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Non-Cohen–Macaulay Projective Monomial Curves with Positive h-Vector

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Abstract. We find an infinite family of projective monomial curves all of which have h-vector with no negative values and are not Cohen–Macaulay.

Let $\mathcal{S} = \{m_0, \dots, m_{p+1}\}$ be a sequence of integers with $0 < m_0 < m_1 < \cdots < m_{p+1}$, $\gcd(\{m_i\}) = 1$ and Γ the monoid generated by the elements of $\mathcal{S}$. Let k be a field and $R = k[u^{m_{p+1}}, t^{m_0}u^{m_{p+1}-m_0}, \dots, t^{m_p}u^{m_{p+1}-m_p}, t^{m_{p+1}}]$. Then R is the homogeneous coordinate ring of the projective monomial curve $C = \operatorname{Proj}(R)$, and is $\mathbb{N}$-graded by assigning degree 1 to its algebra generators. The degree of the curve C is m_{p+1}. The Hilbert function H_R of R is defined by $H_R(n) = \dim_k R_n$. The difference sequence ΔH_R is defined by $\Delta H_R(0) = 1$ and $\Delta H_R(n) = H_R(n) - H_R(n-1)$ for $n \geq 1$. The h-vector of R (or C) is defined to be the second difference sequence $\Delta_R^2 := \Delta(\Delta H_R)$. If R is Cohen–Macaulay then it is immediate that Δ_R^2 has no negative values (being the Hilbert function of $R/(u^{m_{p+1}}, t^{m_{p+1}})$). One might ask about the converse.

Question 1 If R is the homogeneous coordinate ring of a projective monomial curve and $\Delta^2 H_R$ has no negative values, is R Cohen–Macaulay?

Counterexamples to Question 1 appear to be unknown, although a non-monomial counterexample is given in [3, (b)p. 513]. In this note we give an infinite class of counterexamples. Our proof is based on the algorithm for computing ΔH_R which we describe in the next paragraph. More details can be found in [2] or [1]. We learned about the problem from Dilip Patil.

Let Θ_i be the set of all sums (repetitions allowed) of i elements of $\mathcal{S}$ and let

$$\mathfrak{M}_i := \Theta_i \setminus \Bigl(\bigcup_{j<i} \Theta_j\Bigr).$$

The set $\mathfrak{M}_i$ is the set obtained from the set Θ_i by removing those integers which have occurred in Θ_j, for some $j < i$. Starting with $\mathfrak{M}_0 = \Theta_0 = \{0\}$, $\mathfrak{M}_1 = \Theta_1 = \mathcal{S}$, and noting that Θ_{i+1} is the set of all sums of an element of $\mathcal{S}$ with an element of Θ_i, one can find the sets $\mathfrak{M}_i$ recursively. In [1] it was shown that $\Delta H_R(n) = \#\mathfrak{M}_n$ (by defining a ring $\operatorname{gr}(\mathcal{S})$ isomorphic to $R/u^{m_{p+1}}$ which has Hilbert function $H(n) = \#\mathfrak{M}_n$,

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denoting cardinality). Furthermore R is Cohen–Macaulay if and only if addition of m_{p+1} gives a map $\mathfrak{M}_i \to \mathfrak{M}_{i+1}$ for all $i \geq 0$.

For example, if $\mathcal{S} = \{1, 3, 4\}$ we get Macaulay's non-Cohen–Macaulay example $R = k[u^4, tu^3, t^3u, t^4]$. Here $\mathfrak{M}_0 = \{0\}, \mathfrak{M}_1 = \{1, 3, 4\}, \mathfrak{M}_2 = \{2, 5, 6, 7, 8\}, \mathfrak{M}_3 = \{9, 10, 11, 12\}$. The sequence $\Delta H_R(n) = \#\mathfrak{M}_n$ starts out $1, 3, 5, 4 \to$ and the h-vector $\Delta^2 H_R$ is $1, 2, 2, -1, 0 \to$. The quickest way to see that R is not Cohen–Macaulay is to observe that $\Delta^2 H_R(3) = -1$. But one might also observe that $4 + \mathfrak{M}_2 = 4 + \{2, 5, 6, 7, 8\} = \{6, 9, 10, 11, 12\} \not\subseteq \mathfrak{M}_3 = \{9, 10, 11, 12\}$. The simple test using the h-vector seems to work in all familiar examples, but the results of this paper show that it does not suffice in an infinite number of cases.

