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HOMOLOGICAL LINEAR QUOTIENTS AND EDGE IDEALS OF GRAPHS

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Abstract

It is well known that the edge ideal $I(G)$ of a simple graph G has linear quotients if and only if G^c is chordal. We investigate when the property of having linear quotients is inherited by homological shift ideals of an edge ideal. We will see that adding a cluster to the graph G^c when $I(G)$ has homological linear quotients results in a graph with the same property. In particular, $I(G)$ has homological linear quotients when G^c is a block graph. We also show that adding pinnacles to trees preserves the property of having homological linear quotients for the edge ideal of their complements. Furthermore, $I(G)$ has homological linear quotients for every graph G such that G^c is a λ-minimal chordal graph.

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1. Introduction

Let $S = K[x_1, \ldots, x_n]$ be the polynomial ring in the variables $x_1, \ldots, x_n$ over a field K with its natural multigrading. Throughout, a monomial and its multidegree will be used interchangeably and $S(\boldsymbol{x^a})$ denotes the free S-module with one generator of multidegree $\boldsymbol{x^a}$. A monomial ideal $I \subseteq S$ has a (unique up to isomorphism) minimal multigraded resolution

$$\mathbf{F} : 0 \to F_p \to \cdots \to F_1 \to F_0$$

with

$$F_k = \bigoplus_{\boldsymbol{a} \in \mathbb{Z}^n} S(\boldsymbol{x^a})^{\beta_{k,\boldsymbol{a}}}.$$

The kth homological shift ideal of I denoted by $\mathrm{HS}_k(I)$ is the ideal generated by the kth multigraded shifts of I, that is,

$$\mathrm{HS}_k(I) = (\{\boldsymbol{x^a} \mid \beta_{k,\boldsymbol{a}} \neq 0\}).$$

Recently, properties of monomial ideals which are inherited by their homological shift ideals have attracted attention. It is shown in [1, Theorem 3.2] that if I is a matroidal ideal, then so are its homological shift ideals. It is still an open question whether a similar statement holds if one replaces matroidal by polymatroidal. However, there are some partial positive answers for some classes of polymatroidal ideals including polymatroidal ideals satisfying the strong exchange property [13, Corollary 3.6], Veronese-type ideals [13, Theorem 3.3], polymatroidal ideals generated in degree two [7, Theorem 4.5] and for the first homological shift ideal of any polymatroidal ideal [6, Theorem 2.2]. In [3, Proposition 3.1], analogues of these results for the property of being equigenerated squarefree Borel are presented and in [2], a quasi-additive property of homological shift ideals is studied.

Having linear quotients is another property that has received considerable attention. Following [7], we say that a monomial ideal I has homological linear quotients when I has linear quotients and $\mathrm{HS}_k(I)$ inherits this property for every k. It is shown in [3, Theorem 2.4] and [3, Theorems 2.4 and 3.3] that principal Borel ideals as well as squarefree Borel ideals have homological linear quotients (see also [14]). It is shown in [13, Theorem 2.2] that even $\mathbf{c}$-bounded principal Borel ideals have homological linear quotients. It is also proved in [7, Theorem 1.3] that if a monomial ideal I has linear quotients, then $\mathrm{HS}_1(I)$ has the same property.

Regarding having homological linear quotients, we restrict our attention to edge ideals of graphs. Let G be a simple graph on n vertices and $I(G) \subseteq S$ be its edge ideal. From [10] and [12, Theorem 10.2.6], $I(G)$ has linear quotients if and only if G^c is a chordal graph. It is shown in [7, Proposition 3.2] that if $I(G)$ has homological linear quotients, then adding a whisker to G^c gives a graph such that the edge ideal of its complement also has homological linear quotients. As a result, $I(G)$ has homological linear quotients when G^c is a tree. Generalising these two results, we show in Theorem 2.6 that when $I(G)$ has homological linear quotients, then adding clusters to G^c leads to a graph such that the edge ideal of its complement has homological linear quotients. In particular, this implies that $I(G)$ has homological linear quotients when G^c is a block graph (see Corollary 2.7).

Next, we consider another construction of adding pinnacles which preserves the property of having linear quotients for homological shift ideals (see Section 3 for the definition). We will see in Theorem 3.1 that if G^c is obtained by adding pinnacles to a tree, then $I(G)$ has homological linear quotients. Finally, we see in Corollary 3.4 that $I(G)$ has homological linear quotients if G^c is a λ-minimal graph.

2. Block graphs

Throughout, $S = K[x_1, \dots, x_n]$ denotes a polynomial ring over a field K with its natural multigrading. If $u, v \in S$ are monomials, then $u : v$ denotes the monomial $u/\gcd(u, v)$. For a monomial $u \in S$, we set $\max u = \max\{k \mid x_k \text{ divides } u\}$. When $\ell = \max u$, we may sometimes write $x_\ell = \max u$ for ease of use.

