differ by an $\ell^\vee$-covering transformation. Hence, the isomorphism type of the $\boldsymbol{\mu}_6$-orbit space Σ'_ℓ of Σ_ℓ, now viewed as a degree 3 cover of S, is independent of the lift. Note that Σ'_ℓ is a connected oriented surface, not ramified over $S \setminus D$. To see what happens over $z \in D$, we observe that the fiber $\Sigma_\ell(z)$ of $\Sigma_\ell \to S$ over z has the structure of an affine line over $\ell^\vee$ on which $\boldsymbol{\mu}_6$ also acts and in such a manner that the induced action on $\ell^\vee$ is minus the identity. This means that $\boldsymbol{\mu}_6$ acts as transposition on $\Sigma_\ell(z)$: it fixes a point and exchanges the remaining two. It follows that $\Sigma'_\ell \to S$ has no ramification at the image of the fixed point of the lifted $\boldsymbol{\mu}_6$-action, whereas the common image of the remaining pair is a point of simple ramification. It follows that D is the discriminant of $\Sigma'_\ell \to S$ and that Σ'_ℓ has genus 4. This also shows that Σ'_ℓ has no covering transformations. So Σ'_ℓ is unique up to unique isomorphism. □

THEOREM 3.7. *The map which assigns to $[\ell] \in \mathbb{P}(\mathbb{F}_3 \otimes_\mathcal{E} L)$ the cover $\Sigma'_\ell \to S$ defines a* $\mathrm{Mod}(S,D)$*-equivariant bijection*

$$[\ell] \in \mathbb{P}(\mathbb{F}_3 \otimes_\mathcal{E} L) \mapsto r(\Sigma'_\ell/S) \in R(S,D).$$

In particular, $R(S,D)$ thus acquires the structure of the projective space of a 10-dimensional symplectic space over $\mathbb{F}_3$ on which Γ acts through a copy of $\mathrm{Sp}_{10}(\mathbb{F}_3)/\{\pm 1\}$.

Proof. The $\mathrm{Mod}(S,D)$-equivariance of this assignment is clear from the construction. The theorem then follows from the fact that the two sets have the same cardinality and that $\mathrm{Mod}(S,D)$ acts transitively on the target. The rest follows from Lemma 3.5. □

3.5 The confluence divisors

Theorem 3.7 also explains why the preimage of $\mathcal{D} \subset \mathbb{B}_\Gamma$ in $\mathbb{B}_{\Gamma'}$ has three irreducible components, namely the confluence divisors $\mathcal{D}_\mathcal{H}$, $\mathcal{D}_{\mathrm{rm}}$, and $\mathcal{D}_{\mathrm{sg}}$. Let us first observe that a mirror is given by the $\mathcal{E}$-span of (-3)-vector of L and that the mod θ-reduction of the latter determines a line in $\mathbb{F}_3 \otimes_\mathcal{E} L$ and hence a point in $\mathbb{P}(\mathbb{F}_3 \otimes_\mathcal{E} L)$.

PROPOSITION 3.8. *Let ℓ be a line in $\mathbb{F}_3 \otimes_\mathcal{E} L$. Then the three orbits of Γ_ℓ in $\mathbb{P}(\mathbb{F}_3 \otimes_\mathcal{E} L)$ are $[\ell]$, $\mathbb{P}(\ell^\perp) \setminus \{[\ell]\}$, and $\mathbb{P}(\mathbb{F}_3 \otimes_\mathcal{E} L) \setminus \mathbb{P}(\ell^\perp)$ and represent, respectively, the irreducible components $\mathcal{D}_\mathcal{H}$, $\mathcal{D}_{\mathrm{sg}}$, and $\mathcal{D}_{\mathrm{rm}}$ of the preimage of the confluence divisor $\mathcal{D}$ of $\mathbb{B}_\Gamma$ in $\mathbb{B}_{\Gamma_\ell}$.*

