Proceedings of the Edinburgh Mathematical Society (1980), **23**, 225–228 ©

ON THE NON-VANISHING OF POINCARÉ SERIES

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(Received 2nd April 1979)

1.

Let $M = SL(2, \mathbf{Z})$ be the classical modular matrix group. One form of the Poincaré series on M is

$$g_q(z, m) = \tfrac{1}{2} m^{2q-1} \sum_{M_\infty \backslash M} (cz+d)^{-2q} \exp(2\pi i m A z); \tag{1.1}$$

here $z \in \mathbf{H} = \{z = x + iy \colon y > 0\}$, $q \geqq 2$ and $m \geqq 1$ are integers, and the summation is over a complete system of matrices $(ab\colon cd)$ in M with different lower row. The problem of the identical vanishing of the Poincaré series for different values of m and q goes back to Poincaré.

It is known that g_q is a modular form of weight q holomorphic in $\mathbf{H}$, i.e.,

$$g_q(Az, m)(A'z)^q = g_q(z) \;\forall\; A \in M. \tag{1.2}$$

Since the translation $(11\colon 01) \in M$, g_q has a Fourier series

$$g_q(z, m) = \sum_{r=1}^{\infty} c_q(r, m) \exp(2\pi i r z), \tag{1.3}$$

the summation being over $r \geqq 1$ because of $m \geqq 1$.

From the Petersson scalar product formula, we deduce at once that $g_q(z, m) \equiv 0$ if and only if $c_q(m, m) = 0$. On the other hand there is an explicit formula for c_q as an infinite series involving Bessel functions and certain complicated number-theoretic sums (Kloosterman sums).

Very recently R. A. Rankin (**3**) made use of the above to prove the following result. *For each $\varepsilon > o$ there is a $q_0(\varepsilon) > 0$ such that $g_q(z, m) \not\equiv 0$ if*

$$1 \leqq m \leqq q^{2-\varepsilon}, \quad q \geqq q_0. \tag{1.4}$$

(Actually Rankin's result was a bit stronger than this.)

In the present note we wish to generalise the above theorem by replacing M by an arbitrary fuchsian group Γ that acts on $\mathbf{H}$ and has translations. Thus Γ may be infinitely generated and it may be of the second kind. However, the inequality we obtain is weaker than (1.4); the exponent is only $\frac{4}{3}$.

We shall prove:

Theorem. *Let Γ be a fuchsian group acting on $\mathbf{H}$ and possessing translations. Let the*

Poincaré series $g_q(z, m)$ *be defined as in* (1.1). *Then there exist positive constants* q_0 *and* m, *depending only on* Γ, *such that* $g_q(z, m) \not\equiv o$ *for*

$$q \geqq q_0, m \leqq m_{12} q^{4/3}.$$

2.

Let Γ be a fuchsian group acting on $\boldsymbol{H}$ and possessing translations, of which the smallest is $z \to z+\lambda$, $\lambda > 0$. We represent Γ as a group of matrices $G = \{A = (ab\colon cd),\ ad - bc = 1\}$ with real entries, and we assume without loss of generality that $-I = (-1 \quad 0\colon 0 \quad -1) \in G$, so that $G/\pm I \cong \Gamma$. Denote by $G_\infty = <(1\lambda\colon 0 \quad 1)>$ the stabiliser of ∞.

The Poincaré series is defined as before by

$$g_q(z, m = \tfrac{1}{2} m^{2q-1} \sum_{G_\infty \backslash G} (cz+d)^{-2q} \exp(2\pi i m A z), \tag{2.1}$$

with the same conditions on q and m and the same meaning for $G_\infty \backslash G$. It has the Fourier expansion

$$g_q(z, m) = \sum_{r=1}^{\infty} c_q(r, m) \exp(2\pi i r z/\lambda), \tag{2.2}$$

which converges absolutely uniformly on compact subsets of $\boldsymbol{H}$. Before writing down the formula for c_q we introduce some further notation (**1**, 270–71, 297–98).

The set of positive numbers $|c|$ such that $(..\colon c.) \in G$ is discrete: $0 < c_0 < c_1 < \ldots \to \infty$. For $c = c_n > 0$ let

$$D_c = \{d\colon (..; cd) \in G,\ 0 \leqq -d < c\lambda\};$$

this is a finite set. Now define the Kloosterman sum

$$W_c(M, \nu) = \sum_{d \in D_c} \exp\left\{2\pi i \frac{\nu a + m d}{c\lambda}\right\}, \tag{2.3}$$

where $(ab\colon cd) \in G_\infty \backslash G$. Also let

$$J_r(z) = \sum_{n=0}^{\infty} \frac{(-1)^n (z/2)^{r+2n}}{n!\Gamma(r+n+1)}$$

be the Bessel function of the first kind. Then

$$c_q(m, m) = m^{2q-1}\left\{1 + \frac{2\pi(-1)^q}{\lambda} \sum_{n=0}^{\infty} \frac{W_{c_n}(m, m)}{c_n} J_{2q-1}\left(\frac{4\pi m}{c_n \lambda}\right)\right\}. \tag{2.4}$$

The convergence of (2.2) and formula (2.4) are proved in (**1**, 276–7, 295–8) on the assumption that Γ is an H-group, but examination of the proof shows this assumption to be unessential.