Let $I \subseteq S$ be a monomial ideal. We denote its minimal set of monomial generators by $G(I)$. A monomial ideal $I \subseteq S$ is said to have linear quotients if there exists an ordering $u_1, \dots, u_r$ of the elements of $G(I)$, called an admissible order, such that for

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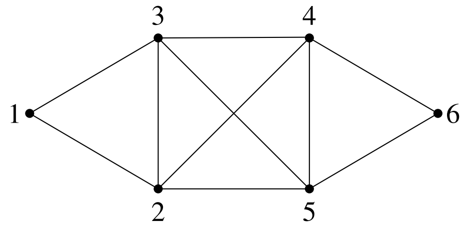

FIGURE 1. A chordal graph such that $\mathrm{HS}_2(I(G^c))$ does not have linear quotients.

each $i = 1, \ldots, r-1$, the colon ideal $(u_1, \ldots, u_i) : (u_{i+1})$ is generated by a subset of $\{x_1, \ldots, x_n\}$. If I has linear quotients with respect to the ordering $u_1, \ldots, u_r$ of $G(I)$, we define

$$\mathrm{set}(u_{i+1}) = \{x_j \mid x_j \in (u_1, \ldots, u_i) : (u_{i+1})\}.$$

REMARK 2.1. Let a monomial ideal $I \subseteq S$ have linear quotients. By [15, Lemma 1.5], a minimal multigraded free resolution $\mathbf{F}$ of I can be described as follows: the S-module F_i in homological degree i of $\mathbf{F}$ is the multigraded free S-module whose basis is formed by the monomials $ux_{\ell_1} \ldots x_{\ell_i}$ for which $u \in \mathrm{G}(I)$ and $x_{\ell_1}, \ldots, x_{\ell_i}$ are distinct elements of $\mathrm{set}(u)$.

Henceforth, all graphs considered in this paper are simple graphs. Let G be a graph on the vertex set $V(G) = \{x_1, \ldots, x_n\}$ with edge set $E(G)$. The ideal

$$I(G) = (x_i x_j \mid \{x_i, x_j\} \in E(G)) \subseteq S$$

is called the edge ideal of G. The *complement* of G, denoted by G^c, is the graph on the vertex set $V(G)$ whose edge set is

$$E(G^c) = \{\{x_i, x_j\} \mid x_i \neq x_j \text{ and } \{x_i, x_j\} \notin E(G)\}.$$

The set of all vertices adjacent to a vertex x_i in G, denoted by $\mathrm{N}_G(x_i)$, is called the neighbourhood of x_i in G. The distance between vertices x_i and x_j of a connected graph G, denoted by (x_i, x_j), is the number of edges in the shortest path connecting them.

A graph G is called a chordal graph if it has no induced cycle of length greater than three. An ordering $x_1 > x_2 > \cdots > x_n$ of vertices of a graph G is called a perfect elimination ordering if whenever a vertex x_i is adjacent to vertices x_j and x_k with $i < j < k$, then x_j and x_k are also adjacent. Chordal graphs are characterised in [5, 11] as those graphs whose vertices admit a perfect elimination ordering.

REMARK 2.2. While it is known by [10] and [12, Theorem 10.2.6] that the edge ideal $I(G)$ of a graph G has linear quotients if and only if G^c is chordal, this property is not inherited by homological shift ideals. For example, consider the graph G presented in Figure 1. Here, the labelling of vertices gives a perfect elimination ordering of vertices with respect to $x_1 > \cdots > x_6$ and even more with respect to $x_6 > \cdots > x_1$. One has

$$I(G^c) = (x_1x_4, x_1x_5, x_1x_6, x_2x_6, x_3x_6),$$

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and

$$\mathrm{HS}_2(I(G^c)) = (x_1x_2x_3x_6, x_1x_4x_5x_6),$$

which does not have linear quotients with respect to any ordering of its generators.

Let $u = x_{i_1} \cdots x_{i_m} \in S$ be a squarefree monomial with $i_1 < \cdots < i_m$. We say that x_{i_t} is a *source variable* of u with respect to a graph G, or shortly a source of u when the graph is clear from the context, if the following conditions hold:

- $1 \leq i_t < \max u$;
- x_{i_t} is adjacent to x_{i_s} in G for $t < s \leq \max u$.

THEOREM 2.3 [13, Theorem 4.1]. *Let G be a chordal graph. Suppose that $x_1 > x_2 > \cdots > x_n$ is a perfect elimination ordering of $V(G)$. Then, for each k,*

$$\mathrm{HS}_k(I(G^c)) = \left(u \,\middle|\, \begin{array}{l} u \text{ is a squarefree monomial of degree } k+2 \\ \text{which has a source with respect to } G^c \end{array} \right).$$

A graph G is said to be a *biconnected graph* if it is connected and nonseparable, that is, if we remove any of its vertices, the graph remains connected. A biconnected component is a maximal biconnected subgraph. A graph G is called a *block graph* if every biconnected component is a clique.