Proof. By Lemma 3.2, we can choose a standard set of arcs $\{\gamma_i\}_{i \in \mathbb{Z}/12}$ connecting the base point o with the points of D such that the associated collection $\{c_i \in \pi_1(S \setminus D, o)\}_{i \in \mathbb{Z}/12}$ is a standard set of generators of $\pi_1(S \setminus D, o)$ (as in the proof of Lemma 3.2) for which the covering $\Sigma' \to S$ has the property that $\rho(c_0) = \rho(c_1) \neq \rho(c_2) = \cdots = \rho(c_{11})$ (so this is different than in the proof of Lemma 3.2). The juxtaposition of γ_{i+1} and the inverse of γ_i is isotopic to an arc δ_i as in §2, and, as explained there, determines a (-3)-vector $a_{\delta_i} \in L$ up to a $\boldsymbol{\mu}_6$-multiple and hence a line ℓ_i in $\mathbb{F}_3 \otimes_\mathcal{E} L$.

Let ℓ be the line in $\mathbb{F}_3 \otimes_\mathcal{E} L$ for which $\Sigma' \cong \Sigma'_\ell$. We claim that $\ell = \ell_0$. Indeed, the covering $\Sigma'_\ell \to S$ is disconnected over $S \setminus \delta_0$. It is then not hard to see that $\Sigma_\ell \to \Sigma$ must be a trivial

$\ell^{\vee}$-covering over the preimage of δ_0. This implies that ℓ has its support over δ_0, so that $\ell = \ell_0$.

It remains to observe that for $i = 0, 1, 2$, the mirror $\mathbb{B}^{T_{\gamma_i}}$ parametrizes the points where the covering $\Sigma'_\ell \to P$, acquires, after shrinking δ_i, respectively, a separating double point, a point of ramification of order 2, and a nonseparating double point. So ℓ_0, ℓ_1, ℓ_2 represent, respectively, $\mathcal{D}_{\mathcal{H}}$, $\mathcal{D}_{\mathrm{rm}}$, and $\mathcal{D}_{\mathrm{sg}}$. We note that ℓ_2 is perpendicular to ℓ_0, but ℓ_1 is not. $\square$

3.6 The strict transform of the Theta-null locus

The goal of this section is to show that the strict transform $\mathcal{M}_4(g_3^1)^{\Theta}$ of $\mathcal{M}_4^{\Theta}$ in $\mathcal{M}_4(g_3^1)$ is also locally a ball quotient in $\mathbb{B}_{\Gamma'}$. This was actually established in the paper with Heckman [4], and the goal of this section is to show how. The generic point of that locus parametrizes the canonical genus 4 curves X that lie on a quadric cone Q^0 in a three-dimensional complex projective space and are transversally cut out on Q^0 by a cubic hypersurface (so not containing the vertex). The base of the cone Q^0 (in other words, the set of *rays* in Q^0, lines on Q^0 that pass through its vertex) is a conic. We denote it by P, because the projection $X \to P$ gives us the unique g_3^1 on X. We assume for now that the discriminant divisor D of this projection is reduced.

The vertex of Q^0 is the unique singular point of Q^0 and is resolved by a single blowup: $\hat{Q}^0 \to Q^0$. Its exceptional set, which we can identify with P, has self-intersection -2. It defines a section of the evident morphism $\hat{Q}^0 \to P$ whose fibers are the rays of Q^0. In other words, $\hat{Q}^0$ is a Hirzebruch surface whose exceptional section is P. Its Picard group is freely generated by P and the class f of fiber. The curve X on $\hat{Q}^0$ has class $3(P + 2f)$ and hence the class of $X + P$ is $3(P + 2f) + P = 4P + 6f$. Since this is 2-divisible, it determines a double covering $E_X \to \hat{Q}^0$ with ramification divisor $X + P$ ([1]). The composite with the projection $\hat{Q}^0 \to P$ defines an elliptic fibration for which the preimage of $P \subset \hat{Q}^0$ in E_X defines the zero section (with self-intersection number -1). The covering transformation of the double cover $E_X \to \hat{Q}^0$ is the fiberwise natural involution with respect to the zero section. Note that D now also appears as the discriminant of the fibration $E_X \to P$. Indeed, all fibers are smooth, except those over D: over each point of D, we find a nodal curve (a Kodaira fiber of type I_1) whose double point lies over the singular point of the projection $X \to P$.

