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A complex nonlinear complementarity problem

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In this paper we study the existence and uniqueness of solutions for the following complex nonlinear complementarity problem: find $z \in S$ such that $g(z) \in S^*$ and $\operatorname{re}\left(g(z), z\right) = 0$, where S is a closed convex cone in C^n, S^* the polar cone, and g is a continuous function from C^n into itself. We show that the existence of a $z \in S$ with $g(z) \in \operatorname{int} S^*$ implies the existence of a solution to the nonlinear complementarity problem if g is monotone on S and the solution is unique if g is strictly monotone. We also show that the above problem has a unique solution if the mapping g is strongly monotone on S.

1. Preliminaries

Let C^n denote the n-dimensional complex space with hermitian norm and the usual inner product. If S denotes a closed convex cone in C^n, the polar of S, denoted by S^*, is defined by

$$S^* = \left\{y \in C^n : \operatorname{re}(x, y) \geq 0 \text{ for all } x \in S\right\} .$$

Given $e \in S^*$ and $r > 0$ we write

$$D_r(e) = \{x \in S : \operatorname{re}(e, x) \leq r\} ,$$

$$D_r^0(e) = \left\{x \in D_r(e) : \operatorname{re}(e, x) < r\right\} ,$$

and

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$$S_r(e) = \{x \in D_r(e) : \text{re}(e, x) = r\} .$$

Note that $D_r(e)$ is the disjoint union of $D_r^0(e)$ and $S_r(e)$. We write

$$S_r = \{x \in S : \|x\| = r\} .$$

A mapping $g : C^n \to C^n$ is said to be monotone on S if $\text{re}\big(g(x)-g(y), x-y\big) \geq 0$ for each $(x, y) \in S \times S$, and strictly monotone if strict inequality holds whenever $x \neq y$. The function g is said to be strongly monotone if there is a constant $c > 0$ such that for each $(x, y) \in S \times S$ we have

$$\text{re}\big(g(x)-g(y), x-y\big) \geq c\|x-y\|^2 .$$

Given a continuous function $g : C^n \to C^n$, the nonlinear complementarity problem in C^n consists of finding a z such that

$$z \in S , \quad g(z) \in S^* ,$$

(1.1) $$\text{and}$$

$$\text{re}\big(g(z), z\big) = 0 ,$$

where S is a closed convex cone in C^n .

Several authors including Bazaraa, Goode, and Nashed [1], Eaves [2], Habetler and Price [3], and Karamardian [5] have discussed complementarity problems in different contexts. In particular Parida and Sahoo in [6] and [7] have considered this problem in the complex case by taking S to be a polyhedral cone. In this paper we study this problem for any closed convex cone in C^n . We show that if g is monotone on S the existence of a $z \in S$ with $g(z) \in \text{int } S^*$ implies the existence of a solution to (1.1) and the solution is unique if g is strictly monotone. We also show that (1.1) has a unique solution if g is strongly monotone on S .

2. Some existence theorems

We start by mentioning a modified version of a lemma of Hartman and Stampacchia [4]. Since the result is known for R^n we only give a brief outline for the sake of completeness (see Eaves [2]).

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PROPOSITION 2.1. *Let* $g : C^n \to C^n$ *be a continuous map on a non-empty, compact, convex set* $K \subset C^n$. *Then there is a* $z_0 \in K$ *such that*

$$\operatorname{re}\left(g\left(z_0\right), z-z_0\right) \geq 0$$

for all $z \in K$.

Proof. For a fixed $u_0 \in K$ consider the function $h : K \to R$ defined by

$$h(w) = \|w-u_0+gu_0\| .$$

Clearly h is continuous on K . Since K is compact, h attains a minimum, say at w_0 . It is also easily verified that w_0 is unique. The correspondence $u_0 \mapsto w_0$ defines a continuous function of K into itself. Using Brouwer's Theorem, we have a fixed point z_0 which is the required point of the proposition.