Let G be a graph and $v \in V(G)$. We say that the graph H is obtained from G by adding a *t-cluster* or simply a *cluster* via v when we add $t-1$ new vertices $y_1, \ldots, y_{t-1}$ to $V(G)$, and add all edges $\{y_iy_j \mid 1 \leq i < j \leq t\}$ to $E(G)$ (note that we set $v = y_t$).

The first statement of the following lemma is a special case of [13, Proposition 1.7].

LEMMA 2.4. *Let $I \subset K[\mathbf{x}] = K[x_1, \ldots, x_n]$ be a monomial ideal that has homological linear quotients and consider the ideal $\mathfrak{m} = (y_1, \ldots, y_m)$ in $K[\mathbf{y}] = K[y_1, \ldots, y_m]$ with m new variables. Then the kth homological shift ideal of $\mathfrak{m}I \subseteq K[\mathbf{x}, \mathbf{y}]$ is*

$$\begin{aligned} \mathrm{HS}_k(\mathfrak{m}I) = (y_1, \ldots, y_m)\mathrm{HS}_k(I) &+ (y_iy_j \mid 1 \leq i < j \leq m)\mathrm{HS}_{k-1}(I) \\ &+ (y_iy_jy_k \mid 1 \leq i < j < k \leq m)\mathrm{HS}_{k-2}(I) + \cdots + (y_1 \cdots y_m)\mathrm{HS}_{k-m+1}(I). \end{aligned}$$

Furthermore, the ideal $\mathrm{HS}_k(\mathfrak{m}I)$ has homological linear quotients for every k.

PROOF. Let $I = \mathrm{HS}_0(I)$ have linear quotients with respect to the ordering $u_1, u_2, \ldots, u_\ell$ of its generators. Then $\mathfrak{m}I$ has simply linear quotients with respect to the order:

$$u_1y_1, u_2y_1, \ldots, u_\ell y_1, u_1y_2, u_2y_2, \ldots, u_\ell y_2, \ldots, u_1y_m, u_2y_m, \ldots, u_\ell y_m.$$

With this ordering of generators,

$$\mathrm{set}(u_iy_j) = \mathrm{set}(u_i) \cup \{y_1, \ldots, y_{j-1}\},$$

where $\mathrm{set}(u_i) = \{x_j \mid x_j \in (u_1, \ldots, u_{i-1}) : (u_i)\}$. Using Remark 2.1 to construct $\mathrm{HS}_k(\mathfrak{m}I)$ gives the conclusions in Table 1.

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TABLE 1. Conclusions for $\mathrm{HS}_k(I)$ in Lemma 2.4.

$y_1\mathrm{HS}_k(I)$	generated by $u_1y_1,\dots,u_\ell y_1$ and their sets
$y_2\mathrm{HS}_k(I)+y_1y_2\mathrm{HS}_{k-1}(I)$	generated by $u_1y_2,\dots,u_\ell y_2$ and their sets
$y_3\mathrm{HS}_k(I)+y_1y_3\mathrm{HS}_{k-1}(I)$ $+y_2y_3\mathrm{HS}_{k-1}(I)+y_1y_2y_3\mathrm{HS}_{k-2}(I)$	generated by $u_1y_3,\dots,u_\ell y_3$ and their sets
$\vdots$	$\vdots$
$y_m\mathrm{HS}_k(I)+\cdots+y_1\dots y_m\mathrm{HS}_{k-m+1}(I)$	generated by $u_1y_m,\dots,u_\ell y_m$ and their sets

The sum of the ideals in the left column of Table 1 gives

$$\begin{aligned}\mathrm{HS}_k(\mathfrak{m}I)=(y_1,\dots,y_m)\mathrm{HS}_k(I)&+(y_iy_j\mid 1\le i<j\le m)\mathrm{HS}_{k-1}(I)\\&+(y_iy_jy_k\mid 1\le i<j<k\le m)\mathrm{HS}_{k-2}(I)+\cdots\\&+(y_1\cdots y_m)\mathrm{HS}_{k-m+1}(I).\end{aligned}$$