If $n \in \Gamma$, then there exists a unique integer i such that $n \in \mathfrak{M}_i$ and we write $\operatorname{ord}_{\mathcal{S}}(n) := i$. An $\mathcal{S}$-expression for n is a way of writing n as the sum of elements of $\mathcal{S}$ and $\operatorname{ord}_{\mathcal{S}}(n)$ is the smallest cardinality of an $\mathcal{S}$-expression of n. Order corresponds to degree in the grading of R, so the two words can be used interchangeably.

Notation 2 Throughout, $\{a, b, c, \dots\}$ is a set of integers whose elements are $a, b, c, \dots$. If S and T are two sets of integers then $S + T$ is $\{s + t | s \in S, t \in T\}$. If S is a set and n is an integer then $n + S$ is $\{n + s | s \in S\}$.

The examples: We did a systematic search for counterexamples to Question 1 using Mathematica programs based on the algorithm described above. We found no counterexamples with $p = 1$ (projective monomial curves in $\mathbb{P}^3$) or more generally for $\mathcal{S}$, an almost arithmetic progression (all but one element of $\mathcal{S}$ in an arithmetic progression). (If $\mathcal{S}$ is an arithmetic progression, then R is Cohen–Macaulay, so no counterexample is possible.) If $p = 2$, out of $91390 = \binom{40}{4}$ cases tested (all cases up to degree 40, of which 5619 are ruled out by not having greatest common divisor 1), we found 230 counterexamples, the smallest degree being 20. If $p = 3$ the smallest degree of a counterexample is 12, for example $\mathcal{S} = \{1, 2, 5, 8, 12\}$. For $p = 4, 5, 6, 7, 8$ the smallest degree of counterexample is respectively 14, 16, 18, 20, 22. (Does this pattern continue? It seems unlikely. In any event enough further cases were tested to show that 12 is the lowest degree of a counterexample.) If $p = 2$ one of the degree 20 examples is $\mathcal{S} = \{5, 9, 11, 20\}$. This is the first of our proposed infinite family of counterexamples, sets $\mathcal{S}_x$ of the form $\{-1 + 6x, 3 + 6x, 5 + 6x, 8 + 12x\}$ for x an integer greater than or equal to 1. Curiously, of the 230 counterexamples with $p = 2$ of degrees between 20 and 40, there are no counterexamples with degrees 21, 25, 26, 27, and only 2 of degree 33 (compared with 14 of degree 34 and 26 of degree 32), so the existence of counterexamples seems to be a bit delicate. Having the sum of the second and third coordinates equal to the fourth seems to be part of what makes our family work. However out of our original 230 counterexamples only 30 have this feature. A further 30 have the sum of the first and second coordinates equal to the fourth. These are obtainable from the previous 30 by interchanging t and u in the corresponding ring R. There are two examples with the sum of the first and third coordinates equal to the fourth.

For the rest of this paper let $\mathcal{S} = \mathcal{S}_x$ as defined above. Then $p = 2$ and $m_{p+1} = m_3 = 8 + 12x$. The sets $\mathfrak{M}_i$ decompose naturally into subsets according to the maximum number of copies of m_{p+1} in a minimal $\mathcal{S}$-expression of an integer $n \in \Gamma$. Based

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on this idea we define sets (i, j) where $0 \le j \le \min(i, 2x + 3)$ as follows:

(1) If $0 \le i \le 2x + 1$, then
   (a) $(i, 0) = \{(8 + 12x)i\}$;
   (b) $(i, 1) = (8 + 12x)i - (9 + 6x) + \{0, 4, 6\}$;
   (c) $(i, 2) = (8 + 12x)i - 2(9 + 6x) + \{0, 4, 6, 8, 12\}$;
   (d) $(i, j) = (8 + 12x)i - j(9 + 6x) + \{0, 4, 6, 8, 12, 6j\}$ for $3 \le j \le i$.

