ADJOINT ABELIAN OPERATORS ON BANACH SPACE

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I. In the first part of this paper we introduce a new class of operators, mentioned in the title. It is easy to say that these are a generalization of self-adjoint operators for Hilbert space. This is deceptive since it implies that the definition of self-adjointness is forced into the unnatural setting of a Banach space. We feel that the definition of adjoint abelian preserves the obvious distinction between a space and its dual. Certain attractive properties of self-adjoint operators have already been singled out and carried over to Banach space. Specifically, we mention the notion of hermitian (see **17**; **11**), and spectral type operators (see **4**). There is some comparison of these concepts later.

Let X be a Banach space. We define a duality map ϕ: $X \rightarrow X^*$ as follows. Given $x \in X$, by the Hahn-Banach theorem, there exists an $x^* \in X^*$ such that $||x^*|| = ||x||$ and $x^*(x) = ||x||^2$. Set $\phi(x) = x^*$, and $\phi(\lambda x) = \bar{\lambda}x^*$, and define ϕ on the rest of X in the same manner. In general, ϕ is not unique, linear or continuous. The duality map ϕ induces a semi-inner product $[\cdot, \cdot]$ if we set $[x, y^*] = y^*(x)$. It is clear that $[\cdot, \cdot]$ has the following properties (see **11** for an elegant discussion of the semi-inner product):

(1) $[x_1 + x_2, y^*] = [x_1, y^*] + [x_2, y^*]$;

(2) $\lambda[x, y^*] = [\lambda x, y^*] = [x, \lambda y^*] = [x, (\bar{\lambda}y)^*]$;

(3) $[x, x^*] \geqq 0$ and equality implies that $x = 0$;

(4) $|[x, y^*]|^2 \leqq [x, x^*][y, y^*]$.

Definition. Let A be a bounded linear operator mapping the Banach space X into itself. If there exists a duality map ϕ: $X \rightarrow X^*$, such that $A^*\phi = \phi A$, then A is adjoint abelian (equivalently, $(Ax)^* = A^*x^*$ for all $x \in X$).

If X is a Hilbert space, then the duality map is unique, and the adjoint abelian operators are precisely the self-adjoint ones.

Despite the generality of the definition, a surprisingly large number of properties of self-adjoint operators carry over to adjoint abelian operators, particularly for reflexive Banach spaces.

From now on, we will assume that X has been equipped with a semi-inner product and duality map.

LEMMA 1. *If A and B are adjoint abelian with respect to the same semi-inner product, and A commutes with B, then AB is adjoint abelian. If A is invertible, then A^{-1} is adjoint abelian.*

Received December 29, 1967 and in revised form, April 25, 1968. The author gratefully acknowledges the support of the National Science Foundation, Grant number NSF GP-8178.

The proof is clear.

Note, however, that if A and B are adjoint abelian, this does not imply that $A + B$ is adjoint abelian. In fact, the adjoint abelian operators are not even invariant under translation by real scalars. We will illustrate this with an example later.

Definition. An operator T on a Banach space X is hermitian if $||I + i\alpha T|| = 1 + o(\alpha)$ for α real. This is equivalent to $W(T)$ real, where

$$W(T) = \{[Tx, x^*]: ||x|| = 1\}$$

and [,] is any semi-inner product on X. If $W(T)$ is real in one semi-inner product, it is real in every semi-inner product. A discussion of the numerical range for a semi-inner product space and the equivalence of the two definitions can be found in (**11**).

LEMMA 2. *If A is adjoint abelian, then A^{2n} is hermitian, and $W(A^2)$ is positive.*

Proof. Let $||x|| = 1$. Then $[A^{2n}x, x^*] = [A^n x, (A^n x)^*] = ||A^n x||^2 \geqq 0$.

COROLLARY 1. *If A is adjoint abelian, then $\sigma(A)$ is real.*

Proof. Since A^2 is hermitian, $\sigma(A^2) \subset$ convex hull $W(A^2)$ is real and positive; see (**11**). The Spectral Mapping Theorem thus implies that $\sigma(A)$ is real.

Even though $\sigma(A)$ is real, the numerical range $W(A)$ need not be. For that reason, we include the next lemma.

LEMMA 3. *Let A be invertible and adjoint abelian. If* $\operatorname{Re} A \geqq 0$ (*i.e.,* $\operatorname{Re} W(A) \geqq 0$), *then* $\operatorname{Re} A^{-1} \geqq 0$.

The proof is clear.

Lemma 3 is not true for operators in general. An example for L_p, $p \neq 2$, can be found in (**18**).