Next we show that $\mathrm{HS}_k(\mathfrak{m}I)$ has linear quotients for every k. Notice that each $\mathrm{HS}_\ell(I)$ has linear quotients by assumption. For each ℓ, we fix an admissible ordering on the minimal set of monomial generators of $\mathrm{HS}_\ell(I)$, and set $u>_\ell v$ for each u and v in $G(\mathrm{HS}_\ell(I))$ if u comes before v in the fixed admissible ordering. Next we show that $\mathrm{HS}_k(\mathfrak{m}I)$ has linear quotients with the following ordering of monomial generators of $\mathrm{HS}_k(\mathfrak{m}I)$: the monomial $y_{i_1}\cdots y_{i_t}u$ with $u\in\mathrm{HS}_{k-t+1}(I)$ comes before $y_{j_1}\cdots y_{j_s}v$ with $v\in\mathrm{HS}_{k-s+1}(I)$ if either $y_{i_1}\cdots y_{i_t}>_{\mathrm{glex}}y_{j_1}\cdots y_{j_s}$ or if $y_{i_1}\cdots y_{i_t}=y_{j_1}\cdots y_{j_s}$ and $u>_{k-t+1}v$. Here $>_{\mathrm{glex}}$ denotes the graded lexicographic order on $K[\mathbf{y}]$ induced by $y_1>\cdots>y_m$. To see why this is an admissible ordering for $\mathrm{HS}_k(\mathfrak{m}I)$, consider the colon

$$w=y_{i_1}\cdots y_{i_t}u:y_{j_1}\cdots y_{j_s}v$$

of elements of the minimal set of monomial generators of $\mathrm{HS}_k(\mathfrak{m}I)$ in which $y_{i_1}\cdots y_{i_t}u$ comes before $y_{j_1}\cdots y_{j_s}v$ in the ordering just described. Suppose that $\deg w>1$. We show that there exists $y_{\ell_1}\cdots y_{\ell_s}\tilde{v}$ in the set of generators which appears before $y_{j_1}\cdots y_{j_s}v$ and

$$y_{\ell_1}\cdots y_{\ell_s}\tilde{v}:y_{j_1}\cdots y_{j_s}v$$

is a degree one monomial which divides w. We consider two cases.

Case 1. Assume that $y_{i_1}\cdots y_{i_t}>_{\mathrm{glex}}y_{j_1}\cdots y_{j_s}$. By Remark 2.1, the element v in the minimal set of monomial generators of $\mathrm{HS}_{k-s+1}(I)$ is a product of an element $\hat{v}$ in the minimal set of monomial generators of I and $k-s+1$ pairwise distinct elements of $\mathrm{set}(\hat{v})$. If $t>s$, then $k-s+1>k-t+1\ge 0$. Thus, $k-s+1\ne 0$. In particular, there exists x_p in the subset of $\mathrm{set}(\hat{v})$ that divides $v/\hat{v}$. Since y_{i_r} divides $y_{i_1}\cdots y_{i_t}:y_{j_1}\cdots y_{j_s}$, it follows that

$$y_{i_r}\Big(y_{j_1}\cdots y_{j_s}\frac{v}{x_p}\Big)$$

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has the desired properties, that is, it comes before $y_{j_1} \cdots y_{j_s} v$ and its colon with respect to $y_{j_1} \cdots y_{j_s} v$ is y_{i_r}.

Otherwise, $t = s$. Suppose that $y_{i_1} \cdots y_{i_s} : y_{j_1} \cdots y_{j_s} = y_{\ell_1} \cdots y_{\ell_p}$ with $\ell_1 < \cdots < \ell_p$. Then

$$y_{\ell_1} \frac{y_{j_1} \cdots y_{j_s}}{y_{j_s}} v,$$

where $j_s = \max(y_{j_1} \ldots y_{j_s})$ has the desired properties.

Case 2. Now assume that $y_{i_1} \cdots y_{i_t} = y_{j_1} \cdots y_{j_s}$ and $u >_{k-s+1} v$. Since HS_{k-s+1} has linear quotients with respect to the ordering given by $>_{k-s+1}$, there exists $\tilde{v}$ in the minimal set of monomial generators of $\mathrm{HS}_{k-s+1}(I)$ such that $\tilde{v} >_{k-s+1} v$ and $\tilde{v} : v = x_p$ for some p with $x_p \mid u : v$. Hence, $y_{j_1} \cdots y_{j_s} \tilde{v}$ is the desired element since it comes before $y_{j_1} \cdots y_{j_s} v$ in the ordering of the generators of $\mathrm{HS}_k(\mathfrak{m}I)$ described before and in addition $y_{j_1} \cdots y_{j_s} \tilde{v} : y_{j_1} \cdots y_{j_s} v = x_p$. □

