Third Meeting, January 9th, 1885.

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On the problem to construct the minimum circle enclosing n given points in a plane.

By Professor Chrystal.

If we consider the circle circumscribing any triangle ABC (see figures 11, 12), and diminish its radius still causing it to pass through A and B; then if ACB be an acute angle, C passes without the circle, but if ACB be an obtuse angle, C remains within the circle. If C be a right angle, the radius of the circle, being $\frac{1}{2}$AB, cannot be farther diminished.

This result, which will be used in what follows, leads in the first place to the result, otherwise obvious, that the minimum circle enclosing three points is the circle through them when they form an acute-angled or right-angled triangle, but the circle having as diameter the line joining the two which are most distant when the triangle which they form is obtuse-angled.

Let there be n points. We can always select m of them which form the vertices of a convex polygon enclosing all the others. The problem therefore reduces to finding the minimum circle enclosing a convex m-gon.

Since the m-gon lies wholly on one side of the unlimited straight line, of which any one of its sides is a part, we may regard this line as a circle of infinite radius wholly enclosing the m-gon. Let us diminish the radius of this circle continuously (always supposing it to pass through two particular vertices of the m-gon). Then one or other of two things will happen. Either the radius will diminish down to its minimum value (half the selected side of the m-gon) before the circle passes over any vertex of the m-gon; in which case the circle on the selected side as diameter contains all the n points, and must be the minimum circle, since no less circle can contain the two vertices started with: or the circle will first pass over a third vertex of the m-gon. In the latter case if the three vertices involved form an acute-angled triangle, the minimum circle is reached, for this circle contains all the n points and is the least that can contain the three in question; if the triangle be obtuse-angled, we observe first that

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the obtuse angle cannot be opposite the selected side, for by supposition we have not reached the minimum radius which corresponds to a right angle, the obtuse angle must therefore be at one or other of the extremities of the selected side. Throwing out the vertex at which the obtuse angle occurs, let us still farther diminish the radius, the circle still passing through the two vertices retained; by this process the neglected vertex passes inside the circle and need not be farther considered; and we shall arrive either as before at a minimum circle on the line joining the two retained vertices as diameter, or at a circle passing through three vertices of the polygon, which will be a minimum circle if these vertices form an acute-angled triangle, and may be farther diminished if this triangle be obtuse-angled, it being remarked as before that the obtuse angle cannot be at the vertex last taken in, and so on.

Since the radius of the circle is being continually diminished this process must come to an end, and that in one or other of two ways, viz:—either the minimum circle must appear as a circle on a side or diagonal of the m-gon as diameter (which side or diagonal will be the greatest distance between any pair of the m points), or it will appear as a circle described through three of the vertices of the m-gon, forming an acute-angled triangle. The minimum circle may in certain limiting cases contain more of the vertices than the two or the three that determine it, but this does not affect the conclusion.

It is interesting topologically to remark that point systems divide themselves into two distinct classes, each of course with limiting cases, according as their minimum circle is determined by two of the points, or by three of them.

The above discussion suggests at once the following practical method for determining the minimum circle of a point system.

Construct by taking m of the points a convex polygon enclosing them all. Take any side of this m-gon, and find the vertex at which it subtends the least angle. If this least angle be right or obtuse, the minimum circle is the circle on the chosen side as diameter. If the triangle formed by the three vertices be acute-angled its circumscribed circle is the minimum circle; if not, take the side opposite the obtuse angle, find the vertex at which it subtends the least angle, and go on as before.

The number of steps in the construction will depend of course on the side first selected. It is clear however that no side or diagonal of the polygon can occur more than once as the chosen side, so that $\frac{1}{2}m(m-1)$ is an upper limit to the number of steps.

In any particular case it might be more expeditious first to try whether a circle on the greatest side or diagonal encloses all the points; if this is not so, then the minimum circle is obtained by finding that triangle formed by three of the m points which is acute-angled and has the greatest circumscribing circle.

The theory given for the case of n points in a plane may be extended to n points in space. Consider any three points A,B,C, and a fourth D. Through the four points we may describe a sphere. Let us suppose the sphere always to pass through ABC, and its radius to vary; the circle circumscribing ABC thus forms the base of a variable spherical segment. Through the axis of this circle we can always draw a plane containing D. This plane will cut the sphere in a great circle, and the circle ABC in two points K and L. The segment KDL will be greater than, equal to, or less than, a semicircle, according as the spherical segment is greater than, equal to, or less than, a hemisphere. In the first case, if we diminish the radius of the sphere, D will pass without the spherical segment. Hence if all the four spherical segments are greater than hemispheres, that is if the centre of the sphere is within the tetrahedron ABCD, the sphere passing through ABCD is the minimum sphere. In the second case the radius cannot be further diminished, unless one of the angles (C say) of the triangle ABC be obtuse, and then the sphere may be reduced to a sphere having the circle described about ABD as a great circle. If the triangle ABD is acute-angled, the sphere now got is the least. If ABD is obtuse angled, the sphere may be finally reduced to the sphere having as diameter the side opposite the obtuse angle. In the last case, if the radius of the sphere be diminished, D will pass within; and the possibility of further reductions may be discussed as in the second case.

We may therefore classify point systems in space of three dimensions into three classes according as the minimum sphere is determined by four, three, or two of the points. A chain of construction for finding the minimum sphere may be given analogous to that for the minimum circle in the case of co-planar points; but the details are naturally more complicated.

An analytical theory of the above geometrical relations doubtless exists, and it might be extended to space of n dimensions.

The plane problem might be generalised in various ways. For example, if we project orthogonally the n co-planar points upon another plane, we have the problem: To find the ellipse of minimum area enclosing n given points, which shall be similar and similarly situated to a given ellipse.

The problem of the circle, or sphere, enclosing n points is interesting in connection with the theory of functions of a complex variable, and in the allied subject of the Theory of the potential. There are also certain questions of a practical nature regarding the representation of observations by curves of a given description which lead to similar problems. Let us suppose for example that the points in figure 13 represent the plotted results of a series of observations, and that we have some reason to suppose that the points would lie on an ellipse whose foci are F and F′, if the results were freed from errors of observation. If now we draw the least ellipse having F and F′ as foci which includes all the points, and the greatest confocal ellipse which excludes them all, we can assign the utmost error that can be incurred by representing the observations by means of an ellipse having F and F′ as foci; and can thus tell whether, in view of the possible errors of observation, such an ellipse is a sufficiently accurate representation or not.

It was a practical question of the above nature that first suggested to me the problem of the minimum circle enclosing n points. I drew Professor Tait's attention to it, and he gave an interesting topological discussion of the case of four points, in an address delivered before this society, and published in the *Philosophical Magazine* for January 1884 (Fifth Series, vol. xvii., pp. 30-46). In consequence of the interest shown in the problem by Professor Hermite, I worked out for myself and sent to him the solution above given. I learned a day or two before communicating it to the Mathematical Society of Edinburgh that the problem had originally been proposed by Professor Sylvester (*Quarterly Journal of Mathematics*, I., p. 79); and that a solution had been given by him in an article in the *Philosophical Magazine* more than twenty years ago (1860, Fourth Series, vol. xx., pp. 206-212). I have since consulted this paper, and find that the solution there given is due to Peirce. It is only briefly indicated, but appears to be substantially identical with the one I have given above. An extension is given to points on the surface of a sphere; but no mention is made of the three dimensional problem. The connection between the minimum circle problem and the main part of Sylvester's article, which deals with the approximate representation of $\sqrt{x^2+y^2}$ and $\sqrt{x^2+y^2+z^2}$ by means of linear functions of x, y, z, is very interesting.

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