THEOREM 1. *If A is adjoint abelian, then*

$$||Ax||^n \leqq ||A^n x||\,||x||^{n-1} \quad \textit{for } n = 1, 2, \ldots .$$

Proof. Note that

$$||Ax||^2 = |[A^2x, x^*]| \leqq ||A^2x||\,||x||.$$

Thus, $||Ax||^{2n} \leqq ||A^2x||^n||x||^n = ||A(Ax)||^n||x||^n \leqq ||A^{n+1}x||\,||Ax||^{n-1}||x||^n$ or $||Ax||^{n+1} \leqq ||A^{n+1}x||\,||x||^n$, which completes the induction argument.

COROLLARY 2. *If A is adjoint abelian, then $||A|| = R_{\mathrm{sp}}(A)$, the spectral radius of A.*

Proof. This follows directly from the fact that $||A^n|| = ||A||^n$.

COROLLARY 3. *If A is adjoint abelian, and $A^n x = 0$ for some $x \in X$, then $Ax = 0$.*

https://doi.org/10.4153/CJM-1969-058-4 Published online by Cambridge University Press

We will present a more complicated version of the next lemma later. Lemma 4 has been included because of its simplicity, and because it avoids an additional hypothesis necessary in the later version (Theorem 5).

LEMMA 4. *Let A be adjoint abelian and let $Ax_i = \lambda_i x_i$ for $i = 1, 2$, where $\lambda_1 \neq \lambda_2$. Then $[x_1, x_2^*] = 0$.*

Proof. Note that

$$\lambda_1[x_1, x_2^*] = [Ax_1, x_2^*] = [x_1, (Ax_2)^*] = [x_1, \lambda_2 x_2^*] = \lambda_2[x_1, x_2^*]$$

since λ_2 is real. Hence, $(\lambda_1 - \lambda_2)[x_1, x_2^*] = 0$, and the conclusion follows.

The following lemma appears in (**12**), with slight modification. The proof is short, and we include it since it contains several useful facts.

LEMMA 5. *Let A be adjoint abelian. Let $f(t)$ be continuous on $\sigma(A^2)$. Then $||f(A^2)|| \leqq 8||f(t)||_\infty$ ($||f(t)||_\infty = \sup\{|f(t)|: t \in \sigma(A^2)\}$).*

Proof. For any operator T, $||T|| \leqq 4|W(T)|$ (the numerical radius of T) (**11**). For B hermitian, $|W(B)| = R_{\text{sp}}(B)$, the spectral radius of B; see (**12**). If B is hermitian and $p(t)$ is a polynomial with real coefficients, then $p(B)$ is hermitian. Let $p(t) = p_{\text{r}}(t) + ip_{\text{i}}(t)$, where p_{r} and p_{i} are the real and imaginary parts of the polynomial p. Let $B = A^2$. Using the above, we see that

$$||p(B)|| \leqq 4[R_{\text{sp}}(p_r(B)) + R_{\text{sp}}(p_{\text{i}}(B))] = 4[||p_{\text{r}}(t)||_\infty + ||p_{\text{i}}(t)||_\infty] \leqq 8||p(t)||_\infty.$$

The passage to continuous functions involves no difficulties.

II. We digress, for a moment, from our discussion of adjoint abelian operators.

Definition. Let $\{x_n\}$ be a sequence in the Banach space X such that $x^*(x_n)$ is a Cauchy sequence for all $x^* \in X^*$. If this implies the existence of an x_0, such that $\{x_n\}$ converges weakly to x_0, then X is weakly complete. Observe that a reflexive Banach space is always weakly complete.

The next theorem is implicit in the combined work of Dunford (**4**), and Bartle, Dunford, and Schwartz (**2**), but does not seem to be written down anywhere;† see also (**7**; **8**; **14**). Since the proof parallels (**4**, Theorem 18), we omit some of the details.

THEOREM 2. *Let T be an operator mapping the weakly complete Banach space X into itself. Let $||f(T)|| \leqq K||f(s)||_\infty$ for some constant K and all $f \in C(S)$, S a compact Hausdorff space ($||f(s)||_\infty = \sup\{|f(s)|: s \in S\}$). Then, T is a scalar operator.*

†*Added in proof.* See Theorem 2.5 of *Unbounded normal operators on Banach spaces* by T. W. Palmer (Trans. Amer. Math. Soc. *133* (1968), 385–414).

Proof. For $f \in C(S)$ and $x \in X$, set $B_f(x) = f(T)x$. For x fixed, $B_f(x)$ is a bounded linear operator, mapping $C(S)$ into X. Hence, by (**2**, Theorems 3.2 and 3.5),

$$B_f(x) = \int f(x)\, du_x(s),$$

where $u_x(s)$ is a measure on Borel sets with values in X.