Let I, J and L be monomial ideals in S such that the minimal set of monomial generators $G(I)$ of I is the disjoint union of $G(J)$ and $G(L)$. Then $I = J + L$ is called a Betti splitting if

$$\beta_{k,\boldsymbol{a}}(I) = \beta_{k,\boldsymbol{a}}(J) + \beta_{k,\boldsymbol{a}}(L) + \beta_{k-1,\boldsymbol{a}}(J \cap L)$$

for all k and all multidegrees $\boldsymbol{a}$. In particular, as noted in [4, 7], if $I = J + L$ is a Betti splitting, then for each k,

$$\mathrm{HS}_k(I) = \mathrm{HS}_k(J) + \mathrm{HS}_k(L) + \mathrm{HS}_{k-1}(J \cap L).$$

THEOREM 2.5 [8, Corollary 2.4]. *Let I, J and L be monomial ideals in S such that $G(I)$ is the disjoint union of $G(J)$ and $G(L)$. If both J and L have linear resolutions, then $I = J + L$ is a Betti splitting.*

THEOREM 2.6. *Let G be a graph, and suppose that the graph H is obtained from G by adding a cluster. If the edge ideal $I(G^c)$ has homological linear quotients, then $I(H^c)$ also has homological linear quotients.*

PROOF. Let $V(G) = \{x_1, x_2 \ldots, x_n\}$ and H be obtained by adding a t-cluster to G via x_n. Suppose that $y_1, \ldots, y_t = x_n$ are vertices of the new clique that is added to G. Then

$$I(H^c) = I(G^c) + (x_i y_j \mid 1 \le i \le n-1 \text{ and } 1 \le j \le t-1).$$

Set $I = I(H^c)$, $J = I(G^c)$ and $L = (x_i y_j \mid 1 \le i \le n-1 \text{ and } 1 \le j \le t-1)$. The ideal L is matroidal. So L has a linear resolution. The ideal J also has a linear resolution by the assumption. Hence, by Theorem 2.5, $I = J + L$ is a Betti splitting. In particular,

$$\mathrm{HS}_k(I) = \mathrm{HS}_k(J) + \mathrm{HS}_k(L) + \mathrm{HS}_{k-1}(J \cap L)$$

for each k. Observe that

$$J \cap L = (x_i x_j y_\ell \mid \{x_i, x_j\} \in E(G^c) \text{ and } 1 \le \ell \le t-1) = (y_1, \ldots, y_{t-1})J.$$

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Thus, by Lemma 2.4, $\mathrm{HS}_{k-1}(J \cap L)$ has linear quotients and

$$\begin{aligned}\mathrm{HS}_{k-1}(J \cap L) = (y_1, \dots, y_{t-1})\mathrm{HS}_{k-1}(J) &+ (y_i y_j \mid 1 \le i < j \le t-1)\mathrm{HS}_{k-2}(J)\\ &+ (y_i y_j y_k \mid 1 \le i < j < k \le t-1)\mathrm{HS}_{k-3}(J) + \cdots\\ &+ (y_1 \cdots y_{t-1})\mathrm{HS}_{k-t+1}(J).\end{aligned}$$

Writing the homological shift ideals of $(x_i \mid 1 \le i \le n-1)$ as Koszul complexes and applying Lemma 2.4 yields

$$\mathrm{HS}_k(L) = \left(x_{i_1} \cdots x_{i_p} y_{j_1} \cdots y_{j_q} \;\middle|\; \begin{matrix} 1 \le i_1 < \cdots < i_p < n \\ 1 \le j_1 < \cdots < j_q < t \end{matrix} \text{ and } p + q = k + 2\right).$$

By our discussion, the ideals $\mathrm{HS}_k(L)$, $\mathrm{HS}_{k-1}(J \cap L)$ and $\mathrm{HS}_k(J)$ have linear quotients. Suppose that they have linear quotients with respect to the following ordering of their minimal set of monomial generators:

- $\mathrm{HS}_k(L) = (u_1, \dots, u_p)$;
- $\mathrm{HS}_{k-1}(J \cap L) = (v_1, \dots, v_q)$;
- $\mathrm{HS}_k(J) = (w_1, \dots, w_r)$.

We claim that $\mathrm{HS}_k(I)$ has linear quotients with respect to the ordering of generators:

$$u_1, \dots, u_p, v_{j_1}, \dots, v_{j_s}, w_1, \dots, w_r. \tag{2.1}$$

Here $1 \le j_1 < \cdots < j_s \le q$ and the elements $v_{j_1}, \dots, v_{j_s}$ are those elements of $G(\mathrm{HS}_{k-1}(J \cap L)) = \{v_1, \dots, v_q\}$ which do not appear among $u_1, \dots, u_p$, that is, those elements of $G(\mathrm{HS}_{k-1}(J \cap L))$ divided by x_n. Let v be a squarefree monomial in $K[\boldsymbol{x}, \boldsymbol{y}]$. Denote by $\deg_{\boldsymbol{y}} v$ the number of y_j which divide v for $j = 1, \dots, t$.