(2) If $i = 2x + 2$, then
   (a) If $0 \le j \le 2x + 1$ then (i, j) is defined as in case (1).
   (b) $(2x + 2, 2x + 2) = (8 + 12x - 9 - 6x)i + \{4, 6, 8, 12, 6j\}$ (same as (i, i) in case (1) except that the smallest element is omitted).

(3) If $i \ge 2x + 3$, then
   (a) If $0 \le j \le 2x + 1$ then (i, j) is defined as in cases (1) and (2).
   (b) If $j = 2x + 2$ then $(i, j) = (8 + 12x)i - j(9 + 6x) + \{6, 8, 12, 6j\}$ (the two smallest elements omitted from previous formulae).
   (c) $(i, 2x + 3) = \{(8 + 12x)i - (9 + 6x)j + 12\}$ (a singleton).

We sometimes refer to the (i, j) as *blocks* because (i, j) is (almost) a consecutive set if the elements of $\{\bigcup_j (i, j)\}$ are listed by increasing magnitude. (This is illustrated by Example 5 below.) The definitions of the (i, j) were formulated by studying the $\mathfrak{M}_i$ for various values of x, but so far they are just sets defined as above. We relate them to the $\mathfrak{M}_i$ by the following lemma.

Lemma 3 *The sets (i, j) are pairwise disjoint, and $\mathfrak{M}_i = \bigcup_j (i, j)$.*

Assuming Lemma 3, we can prove our main result.

Theorem 4 *For all integers $x \ge 1$, $\mathcal{S}_x = \{-1 + 6x, 3 + 6x, 5 + 6x, 8 + 12x\}$ gives a counterexample to Question 1. More explicitly, the corresponding homogeneous coordinate ring R has h-vector $1\ 3\ 5\ 6\ \cdots\ 6\ 5\ 0$ (with 6 in degrees $3 \le n \le 2x + 1$) and is not Cohen–Macaulay.*

Proof From the definitions $m_{p+1} + (i, j) = (i + 1, j)$, with the one exception $i = j = 2x + 2$. Given Lemma 3, this means (with one exception) that $\Delta^2 H_R(n)$ is the number of elements in the new block (n, n) that appears in degree n. $(0, 0)$ contains 1 element (namely 0); $(1, 1) = \{-1 + 6x, 3 + 6x, 5 + 6x\}$ contains 3 elements, and $(2, 2)$ contains 5 elements. For $3 \le n \le 2x + 1$, (n, n) contains 6 elements (and since $x \ge 1$, the interval $3 \le n \le 2x + 1$ is non-empty). The block $(2x + 2, 2x + 2)$ contains 5 elements. The block $(2x + 3, 2x + 2)$ contains 4 elements, so in passing from $(2x + 2, 2x + 2)$ to $(2x + 3, 2x + 2)$ one element disappears. But, it is replaced by the one element in $(2x + 3, 2x + 3)$ so $\Delta^2 H_R(2x + 3) = 0$ and the h-vector of R is $1\,3\,5\,6\ \cdots\ 6\,5\,0 \to$ (with 6 in degrees $3 \le n \le 2x+1$). In particular $\Delta^2 H_R(n)$ has no negative values. But R is not Cohen–Macaulay because adding m_{p+1} to $(2x+2, 2x+2)$ is not a map into $(2x + 3, 2x + 2)$. In the language of [1, (1.6)], Γ has one unstable element $(8 + 12x)(2x + 2) - (9 + 6x)(2x + 2) + 4 = (-1 + 6x)(2x + 2) + 4$ in degree $2x + 2$. ■

