THE ERROR TERM FOR THE SQUAREFREE INTEGERS

L. Moser and R. A. MacLeod

(received February 7, 1966)

Let $Q(x)$ denote the number of squarefree integers $\leq x$. Recently K. Rogers [1] has shown that $Q(x) \geq 53x / 88$ for all x, with equality only at $x = 176$. Define $R(x)$ to be $Q(x) - 6/\pi^2 x$. (We observe that $53/88 \doteq 0.6023$ and $6/\pi^2 \doteq 0.6079$.) Our objective will be to examine $R(x)$. In particular, we show that $|R(x)| < \sqrt{x}$ for all x and observe that $|R(x)| < 1/2 \sqrt{x}$ for $x \geq 8$. The best result of this type obtained by Rogers was

$$(1) \qquad |R(x)| < 12/\pi^2 \sqrt{x} + 8/3\, x^{1/4}.$$

We have

$$Q(x) = \sum_{d \leq x} |\mu(d)| = \sum_{n \leq x} \sum_{d^2 | n} \mu(d) = \sum_{d^2 \leq x} \mu(d) \sum_{\substack{n \leq x \\ d^2 | n}} 1$$

$$= \sum_{d^2 \leq x} \mu(d) \, [x/d^2],$$

so that

$$(2) \qquad \left|Q(x) - \frac{6}{\pi^2} x\right| \leq \left| \sum_{d^2 \leq x} \mu(d) \left(\frac{x}{d^2} - \left[\frac{x}{d^2} \right] \right) \right| + x \left| \sum_{d^2 > x} \frac{\mu(d)}{d^2} \right|,$$

where $x - [x]$ denotes the fractional part of x. Thus, we have almost immediately

$$(3) \qquad |R(x)| \leq 2\sqrt{x} + 1.$$

We direct our attention to improving (3). Let

Canad. Math. Bull. vol. 9, no. 3, 1966.

https://doi.org/10.4153/CMB-1966-039-4 Published online by Cambridge University Press

$$A(x) \sum_{\substack{d \le x \\ \mu(d)=1}} 1, \quad B(x) = \sum_{\substack{d \le x \\ \mu(d)=-1}} 1, \quad C(x) = \max(A(x), \ B(x)) .$$

Then clearly

$$\left| \sum_{d \le x} \mu(d)\,(g(x, d) - [g(x, d)]) \right| \le C(x) \quad \text{for any } g .$$

Let $M(x) = \sum_{d \le x} \mu(d)$. Then

$$C(x) = \frac{1}{2}(A(x) + B(x)) + \frac{1}{2}\left|A(x) - B(x)\right| = \frac{1}{2} Q(x) + \frac{1}{2}\left|M(x)\right| .$$

Since every fourth integer is divisible by 4, and every ninth by 9, we have

$$(4) \qquad Q(x) \le [x] - [\tfrac{x}{4}] - [\tfrac{x}{9}] + [\tfrac{x}{36}] \le \frac{2}{3}x + \frac{4}{3} \quad \text{for all } x .$$

Let $f(x) = [x] - [\frac{x}{2}] - [\frac{x}{3}] - [\frac{x}{5}] + [\frac{x}{30}]$. Since $\sum_{d \le x} \mu(d)[\frac{x}{d}] = 1$,

$\sum_{d \le x} \mu(d)\,[\frac{x}{md}] = 1$ for $x \ge m$, and $\sum_{d \le x} \mu(d) f(\frac{x}{d}) = -1$ for $x \ge 30$.

Now, $f(x) = 1$ for $1 \le x < 6$ and 0 or 1 for $x \ge 6$, so that

$$(5) \qquad \begin{aligned} \left|M(x) + 1\right| &= \left|\sum_{d \le x} \mu(d)\{1 - f(\tfrac{x}{d})\}\right| \le \sum_{d \le \frac{x}{6}} |\mu(d)| \\ &= Q(\tfrac{x}{6}) \le \frac{1}{9}x + \frac{4}{3} \quad \text{for } x \ge 30 . \end{aligned}$$

Hence

$$(6) \qquad \left|M(x)\right| \le \frac{1}{9}x + \frac{7}{3} \quad \text{for } x \ge 30 .$$

One readily checks that (5) holds for $x \ge 2$ and (6) for $x \ge 1$. Thus,

$$(7) \qquad C(x) \le \frac{1}{2}\left(\frac{2}{3}x + \frac{4}{3}\right) + \frac{1}{2}\left(\frac{1}{9}x + \frac{4}{3} + 1\right) = \frac{7}{18}x + \frac{11}{6} \quad \text{for } x \ge 1 ,$$

and $\left| \sum_{d^2 \le x} \mu(d) \left(\frac{x}{d^2} - \left[\frac{x}{d^2} \right] \right) \right| \le C(\sqrt{x}) \le \frac{7}{18}\sqrt{x} + \frac{11}{6}$ for $x \ge 1$.