First consider $u : v_{j_i}$ for some $i = 1, \dots, s$ and $u \in \{u_1, \dots, u_p, v_1, \dots, v_{j_{i-1}}\}$. Let z be a variable dividing $u : v_{j_i}$. Then

$$\tilde{u} = \frac{v_{j_i}}{x_n} z$$

is a monomial appearing among $u_1, \dots, u_p$ in (2.1) and $\tilde{u} : v_{j_i} = z$.

Next consider $u : w_j$ for some $j = 1, \dots, r$ and $u \in \{u_1, \dots, u_p, v_1, \dots, v_{j_s}\}$ (see (2.1)). Since $\deg_{\boldsymbol{y}} u \ge 1 > \deg_{\boldsymbol{y}} w_j$, one deduces that y_{j_ℓ} divides $u : w_j$ for some ℓ. So $u = (w_j / \max w_j) y_{j_\ell}$ is simply an element of $\{u_1, \dots, u_p, v_1, \dots, v_{j_s}\}$ with $u : w_j = y_{j_\ell}$, as desired. □

Corollary 2.7. *Let G be a block graph. Then the edge ideal $I(G^c)$ has homological linear quotients.*

Corollary 2.8 [7, Corollary 3.3]. *Let G be a tree. Then the edge ideal $I(G^c)$ has homological linear quotients.*

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3. λ-minimal graphs

Let $e = \{x_i, x_j\}$ be an edge of a graph G. By *adding a pinnacle on* e, we mean adding a new vertex y, and edges $\{x_i, y\}$ and $\{x_j, y\}$ to G. We call the subgraph induced on these two new edges a *pinnacle* and the vertex y its *tip* (see Figure 2).

Herzog and Ficarra, using an inductive argument by adding whiskers, showed in [7] that if G is a tree, then the edge ideal $I(G^c)$ has homological linear quotients. Here, generalising their result, we determine a labelling on the vertices of trees with some pinnacles to find an admissible ordering of generators for every $\mathrm{HS}_k(I(G^c))$.

THEOREM 3.1. *Let G be either a tree or obtained by adding some pinnacles to a tree. Then the edge ideal $I(G^c)$ has homological linear quotients.*

PROOF. We may assume that $\{x_1, x_2, \ldots, x_n\}$ is the vertex set of G such that for some t, the induced subgraph H on $\{x_t, x_{t+1}, \ldots, x_n\}$ is a tree and G is obtained by adding some pinnacles to H with tips $\{x_1, x_2, \ldots, x_{t-1}\}$. By a suitable relabelling of vertices, we may also assume that if $t \leq i, j \leq n$ and $\dot{(}x_j, x_n) < \dot{(}x_i, x_n)$, then $i < j$.

One can see that the labelling described above gives a perfect elimination ordering on the vertices of G. In fact, if x_i is the tip of a pinnacle on an edge $\{x_{j_1}, x_{j_2}\} \in E(H)$, then

$$\{x_j \in \mathrm{N}_G(x_i) \mid j > i\} = \{x_{j_1}, x_{j_2}\}$$

is a clique. Otherwise, if x_i is a vertex of the tree H with $i < n$, the set

$$\{x_j \in \mathrm{N}_G(x_i) \mid j > i\}$$

has exactly one element. In contrast, assume that distinct elements x_{j_1} and x_{j_2} belong to $\{x_j \in \mathrm{N}_G(x_i) \mid j > i\}$. Then by labelling the vertices as described above, both $d(x_{j_1}, x_n)$ and $d(x_{j_2}, x_n)$ are less than or equal to $d(x_i, x_n)$. Hence, there exist a path P_1 from x_{j_1} to x_n and a path P_2 from x_{j_2} to x_n neither of which contains x_i. This yields the existence of two paths from x_i to x_n, one via the adjacent vertex x_{j_1} and P_1, and the other via the adjacent vertex x_{j_2} and P_2, a contradiction to the fact that H is a tree. Thus, the labelling of $V(G)$ gives a perfect elimination ordering and, by Theorem 2.3, for each k, the kth homological shift ideal of $I = I(G^c)$ is

$$\mathrm{HS}_k(I) = \left(u \,\middle|\, \begin{array}{l} u \text{ is a squarefree monomial of degree } k+2 \\ \text{which has a source with respect to } G^c \end{array} \right). \tag{3.1}$$

Fix k in the set $\{0, \ldots, \operatorname{proj\,dim} I\}$. We will show that $\mathrm{HS}_k(I)$ has linear quotients with respect to the lexicographic ordering of generators with $x_1 > \cdots > x_n$. For this purpose, suppose that u and v are two monomials in the minimal set of monomial generators of $\mathrm{HS}_k(I)$, $u >_{\mathrm{lex}} v$, and

$$u : v = x_{i_1} \cdots x_{i_p}$$

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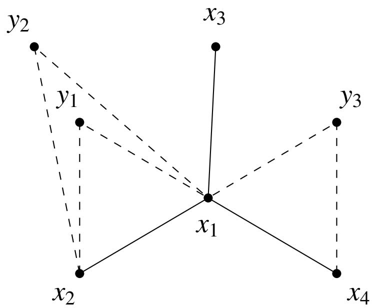