By taking the j-invariant of the fibers, we obtain a morphism

$$j : P \to \mathrm{PSL}_2(\mathbb{Z}) \backslash \mathbb{H}^* = \big(\mathrm{PSL}_2(\mathbb{Z}) \backslash \mathbb{H} \big) \cup \{\infty\} = \overline{\mathcal{M}}_{1,1},$$

where $\mathbb{H}^* := \mathbb{H} \cup \mathbb{P}^1(\mathbb{Q})$ is endowed with the horocyclic topology. This morphism has degree 12 and $D = j^*(\infty)$. In [4], it was observed that the D that thus appear can be characterized in $\mathbb{B}_{\Gamma}$ as hyperball quotient as follows. The abelianization of $\mathrm{PSL}_2(\mathbb{Z})$ is naturally isomorphic with $\boldsymbol{\mu}_6$ and hence defines a ramified $\boldsymbol{\mu}_6$-cover $\mathrm{PSL}_2(\mathbb{Z})' \backslash \mathbb{H} \to \mathrm{PSL}_2(\mathbb{Z}) \backslash \mathbb{H}$. This morphism extends across the cusp ∞ with total ramification, so that we get a $\boldsymbol{\mu}_6$-cover

$$\overline{\mathcal{M}}'_{1,1} := \mathrm{PSL}_2(\mathbb{Z})' \backslash \mathbb{H}^* \to \mathrm{PSL}_2(\mathbb{Z}) \backslash \mathbb{H}^* = \overline{\mathcal{M}}_{1,1}.$$

The left-hand side can be identified with the (Eisenstein) elliptic curve $\mathbb{C}/\mathcal{E}$ whose origin is the unique point over ∞ and on which $\boldsymbol{\mu}_6$ acts in the standard manner. The differential

---

[1] Kondō [5] considers instead the $\boldsymbol{\mu}_3$-cover of $\hat{Q}^0$ ramified along X, which produces a K3-surface with $\boldsymbol{\mu}_3$-action.

on $\mathbb{C}/\mathcal{E}$ defined by dz transforms under the $\boldsymbol{\mu}_6$-action via the character χ^{-1} and hence $d\bar{z}$ transforms with the character χ. So $H^1(\overline{\mathcal{M}}'_{1,1};\mathbb{C})_\chi$ is of Hodge type $(0,1)$.

The $\boldsymbol{\mu}_6$-cover $C \to P$ with discriminant D is then the pullback of the above $\boldsymbol{\mu}_6$-cover under the j-morphism:

$$
\begin{array}{ccc}
C & \xrightarrow{\ j'\ } & \overline{\mathcal{M}}'_{1,1} \\
{\scriptstyle\boldsymbol{\mu}_6}\downarrow & & {\scriptstyle\boldsymbol{\mu}_6}\downarrow \\
P & \xrightarrow{\ j\ } & \overline{\mathcal{M}}_{1,1}
\end{array}
$$

Since j is of degree 12, so is j'.

PROPOSITION 3.9. *The map j'^* embeds $H^1(\overline{\mathcal{M}}'_{1,1})$ in $H^1(C)^\circ$ as a primitive rank one $\mathcal{E}$-submodule that is generated by a vector ε with $h(\varepsilon,\varepsilon) = -6$.*

Proof. This is proved in [4], but let us give here a simpler argument. First, note that since $H^1(\overline{\mathcal{M}}'_{1,1}) = H^1(\overline{\mathcal{M}}'_{1,1})^\circ$, j'^* takes values in $H^1(C)^\circ$. If $a \in H^1(\overline{\mathcal{M}}'_{1,1})$ is primitive, then $a \cdot \tau a = 1$ and hence if we put $\varepsilon := j'^* a$, then

$$h(\varepsilon,\varepsilon) = -\tfrac{1}{2}(\varepsilon,\tau\varepsilon) = -\tfrac{1}{2}.12(a \cdot \tau a) = -6,$$

where we used that j' is of degree 12. This also implies that ε is primitive: if it is divisible by $\lambda \in \mathcal{E}$: $\varepsilon = \lambda \varepsilon'$ with $\varepsilon' \in H^1(C)^\circ$, then $\lambda\bar{\lambda}h(\varepsilon',\varepsilon') = -6$. But $\lambda\bar{\lambda}$ is a positive integer that cannot take the value 2 and $h(\varepsilon',\varepsilon') \in \mathcal{E}\theta \cap \mathbb{Q} = 3\mathbb{Z}$. This implies that $\lambda\bar{\lambda} = 1$, meaning that λ is a unit. □