Also, $$\sum_{d>x} \frac{\mu(d)}{d^2} = \sum_{d>x} \frac{\{M(d)+1\} - \{M(d-1)+1\}}{d^2}$$

$$= \sum_{d>x} \{M(d)+1\} \left(\frac{1}{d^2} - \frac{1}{(d+1)^2} \right) - \frac{M(x)+1}{([x]+1)^2} .$$

Hence
$$\left| \sum_{d>x} \frac{\mu(d)}{d^2} \right| \le \frac{1}{9} \sum_{d>x} d\left(\frac{1}{d^2} - \frac{1}{(d+1)^2} \right) + \frac{4}{3} \sum_{d>x} \left(\frac{1}{d^2} - \frac{1}{(d+1)^2} \right) +$$

$$\frac{1}{9}\frac{1}{x} + \frac{4}{3}\frac{1}{x^2} \le \frac{1}{3}\frac{1}{x} + \frac{8}{3}\frac{1}{x^2} \quad \text{for} \quad x \ge 2;$$

i.e.

(8) $$x \left| \sum_{d^2>x} \frac{\mu(d)}{d^2} \right| \le \frac{1}{3}\sqrt{x} + \frac{8}{3} \quad \text{for} \quad x \ge 4$$

and

$$|R(x)| \le \frac{13}{18}\sqrt{x} + \frac{9}{2} \quad \text{for} \quad x \ge 4,$$

whence,

(9) $$|R(x)| < \sqrt{x} \quad \text{for} \quad x \ge 263.$$

We note that $R(x) = Q(x) - \frac{6}{\pi^2} x$ is a decreasing function for $n \le x < n+1$ and positive for $x = 1, 2, \ldots, 263$, except at 28, 56, 153, 172, 173, 175, 176, 177, 180 and 181, so that it suffices to check that (9) holds for $x = 1, 2, \ldots, 263$ and to examine $R(x)$ near the ten integers x where it is negative. This we have done, and (9) holds for all $x \ge 1$.

We observe that, using better bounds for $M(x)$ in the above argument, one readily shows

(10) $$|R(x)| \le \sqrt{3}\left(1 - \frac{6}{\pi^2}\right)\sqrt{x} = (0.6792\ldots)\sqrt{x}$$

https://doi.org/10.4153/CMB-1966-039-4 Published online by Cambridge University Press

for all x, with equality only at $x = 3$, and

$$|R(x)| < \frac{1}{2}\sqrt{x} \quad \text{for} \quad x \geq 8. \tag{11}$$

We note that the strong form of the prime number theorem, which is equivalent (see e.g. Landau [2], p. 157) to

$$M(x) = 0\left(\frac{x}{\log^{\alpha} x}\right) \quad \text{for each } \alpha,$$

implies

$$R(x) = 0\left(\frac{\sqrt{x}}{\log^{\alpha} x}\right) \quad \text{for each } \alpha, \tag{12}$$

while the Riemann Hypothesis implies (Landau [2], p. 161)

$$M(x) = 0(x^{1/2 + \epsilon}) \quad \text{for each } \epsilon > 0,$$

which yields $R(x) = 0(x^{2/5 + \epsilon})$ for each $\epsilon > 0$. In the other direction, Evelyn and Linfoot [3] have proved $R(x) \neq o(x^{1/4})$.

REFERENCES

1. K. Rogers, The Schnirelmann density of the squarefree integers, Proc. Amer. Math. Soc. 15(1964), 515-516.

2. E. Landau, Vorlesungen Uber Zahlentheorie, II, Chelsea (1947).

3. C. J. A. Evelyn and E. H. Linfoot, On a problem in the additive theory of numbers (Fourth paper). Ann. of Math. 32(1931) 261-270.

University of Alberta, Edmonton

https://doi.org/10.4153/CMB-1966-039-4 Published online by Cambridge University Press