LEMMA 2.2. *Let* S *be a closed convex cone of* C^n *and let* $e \in \operatorname{int} S^*$. *Then the set* $D_r(e)$ *is compact.*

Proof. Let $f : S \to R$ be the continuous function defined by

$$f(z) = \operatorname{re}(e, z) .$$

Then $D_r(e) = f^{-1}[0, r]$. Thus $D_r(e)$ is closed. Note also that for any k with $0 \leq k \leq 1$, $kD_r(e) \subset D_r(e)$.

It will now suffice to prove that $D_r(e)$ is bounded. Suppose to the contrary that it is not; then we can choose a sequence $\{z_n\}$ of isolated points in $D_r(e)$ satisfying

(i) $\|z_n\| \geq 1$ for all n , and

(ii) $\|z_n\| \to \infty$ as $n \to \infty$.

Let $y_n = \left(z_n/\|z_n\|\right)$; then $\|y_n\| = 1$ and $y_n \in D_r(e)$ for all n . We therefore have a convergent subsequence $y_{n_k} \to y$. Since $D_r(e)$ is

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closed, $y \in D_r(e)$. Moreover,

$$\mathrm{re}(e, y) = \lim_{k\to\infty} \mathrm{re}\left(e, y_{n_k}\right) = \lim_{k\to\infty} \mathrm{re}\left(e, \left(z_{n_k}/\|z_{n_k}\|\right)\right) .$$

But

$$\mathrm{re}\left(e, \left(z_{n_k}/\|z_{n_k}\|\right)\right) \le \left(r/\|z_{n_k}\|\right) \to 0 \quad \text{as} \quad k \to \infty .$$

Thus $\mathrm{re}(e, y) = 0$. Since $e \in \mathrm{int}\, S^*$ we conclude that $y = 0$. This is a contradiction in view of the fact that $\|y_{n_k}\| = 1$ for every k .

LEMMA 2.3. *Let* $x_0 \in C^n$, $e \in S^*$, *and* $r > 0$ *be given. If there is a* $z_0 \in D_r^0(e)$ *such that*

(2.1) $$\mathrm{re}\left(x_0, z-z_0\right) \ge 0$$

for all $z \in D_r(e)$, *then* (2.1) *holds for all* $z \in S$.

Proof. Let $z \in S$. Write $u = \lambda z + (1-\lambda)z_0$, $0 < \lambda < 1$. We can choose λ sufficiently small so that u will lie in $D_r(e)$. Then

$$0 \le \mathrm{re}\left(x_0, u-z_0\right) = \lambda\, \mathrm{re}\left(x_0, z-z_0\right) .$$

The result, therefore, follows.

PROPOSITION 2.4. *Let* $g : C^n \to C^n$ *be a continuous map on a closed convex cone* S . *Let* $e \in \mathrm{int}\, S^*$ *and* $r > 0$ *be given. If there exists* $u \in D_r^0(e)$ *such that*

$$\mathrm{re}\left(g(z), z-u\right) \ge 0$$

for all $z \in S_r(e)$, *then there exists* $z_0 \in D_r(e)$ *such that*

(2.2) $$\mathrm{re}\left(g\left(z_0\right), z-z_0\right) \ge 0$$

for all $z \in S$.

Proof. $D_r(e)$ is clearly convex; moreover, by Lemma 2.2, it is compact. Therefore, by Proposition 2.1, there is a $z_0 \in D_r(e)$ satisfying

(2.2) for all $z \in D_r(e)$. If $z_0 \in D_r^0(e)$ then we can obtain the result by taking $x_0 = g(z_0)$ in Lemma 2.3. Suppose that $z_0 \in S_r(e)$; then by hypothesis there is a $u \in D_r^0(e)$ satisfying

$$\operatorname{re}(g(z_0), z_0-u) \geq 0 .$$

Therefore we have

$$\operatorname{re}(g(z_0), z-u) \geq 0$$

for all $z \in D_r(e)$. Now applying Lemma 2.3 with $x_0 = g(z_0)$ we get

$$\operatorname{re}(g(z_0), z-u) \geq 0 \tag{2.3}$$

for all $z \in S$. We also have that

$$\operatorname{re}(g(z_0), u-z_0) \geq 0 . \tag{2.4}$$

The result follows from (2.3) and (2.4).

We are now ready to prove our existence theorems.