FIGURE 2. A tree on the vertex set $\{x_1, \ldots, x_4\}$ with three pinnacles.

with $p > 1$ and $i_1 < \cdots < i_p$. Since $\mathrm{HS}_k(I)$ is generated in a single degree, the monomial $v : u$ is also of degree p, say

$$v : u = x_{\ell_1} \cdots x_{\ell_p} \quad \text{with } \ell_1 < \cdots < \ell_p.$$

Notice that $u >_{\mathrm{lex}} v$ implies that

$$i_1 < \ell_1. \tag{3.2}$$

We show that there exists a monomial w in the minimal set of monomial generators of $\mathrm{HS}_k(I)$, such that $w >_{\mathrm{lex}} v$, and

$$w : v = x_{i_s}$$

for some $s = 1, \ldots, p$.

First, suppose that $i_1 \geq t$, so that x_{i_1} is a vertex of the tree H. As discussed above, the vertex x_{i_1} is adjacent to at most one vertex x_j of the tree H with $j > i_1$. From (3.2), x_{i_1} is adjacent to at most one of x_{ℓ_1} or x_{ℓ_2}; say

$$\{x_j \in \mathrm{N}_G(x_{i_1}) \mid j > i_1\} \cap \{\ell_1, \ell_2\} \quad \text{is either } \emptyset \text{ or } \{\ell_1\}.$$

Then the variable x_{i_1} becomes a source of $w = (v/x_{\ell_1})x_{i_1}$ with respect to G^c; here, $i_1 < \ell_2$ guarantees that $i_1 \neq \max w$. Furthermore, by (3.2), $w >_{\mathrm{lex}} v$. Thus, w is a monomial with the desired properties.

Next, suppose that $i_1 < t$, so that x_{i_1} is the tip of a pinnacle. From the labelling given to the vertices of G,

$$\{x_j \in \mathrm{N}_G(x_{i_1}) \mid j > i_1\} = \{x_{t_1}, x_{t_2}\} \tag{3.3}$$

for vertices x_{t_1} and x_{t_2} on an edge of the tree H which is on a pinnacle with the tip x_{i_1}. We consider three cases.

Case 1. If neither of the vertices x_{t_1} and x_{t_2} divides v, then x_{i_1} is a source of the monomial $w = (v/x_{\ell_1})x_{i_1}$ with respect to G^c. By (3.2), $i_1 < \ell_2 \leq \max w$. Hence, the squarefree monomial $w = (v/x_{\ell_1})x_{i_1}$ of degree $k+2$ is an element of $\mathrm{HS}_k(I)$ by (3.1). Furthermore, $w : v = x_{i_1}$ and, by (3.2), $w >_{\mathrm{lex}} v$, as desired.

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Case 2. Assume that exactly one of the variables x_{t_1} and x_{t_2} divides v, say x_{t_1}. Then by (3.3), the variable x_{i_1} is a source of the monomial $w = (v/x_{t_1})x_{i_1}$ with respect to G^c. Here, $i_1 < \max w$ is a consequence of $i_1 < \ell_1 < \ell_2 \leq \max v$ by (3.2). Now since $w = (v/x_{t_1})x_{i_1}$ has a source with respect to G^c, this squarefree monomial of degree $k + 2$ is an element of $\mathrm{HS}_k(I)$. Moreover, from the labelling of vertices, $i_1 < t_1$ because i_1 is a tip, while t_1 is a vertex of the tree H. So $w >_{\mathrm{lex}} v$ and w is a desired monomial.

Case 3. Finally, assume that x_{t_1} and x_{t_2} both divide v. Suppose that $t_1 < t_2$. Since v belongs to the minimal set of monomial generators of $\mathrm{HS}_k(I)$, by (3.1), it has a source variable with respect to G^c. Suppose that x_ℓ is a source of v for some ℓ. Since $\{x_{t_1}, x_{t_2}\}$ is an edge of H and we have assumed that $t_1 < t_2$, it follows that x_{t_1} is not a source of v with respect to G^c. In particular, $x_{t_1} \neq x_\ell$. We show that if either $t_1 < \ell$ or $\ell < t_1$, the variable x_ℓ remains a source in $w = (v/x_{t_1})x_{i_1}$. When $t_1 < \ell$, it is clear that x_ℓ is still a source of $w = (v/x_{t_1})x_{i_1}$ because the replacement of x_{i_1} by x_{t_1} in w occurs before x_ℓ. However, if $\ell < t_1$, then ℓ is not adjacent to i_1 because i_1 is a tip in G with $N_G(x_{i_1}) = \{x_{t_1}, x_{t_2}\}$. So the set

$$\{x_j \mid x_j \text{ divides } w \text{ and } \ell < j\}$$

is still the empty set. Moreover, $x_\ell \neq \max w$ because x_{t_2} divides w. Thus, x_ℓ is a source of w as well. Again, note that we have set the tip x_{i_1} lexicographically greater than the vertex x_{t_1} of H. Hence, $w >_{\mathrm{lex}} v$, as desired. □