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The proof of Lemma 3 now proceeds in steps. Steps 1–3 show that the blocks (i, j) are pairwise disjoint. Step 4 shows that if $n \in (i, j)$ then $\operatorname{ord}_{\mathcal{S}}(n) \leq i$. Steps 5 and 6 are the technical ingredients for showing in step 7 that the reverse implication holds, *i.e.*, if $\operatorname{ord}_{\mathcal{S}}(n) = i$ then $n \in (i', j')$ with $i' \leq i$, and j' an integer such that $0 \leq j' \leq \min(i, 2x + 3)$. Lemma 3 is then proved in a brief final paragraph.

Step 1: First arrange the (i, j) in columns where, as we go down one step in each column, i is increased by 1 and j is increased by 2. Thus if $0 \leq i \leq 2x + 3$ column i has (i, i) at the bottom and for $i = 2\ell$ even, $(\ell, 0)$ at the top, for $i = 2\ell + 1$ odd, $(\ell + 1, 1)$ at the top. Column i contains $\ell + 1$ elements if $0 \leq i \leq 2x + 3$. If $i > 2x + 3$ and $i = 2\ell$ is even, then $(\ell, 0)$ is at the top of column i and $(\ell + x + 1, 2x + 2)$ is at the bottom. If $i > 2x + 3$ and $i = 2\ell + 1$ is odd, then $(\ell + 1, 1)$ is at the top of column i and $(\ell + x + 2, 2x + 3)$ is at the bottom of column i. If $i > 2x + 3$, then column i contains $x + 2$ elements.

Example 5 To help the reader visualize this setup, we illustrate it with

$$\mathcal{S} = \{5, 9, 11, 20\}$$

corresponding to $x = 1$. The columns have been shifted down as we proceed to the right, so that each row contains elements with the same value of i. Thus, after establishing Lemma 3, the rows turn out to be the $\mathfrak{M}_i$. The diagram is given twice, once with the names (i, j), and a second time with the actual sets. The columns begin at the left with column 0 and continue indefinitely to the right, with columns up to 8 shown. Because of space limitations the second diagram is split in two, with columns 0–4 in the first part and 5–8 in the second, the asterisk indicating where the two halves fit together. The blocks with the largest interval of their column are indicated in bold type (see step 3).

Diagram of indices (i, j):

$$\begin{array}{ccccccccc}
(\mathbf{0},\mathbf{0}) &&&&&&&& \\
& (\mathbf{1},\mathbf{1}) & (1,0) &&&&&& \\
&& (\mathbf{2},\mathbf{2}) & (2,1) & (2,0) &&&& \\
&&& (\mathbf{3},\mathbf{3}) & (3,2) & (3,1) & (3,0) && \\
&&&& (\mathbf{4},\mathbf{4})* & *(\mathbf{4},\mathbf{3}) & (4,2) & (4,1) & (4,0) \\
&&&&& (5,5) & (\mathbf{5},\mathbf{4}) & (\mathbf{5},\mathbf{3}) & (5,2) \\
&&&&&&& (6,5) & (\mathbf{6},\mathbf{4})
\end{array}$$

Corresponding diagram of the sets (i, j):

$$\begin{array}{ccccc}
\{\mathbf{0}\} &&&& \\
& \{\mathbf{5},\mathbf{9},\mathbf{11}\} & \{20\} && \\
&& \{\mathbf{10},\mathbf{14},\mathbf{16},\mathbf{18},\mathbf{22}\} & \{25,29,31\} & \{40\} \\
&&& \{\mathbf{15},\mathbf{19},\mathbf{21},\mathbf{23},\mathbf{27},\mathbf{33}\} & \{30,34,36,38,42\} \\
&&&& \{\mathbf{24},\mathbf{26},\mathbf{28},\mathbf{32},\mathbf{44}\}*
\end{array}$$