3.

In showing that $c_q \neq 0$ for certain values of m, Rankin was able to use very accurate estimates for the Kloosterman sum. But for the general fuchsian group only the trivial estimate, obtained by setting each exponential equal to 1 in (2.3), is available. We shall employ the following

Lemma. *Let* $\sum_{\alpha,\beta}$ *denote a sum over*

$$\alpha \leqq c < \beta, \quad 0 \leqq -d < c\lambda.$$

Then

$$\sum_{\alpha,\beta} c^{-r} < m_1\alpha^{2-r} - m_2\beta^{2-r}, \quad r > 2 \tag{3.1}$$

$$\sum_{\alpha,\beta} c^{-r} < m_3\beta^{2-r} - m_4\alpha^{2-r}, \quad r < 2 \tag{3.2}$$

where $m_1, \dots$ *are positive constants depending only on* Γ.

This result appears as a lemma in (**2**), p. 400. Again it is proved under the assumption that Γ is an H-group and again this hypothesis is unnecessary.

Some needed results on Bessel functions can be quoted directly from (**3**). Write

$$x_\nu = (1-\sigma^4)^{1/2}, \; y_\nu = (1+\sigma^4)^{1/2}, \quad \sigma = \nu^{-1/6};$$

$A_1, A_2, \dots$ are absolute constants.

For all $x \geqq 0$ and $\nu \geqq 1$,

$$|J_\nu(\nu x)| \leqq (2\pi\nu^{-1/2})(ex/2)^\nu. \tag{3.3}$$

This is Lemma 4.1. From Lemma 4.3:

$$|J_\nu(\nu x)| \leqq A_2(x^2-1)^{-1/4}\nu^{-1/2}, \quad x \geqq y_\nu, \quad \nu \geqq 15. \tag{3.4}$$

Finally, combining (4.9), (4.10) of (**3**) we get

$$|J_\nu(\nu x)| \leqq A_3\nu^{-1/3}, \quad 2 \geqq x \geqq 2/e. \tag{3.5}$$

Let $q \geqq 8$,

$$S = \sum_{c>0} \frac{W_c(m,m)}{c} J_{2q-1}(4\pi(m/c\lambda)),$$

and write

$$\nu = 2q-1 \geqq 15, \quad Q = 4\pi m/\nu\lambda, \quad x = Q/c\lambda.$$

Then

$$\left|S\right| \leqq \sum_{c>0} 1/c \left| J_\nu(\nu(Q/c)) \right| \sum_{0 \leqq -d < c\lambda} 1 = \sum_{0,\infty} 1/c \left| J_\nu(\nu(Q/c)) \right|$$

$$= \sum_{0,Q_1} + \sum_{Q_1,Q_2} + \sum_{Q_2,\infty} = S_1 + S_2 + S_3,$$

with

$$Q_1 = Q/2\lambda, \quad Q_2 = eQ/2\lambda.$$

In S_3 we have $x = Q/c\lambda \geqq 0$, so from (3.3) and (3.1) with $\beta = \infty$, we obtain

$$|S_3| \leqq A_4 \nu^{-1/2} \sum_{Q_2, \infty} 1/c\left(\frac{eQ}{2c\lambda}\right)^{\nu} = A_4 \nu^{-1/2}\left(\frac{eQ}{2\lambda}\right)^{\nu} \Sigma \frac{1}{c^{\nu+1}}$$

$$\leqq m_5 \nu^{-1/2}\left(\frac{eQ}{2\lambda}\right)^{\nu}\left(\frac{eQ}{2\lambda}\right)^{1-\nu} = m_6 \nu^{-1/2} Q \tag{3.6}$$

$$= m_7 m \nu^{-3/2}.$$

In S_1, $x > 2 > y_\nu$, so (3.4) and (3.2) give

$$|S_1| \leqq A_2 \nu^{-1/2}(2^2-1)^{-1/4} \sum_{0, Q_1} c^{-1} \leqq m_8 \nu^{-1/2} Q_1 = m_9 m \nu^{-3/2}. \tag{3.7}$$

Finally $2/e < x \leqq 2$ in S_2; hence by (3.5) and (3.2),

$$|S_2| \leqq A_3 \nu^{-1/3} \sum_{Q_1, Q_2} c^{-1} \leqq m_{10} \nu^{-1/3} \frac{eQ}{2\lambda} = m_{11} m \nu^{-4/3}.$$

It follows that $|S| < \lambda/2\pi$ for $q \geqq q_0$, $m \leqq m_{12} q^{4/3}$. Then (2.4) shows that

$$|c_q(m, m)| m^{1-2q} \geqq 1 - \frac{2\pi}{\lambda}|S| > 0,$$

completing the proof of the THEOREM of Section 1.

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https://doi.org/10.1017/S0013091500003102 Published online by Cambridge University Press