PROPOSITION 3.2. *Let G be either the complete graph K_3 or obtained by adding some pinnacles to K_3. Then the edge ideal $I(G^c)$ has homological linear quotients.*

PROOF. Set $I = I(G^c)$. Assume that $V(G) = \{x_1, \ldots, x_n\}$ for some $n \geq 3$, the subgraph H induced on $\{x_{n-2}, x_{n-1}, x_n\}$ is a 3-clique, and G is constructed by adding pinnacles with tips $\{x_1, \ldots, x_{n-3}\}$. Fixing $0 \leq k \leq \operatorname{proj\,dim} I(G^c)$, we are going to show that $\mathrm{HS}_k(I(G^c))$ has linear quotients with respect to the lexicographic ordering of its minimal set of monomial generators induced by $x_1 > \cdots > x_n$. For this purpose, first we see that if w is an element of the minimal set of monomial generators of $\mathrm{HS}_k(I)$, then at most one of the vertices x_{n-2}, x_{n-1}, x_n can divide w. Indeed, the neighbourhood of each vertex of G intersects $\{x_{n-2}, x_{n-1}, x_n\}$ exactly in two vertices, and if more than one variable among x_{n-2}, x_{n-1} or x_n divides w, then w does not have a source with respect to G^c, which is a contradiction. (See Theorem 2.3 where the generators of $\mathrm{HS}_k(I)$ are described.)

Next, let u and v be two monomials in the minimal set of monomial generators of $\mathrm{HS}_k(I)$, $u >_{\mathrm{lex}} v$, and $u : v = x_{i_1} \cdots x_{i_p}$ with $p > 1$ and $i_1 < \cdots < i_p$. Since $\mathrm{HS}_k(I)$ is generated in a single degree, we may write

$$v : u = x_{\ell_1} \cdots x_{\ell_p} \quad \text{with } \ell_1 < \cdots < \ell_p.$$

Now on the one hand, $u >_{\mathrm{lex}} v$ implies that $i_1 < \ell_1 < \ell_2$. On the other hand, since at most one of the variables x_{n-2}, x_{n-1} or x_n divides v, we deduce that $\ell_1 \leq n - 3$. Hence, $i_1 < n - 3$. In particular, the vertices x_{ℓ_1} and x_{i_1} are the tips of two pinnacles in G.

Consequently, if $m = \max v$, then x_{i_1} is a source of $w = (v/x_m)x_{i_1}$. To see this, note that $i_1 < \ell_1 < \ell_2 \leq \max v$ and, by removing x_m from v, the monomial w can have only variables corresponding to some tips in its support. So x_{i_1} is not adjacent to any of the x_j in the support of v/x_m. So w is a monomial in $\mathrm{HS}_k(I)$, as described in Theorem 2.3, with $w >_{\mathrm{lex}} v$ and $w : v = x_{i_1}$. □

The statement of Proposition 3.2 does not hold if we replace K_3 by an arbitrary complete graph. For example, consider the graph G in Figure 1 obtained by adding two pinnacles to K_4, and refer to Remark 2.2 where $\mathrm{HS}_2(I(G^c))$ is determined.

Let G be a graph and k be a positive integer. A *k-colouring* of G is a mapping from $V(G)$ to $[k]$. If f is a k-colouring of G, then the colour of each edge $\{x_i, x_j\}$ is defined to be $\{f(x_i), f(x_j)\}$. A k-colouring f of the graph G is called a line-distinguishing colouring if every two distinct edges of G have distinct colours. The minimum number k for which G has a line-distinguishing k-colouring, denoted by $\lambda(G)$, is called the line-distinguishing chromatic number of G. The graph G is called λ-minimal in [9] if $\lambda(G - e) = \lambda(G) - 1$ for each edge e.

THEOREM 3.3 [16, Theorem 2.4]. *Let G be a chordal graph. Then G is λ-minimal if and only if G is either constructed by adding at least one pinnacle to each edge of a star or constructed by adding at least one pinnacle to each edge of the complete graph K_3.*

COROLLARY 3.4. *Let G be a λ-minimal chordal graph. Then the edge ideal $I(G^c)$ has homological linear quotients.*

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