$$\begin{array}{cccc}
\{45,49,51\} & \{60\} && \\
*\{\mathbf{35},\mathbf{39},\mathbf{41},\mathbf{43},\mathbf{47},\mathbf{53}\} & \{50,54,56,58,62\} & \{65,69,71\} & \{80\} \\
\{37\} & \{\mathbf{46},\mathbf{48},\mathbf{52},\mathbf{64}\} & \{\mathbf{55},\mathbf{59},\mathbf{61},\mathbf{63},\mathbf{67},\mathbf{73}\} & \{70,74,76,78,82\} \\
&& \{57\} & \{\mathbf{66},\mathbf{68},\mathbf{72},\mathbf{84}\}
\end{array}$$

Step 2: The sets (i, j) in each column are pairwise disjoint. To see this, it is convenient to replace the set (i, j) by the (possibly larger) set $(i, j)'$ where if $j = 0, 1$ then $(i, j)' = (i, j)$ and if $j \geq 2$ then $(i, j)' = (8 + 12x)i - j(9 + 6x) + \{0, 4, 6, 8, 12, 6j\}$.

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(If $j = 2$ the element $6j$ coincides with one of the other elements. To form the set $(i, 2)'$, first use this formula, then discard the duplicate.) As i increases by 1 and j increases by 2 the integer $(8 + 12x)i - j(9 + 6x)$ decreases by 10 and the $6j$ term (always the largest) increases by 12 (which after subtracting 10 is a net increase of 2). Since the integers $0, 4, 6, 8, 12$ are in distinct congruence classes mod 10, the sets $(i, j)'$ corresponding to a particular column are pairwise disjoint, hence *a fortiori* so are the sets (i, j) in that column.

Step 3: The sets (i, j) are pairwise disjoint. Let $[i, j]$ be the set of all integers n such that $a \le n \le b$, where a is the smallest integer of (i, j) and b is the largest. ($[i, j]$ will be referred to as the interval of (i, j).) From the definition of the (i, j) and the discussion in Step 2, we see that as we go down one step in a column the intervals $[i, j]$ get larger with the upper bound increasing by 2 and the lower bound decreasing by 10 (or 6 in the bottom step of column $2x + 2$, 4 in the bottom step of columns i, i even with $i > 2x + 2$) except that the last interval $[\ell + x + 2, 2x + 3]$ in column i ($i = 2\ell + 1 > 2x + 2$) is a singleton which is contained in the previous interval $[\ell + x + 1, 2x + 1]$ (the smallest element of the latter being two less than the singleton). Notice that the parity of all integers in (i, j) is that of j, so all integers in a block of column i are even if i is even and odd if i is odd. Therefore to show that the (i, j) are disjoint it suffices to observe that the largest intervals in columns of the same parity do not overlap. This is illustrated by the following table. $\mathrm{LI}(i)$ indicates the largest interval in column i. The gap column is the smallest element of the largest interval in column i minus the largest element in the largest interval of column $i - 2$.

i	$\mathrm{LI}(i-2)$	$\mathrm{LI}(i)$	gap
$2 \le i \le 2x+1$	$[i-2, i-2]$	$[i, i]$	$12x - 6i + 10$
$2x+2$	$[2x, 2x]$	$[2x+2, 2x+2]$	2
$2x+3$	$[2x+1, 2x+1]$	$[2x+2, 2x+1]$	2
$i = 2\ell \ge 2x+4$	$[\ell+x, 2x+2]$	$[\ell+x+1, 2x+2]$	2
$i = 2\ell+1 > 2x+4$	$[\ell+x, 2x+1]$	$[\ell+x+1, 2x+1]$	2

The gap in the first row is greater than 0 since in this row $i \le 2x + 1$. Since the gaps are all positive Step 3 follows.