THEOREM 2.5. *Let* $g : C^n \to C^n$ *be a continuous monotone function on a closed convex cone* S *such that there is a* $u \in S$ *with* $g(u) \in \operatorname{int} S^*$; *then* (1.1) *has a solution* $z_0 \in S$. *Moreover, if* g *is strictly monotone, then the solution is unique.*

Proof. Suppose that there is a $u \in S$ with $g(u) \in \operatorname{int} S^*$. Choose $r > \operatorname{re}(g(u), u) > 0$. Now $u \in D_r^0(g(u))$. Since g is monotone on S we have

$$\operatorname{re}(g(z), z-u) \geq \operatorname{re}(g(u), z-u) > 0$$

for all $z \in S_r(g(u))$. By Proposition 2.4, there is a $z_0 \in D_r(g(u))$ such that (2.2) holds for all $z \in S$. Thus we have

$$\operatorname{re}(g(z_0), z) \geq \operatorname{re}(g(z_0), z_0)$$

for all $z \in S$. In particular,

$$\operatorname{re}(g(z_0), z_0+z_0) \geq \operatorname{re}(g(z_0), z_0) ,$$

and consequently,

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(2.5) $$\operatorname{re}\left(g\left(z_0\right),\ z_0\right) \geq 0 .$$

Note that $0 \in S$, so it follows from (2.2) that

(2.6) $$\operatorname{re}\left(g\left(z_0\right),\ z_0\right) \leq 0 .$$

From (2.5) and (2.6) we conclude that z_0 is a solution to (1.1).

Suppose now that g is strictly monotone. If z_0 and w_0 are solutions to (1.1), then

$$\operatorname{re}\left(g\left(z_0\right)-g\left(w_0\right),\ z_0-w_0\right) = -\operatorname{re}\left(g\left(z_0\right),\ w_0\right) - \operatorname{re}\left(g\left(w_0\right),\ z_0\right) \leq 0 .$$

Since g is strictly monotone, this is impossible unless $z_0 = w_0$. Thus the solution is unique.

THEOREM 2.6. *Let $g : C^n \to C^n$ be a continuous strongly monotone function on a pointed closed convex cone S. Then there is a unique solution to* (1.1).

Proof. Since g is strongly monotone on S, there is a constant $c > 0$ such that for any $z \in S$,

$$\operatorname{re}\left(g(z),\ z\right) \geq \operatorname{re}\left(g(0),\ z\right) + c\|z\|^2 ,$$

and hence

$$\frac{\operatorname{re}\left(g(z), z\right)}{\|z\|} \geq \operatorname{re}\left(g(0),\ \frac{z}{\|z\|}\right) + c\|z\| .$$

The continuous function $\theta : S_1 \to R$ defined by

$$\theta(w) = \operatorname{re}\left(g(0),\ w\right)$$

attains its bounds. Let m be its lower bound. Then for all $z \in S$,

$$\frac{\operatorname{re}\left(g(z), z\right)}{\|z\|} \geq m + c\|z\| .$$

Thus if $\|z\| > \frac{|m|}{c}$, then $\operatorname{re}\left(g(z),\ z\right) > 0$. Let $d > \frac{|m|}{c}$. Since S is pointed, we can choose a hyperplane H in C^n satisfying the following two conditions:

(i) the distance of H from the origin is d, and

(ii) H meets all the generators of the cone S.

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It is then clear that the set $B = H \cap S$ is a non-empty, compact, convex set. Moreover, since $\|z\| \geq d$, $\operatorname{re}(g(z), z) > 0$ for every $z \in B$. We can now apply Proposition 2.1 to get a $z_0 \in B$ such that (2.2) holds for all $z \in B$. But every nonzero vector of S is a scalar multiple of a vector in B, so that z_0 satisfies (2.2) for all $z \in S - \{0\}$. Thus

$$\operatorname{re}(g(z_0), z) \geq \operatorname{re}(g(z_0), z_0) > 0$$

for all $z \in S - \{0\}$, showing that $g(z_0) \in \operatorname{int} S^*$. Since g is strongly montone, it is strictly monotone and the result follows from Theorem 2.5.

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