Assertions A4 and A6 of [4] state that any (-6)-vector in L can be written as the sum of two (-3)-vectors with inner product θ and that the (-6)-vectors make up a single Γ-orbit. In particular, the hyperballs in $\mathbb{B}$ defined by such vectors define an irreducible totally geodesic divisor $\mathcal{D}^\Theta$ in $\mathbb{B}_\Gamma$. We can therefore complete Theorem 3.4 as follows (see also [4, Th. 8.2]).

THEOREM 3.10. *The isomorphism in Theorem 3.4 takes the Theta-null locus $\mathcal{M}_4(g_3^1)^\Theta$ onto $\mathcal{D}^\Theta \setminus \mathcal{D}_{\mathrm{sg}}$. In particular, $\mathcal{M}_4(g_3^1)^\Theta$ is a totally geodesic hypersurface in $\mathcal{M}_4(g_3^1)$.*

REMARK 3.11. One would expect a stronger assertion, namely that the involution of $\mathcal{M}_4(g_3^1)$ that assigns to (X,P) the residual pair (X,P') lifts to a reflection in Γ in a (-6)-vector ε, in other words, is given by $x \in L \mapsto x + \tfrac{1}{3}h(x,\varepsilon)\varepsilon$. But such a reflection will not preserve L, as we must then have $h(x,\varepsilon) \in 3\mathcal{E}$. Indeed, since the (-6)-vectors lie in a single $\mathrm{Aut}(L)$-orbit, it suffices to check this for one such vector, say $\varepsilon := a_{\delta_0} + a_{\delta_1}$. But $h(\varepsilon, a_{\delta_2}) = \theta$, which in $\mathcal{E}$ is not divisible by 3. It is surprising (and still a bit mystifying to us) that $\mathbb{B}_{\Gamma'} \setminus \mathcal{D}_{\mathrm{sg}}$ comes with an involution whose fixed point set is $\mathcal{D}^\Theta \setminus \mathcal{D}_{\mathrm{sg}}$, but that is *not* obtained from an index two subgroup of Γ'.

REFERENCES

[1] N. A'Campo, *Tresses, monodromie et le groupe symplectique*, Comment. Math. Helv. **54** (1979), 318–327.

[2] D. Allcock, *The Leech lattice and complex hyperbolic reflections*, Invent. Math. **140** (2000), 283–301.

[3] P. Deligne and G. D. Mostow, *Monodromy of hypergeometric functions and nonlattice integral monodromy*, Inst. Hautes Études Sci. Publ. Math. **63** (1986), 5–89.

[4] G. Heckman and E. Looijenga, "The moduli space of rational elliptic surfaces" in *Algebraic Geometry 2000, Azumino (Hotaka)*, Adv. Stud. Pure Math. **36**, Math. Soc. Japan, Tokyo, 2002, 185–248.

[5] S. Kondō, "The moduli space of curves of g enus 4 and Deligne – Mostow's complex reflection groups" in *Algebraic Geometry 2000, Azumino (Hotaka)*, Adv. Stud. Pure Math. **36**, Math. Soc. Japan, Tokyo, 2002, 383–400.

[6] G. D. Mostow, *Generalized Picard lattices arising from half-integral conditions*, Inst. Hautes Études Sci. Publ. Math. **63** (1986), 91–106.

[7] H. Xue, *A quadratic point on the Jacobian of the universal genus four curve*, Math. Res. Lett. **22** (2015) 1563–1571.

[8] H. Xue, https://www.math.arizona.edu/xuehang/thesis.pdf.

Eduard Looijenga
Department of Mathematics
University of Chicago
Chicago, Illinois 60637
USA
and
Universiteit Utrecht
Utrecht
Netherlands
e.j.n.looijenga@uu.nl