The calculations are all straightforward using the above description of the largest interval in each column, and the definitions of the (i, j). As one example we will work through the $i = 2\ell \ge 2x + 4$ row. If $i = 2\ell$ is even and $i \ge 2x + 4$ then the largest interval in column $i - 2$ is $[\ell + x, 2x + 2]$ and the largest interval in column i is $[\ell + x + 1, 2x + 2]$. These intervals respectively have largest value $(8 + 12x)(\ell + x) - (2x+2)(9+6x)+6(2x+2)$ and smallest value $(8+12x)(\ell+x+1)-(9+6x)(2x+2)+6$ with difference $(8 + 12x)(\ell + x + 1) - (9 + 6x)(2x + 2) + 6 - (8 + 12x)(\ell + x) + (9 + 6x)(2x + 2) - 6(2x + 2) = (8 + 12x) + 6 - 6(2x + 2) = 2$. As an illustration, if $x = 1$, $i = 6$, this yields $\ell = 3$, blocks $(5, 4)$, $(4, 4)$ and difference $46 - 44 = 2$ in Example 5.

Step 4: Every element in (i, j) is the sum of i elements of $\mathcal{S}$. This is obvious for $i = 0, 1$. If $i > 1$ then it is immediate from the definitions that

(a) if $j < i$, then $(i, j) \subseteq 8 + 12x + (i - 1, j)$ (recall that $8 + 12x = m_{p+1} = m_3$);

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(b) if $i \geq 1$, $(i,i) \subseteq (1,1) + (i-1,i-1)$.

The claim of this step then follows by induction.

Step 5: $8 + 12x + (i,j) = (i+1,j)$ with the exception that $(8+12x)(2x+2) - (9+6x)(2x+2) + 4 \in (2x+2, 2x+2)$ but $(8+12x)(2x+3) - (9+6x)(2x+2) + 4 = (8+12x)(2x+2) - (9+6x)(2x+2) + 6(2x+2) \in (2x+2, 2x+2)$ (this is the unstable element referred to in the proof of Theorem 4. In Example 5, 20+24=44).

Step 6: For $1 \leq i \leq 2x+3$, $(i,i) + (1,1) \subseteq \bigcup_{i' \leq i+1} (i', j')$. Recall that $(1,1) = (8+12x) - (9+6x) + \{0,4,6\}$. We check the claim for successive i. First of all $(1,1) + (1,1) = 2(8+12x) - 2(9+6x) + \{0,4,6,8,10,12\}$. The only element in $(1,1) + (1,1)$ that is not in $(2,2)$ is $2(8+12x) - 2(9+6x) + 10$, which equals $8 + 12x \in (1,0)$.

Similarly $(1,1) + (2,2) = (8+12x) - (9+6x) + \{0,4,6\} + 2(8+12x) - 2(9+6x) + \{0,4,6,8,12\} = 3(8+12x) - 3(9+6x) + \{0,4,6,8,10,12,14,16,18\}$. The only elements in $(1,1)+(2,2)$ but not in $(3,3)$ are $3(8+12x) - 3(9+6x) + 10 = 2(8+12x) - (9+6x) \in (2,1)$, $3(8+12x) - 3(9+6x) + 14 = 2(8+12x) - (9+6x) + 4 \in (2,1)$, and $3(8+12x) - 3(9+6x) + 16 = 2(8+12x) - (9+6x) + 6 \in (2,1)$.

Now suppose that $3 \leq i \leq 2x$. Then $(1,1) + (i,i) = (8+12x) - (9+6x) + \{0,4,6\} + (8+12x)i - (9+6x)i + \{0,4,6,8,12,6i\} = (8+12x)(i+1) - (9+6x)(i+1) + \{0,4,6,8,10,12,14,16,18,6i,6i+4,6(i+1)\}$. The elements not in $(i+1,i+1)$ come from the $10,14,16,18,6i,6i+4$ in the brackets. We have $(8+12x)(i+1) - (9+6x)(i+1) + 10 = (8+12x)i - (9+6x)(i-1) \in (i,i-1)$, $(8+12x)(i+1) - (9+6x)(i+1) + 14 = (8+12x)i - (9+6x)(i-1) + 4 \in (i,i-1)$, $(8+12x)(i+1) - (9+6x)(i+1) + 16 = (8+12x)i - (9+6x)(i-1) + 6 \in (i,i-1)$, and $(8+12x)(i+1) - (9+6x)(i+1) + 18 = (8+12x)i - (9+6x)(i-1) + 8 \in (i,i-1)$. If $i = 3$ then $(8+12x)(i+1) - (9+6x)(i+1) + 6i$ is the same as the 18 case just covered. If $i = 4$ then $(8+12x)(i+1) - (9+6x)(i+1) + 6i = (8+12x)(i-1) - (9+6x)(i-3) + 4 \in (3,1)$. If $i > 4$ then $(8+12x)(i+1) - (9+6x)(i+1) + 6i = (8+12x)(i-2) - (9+6x)(i-5) + 6(i-5) \in (i-2,i-5)$. Finally $(8+12x)(i+1) - (9+6x)(i+1) + 6i + 4 = (8+12x)i - (9+6x)(i-1) + 6(i-1) \in (i,i-1)$.

Now consider $i = 2x+1$. Then $(1,1) + (2x+1,2x+1) = (8+12x) - (9+6x) + \{0,4,6\} + (8+12x)(2x+1) - (9+6x)(2x+1) + \{0,4,6,8,12,6(2x+1)\} = (8+12x)(2x+2) - (9+6x)(2x+2) + \{0,4,6,8,10,12,14,16,18,6(2x+1),6(2x+1)+4,6(2x+2)\}$. The elements of $(1,1) + (2x+1,2x+1)$ not in $(2x+2,2x+2)$ come from the $0,10,14,16,18,6(2x+1),6(2x+1)+4$. The $10,14,16,18,6(2x+1)+4$ are handled in the same way as the previous case, yielding an element of $(2x+1,2x)$. If $x = 1$ then the $6(2x+1)$ case coincides with the 18 case. If $x > 1$ then $(8+12x)(2x+2) - (9+6x)(2x+2) + 6(2x+1) = (8+12x)(2x-1) - (9+6x)(2x-4) + 6(2x-4) \in (2x-1,2x-4)$. Finally the 0 case. $(8+12x)(2x+2) - (9+6x)(2x+2) = (8+12x)(2x-1) - (9+6x)(2x-2) + 6(2x-2) \in (2x-1,2x-2)$.

Now consider $i = 2x+2$. Then $(1,1) + (2x+2,2x+2) = (8+12x) - (9+6x) + \{0,4,6\} + (8+12x)(2x+2) - (9+6x)(2x+2) + \{4,6,8,12,6(2x+2)\} = (8+12x)(2x+3) - (9+6x)(2x+3) + \{4,6,8,10,12,14,16,18,6(2x+2),6(2x+2)+4,6(2x+3)\}$. Of

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these elements only $(8+12x)(2x+3)-(9+6x)(2x+3)+12 \in (2x+3, 2x+3)$ so we have to say where all the other elements are. Of the integers in brackets, 10,14,16,18, and $6(2x+2)+4$ all yield an element of $(2x+2, 2x+1)$ as in the discussion of the $3 \le i \le 2x$ case, replacing i by $2x+2$ in the argument for $10, 14, 16, 18, 6i+4$. If $x = 1, (8+12x)(2x+3)-(9+6x)(2x+3)+6(2x+2) = 49 \in (3, 1)$. If $x > 1$, $(8+12x)(2x+3)-(9+6x)(2x+3)+6(2x+2) \in (2x, 2x-3)$ by the argument for $6i$ in the $3 \le i \le 2x$ case, replacing i by $2x+2$. And $(8+12x)(2x+3)-(9+6x)(2x+3)+6(2x+3) = (8+12x)(2x+3)-(9+6x)(2x+1) \in (2x+3, 2x+1)$ leaving only 4,6,8. We have $(8+12x)(2x+3)-(9+6x)(2x+3)+4 = (8+12x)(2x-1)-(9+6x)(2x-3)+6(2x-3) \in (2x-1, 2x-3)$, $(8+12x)(2x+3)-(9+6x)(2x+3)+6 = (8+12x)2x-(9+6x)(2x-1)+6(2x-1) \in (2x, 2x-1)$, and $(8+12x)(2x+3)-(9+6x)(2x+3)+8 = (8+12x)(2x+1)-(9+6x)(2x+1)+6(2x+1) \in (2x+1, 2x+1)$. If $x = 1$ we need a separate calculation for $(8+12x)(2x+3)-(9+6x)(2x+3)+4 = 29 = 9+20 \in (2, 1)$.

Now consider $i = 2x+3$. We have $(1,1)+(2x+3, 2x+3) = (8+12x)-(9+6x)+\{0,4,6\}+(8+12x)(2x+3)-(9+6x)(2x+3)+12 = (8+12x)(2x+4)-(9+6x)(2x+4)+\{12,16,18\}$. But $(2x+4, 2x+4)$ is empty so we have to find where all three elements are. As in the $3 \le i \le 2x$ case we have $(8+12x)(2x+4)-(9+6x)(2x+4)+16 = (8+12x)(2x+3)-(9+6x)(2x+2)+6 \in (2x+3, 2x+2)$ and $(8+12x)(2x+4)-(9+6x)(2x+4)+18 = (8+12x)(2x+3)-(9+6x)(2x+2)+8 \in (2x+3, 2x+2)$. For the 12 case we have $(8+12x)(2x+4)-(9+6x)(2x+4)+12 = (8+12x)(2x+1)-(9+6x)(2x)+6(2x) \in (2x+1, 2x)$.

This completes the proof of Step 6.

Step 7; Every element $n \in \Gamma$ with $\operatorname{ord}_{\mathcal{S}}(n) = i$ occurs in $\bigcup_{i' \le i}(i', j')$. To see this, suppose we have such an n. Choose an $\mathcal{S}$-expression for n of length i with as many as possible copies of $8+12x$, say $n = a_1 + a_2 + \cdots + a_{i-j} + j(8+12x)$ with all $a_\ell \in (1,1)$. If $j = i$ then $n \in (i, 0)$. If $\sum_{\ell=1}^{i-j} a_\ell \in (i-j, i-j)$ Step 7 follows from Step 5. Otherwise suppose there is some integer k, $2 \le k \le i-j$ such that $\sum_{\ell=1}^{k} a_\ell \notin (k,k)$. ($k = 1$ is impossible since $a_1 \in (1,1)$.) Pick the smallest such k. Then by Step 6, $\sum_{\ell=1}^{k} a_\ell \in (k', j')$ with either $k' < k$ or $k' = k$ and $j' < k$. If $k' < k$ then by Step 4 we have an $\mathcal{S}$-expression for n of length less than i. If $k' = k$ and $j' < k$ then by step 5, n can be rewritten as the sum of i elements of $\mathcal{S}$ with more than j copies of $8+12x$, both of which yield a contradiction. This completes the proof of Step 7.

We can now prove Lemma 3. Disjointness of the (i, j) was established in Step 3. Step 7 says that $\mathfrak{M}_i \subseteq \bigcup_{i' \le i}(i', j)$. Suppose $x \in \mathfrak{M}_i$. Then $x \in (i', j)$ for unique i', j. If $i' < i$ then by Step 4, $x \in \mathfrak{M}_{i''}$ for $i'' \le i'$ (hence $i'' < i$). This is a contradiction, and Lemma 3 follows. Hence also does Theorem 4.

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