For each Borel set δ and $x \in X$, set $E(\delta)x = u_x(\delta)$. The linearity of the operator $E(\delta)$ follows from the linearity of $B_f(x)$ in x, and the uniqueness of $u_x(s)$. Since

$$||E(\delta)x|| \leqq \text{total var}(u_x(\delta)) \leqq \sup_{\|f\|_\infty = 1} ||B_f(x)|| \leqq K||x||\ ||f||_\infty,$$

$E(\delta)$ is bounded. The fact that $B_{fg}(x) = B_f(B_g(x))$ and $u_x(\delta)$ is unique and regular implies that $E(\delta \cap \beta) = E(\delta)E(\beta)$ for Borel sets δ and β.

Thus, T is a scalar operator with the representation $T = \int_S s\, dE(s)$. (Of course, the measure $E(\delta)$ must be carried on $\sigma(T)$.)

For the case when X is reflexive, the above argument can also be found in (**10**, p. 93). Note that Theorem 3 is not true without some condition on X. For an example, see the discussion in (**12**, p. 84).

III. This section contains the main results on the spectral decomposition of adjoint abelian operators.

THEOREM 3. *Let A be an adjoint abelian operator on a Banach space X. Then*
(1) *$(A^2)^*$ is a scalar operator (of class X),*
(2) *A^2 is a scalar operator, if X is weakly complete.*

Proof. Lemma 5 and (**4**, Theorem 18) imply (1). The validity of (2) follows from Lemma 5 and Theorem 2.

COROLLARY 4. *Let A be adjoint abelian on the weakly complete Banach space X. If one of the following holds:*
(a) *A is invertible;*
(b) *0 is an isolated point of $\sigma(A)$;*
(c) *$\sigma(A)$ is non-negative (or non-positive);*
(d) *$(0, \delta)$ or $(-\delta, 0)$ are in the resolvent set of A, for $\delta > 0$,*
then A is a scalar operator.

Proof. If (b) holds, then $A = S + N$, where $SN = NS$, S is scalar, and N is nilpotent of order 2; see (**15**). Moreover, $E(0)N = N$ $(S = \int s\, dE(s))$. If $x \in E(0)X$, then $A^2x = N^2x = 0$, which implies that $Ax = Nx = 0$ by Corollary 3. Thus $N = 0$.

The proof of (c) is similar once we invoke (**6**, Theorem 7). To prove (d), we simply apply (c) and the next theorem. The conditions (a)–(d) are not independent, e.g., (a) implies (d).

COROLLARY 5. *Let $A \neq \lambda I$ be an adjoint abelian operator on a weakly complete Banach space X. If the linear operator $T: X \to X$ commutes with A, then T has a proper invariant subspace.*

THEOREM 4. *Let A be an adjoint abelian operator on a weakly complete Banach space X, and let $\epsilon > 0$. Then there exist projections P, Q, with $P + Q = I$, such that*

$$A = PAP + QAQ,$$

where PAP is scalar, and $||QAQ|| < \epsilon$.

Proof. Let $A^2 = \int s\, dE(s)$. Since A commutes with A^2, it commutes with $E(\cdot)$. Let M be the uniform bound on $E(\cdot)$ and set $\delta = \epsilon^2/M$. Let $Q = E(K)$ and $P = E(K')$, where $K = (-\delta, \delta)$. Then $S = PAP$ is an invertible operator on PX, and since S^2 is scalar, so is S. Note that $||A^2x|| \leqq \delta M$ for $x \in QX$. Hence, by Theorem 1,

$$||QAQx||^2 = ||Ax||^2 \leqq ||A^2x||\,||x|| \leqq \delta M||x||^2 = \epsilon^2||x||^2$$

for $x \in QX$, which completes the proof.

COROLLARY 6. *Let A be an adjoint abelian operator on a weakly complete Banach space X. Then* $\mathrm{Null}(A) + \mathrm{Range}(A)$ *is dense in X.*

We have been unable to decide whether every adjoint abelian operator on a weakly complete Banach space is scalar, and we leave this as an open question.

IV. We will now consider adjoint abelian operators on an arbitrary Banach space X.

Notation. Given an operator T, $d_\lambda = \mathrm{dist}[\lambda, \sigma(T)]$.

THEOREM 5. *Let A be adjoint abelian on X. Then $||(A - \lambda I)^{-1}|| \leqq Kd_\lambda^{-4}$ in a neighbourhood of $\sigma(T)$. If $[a, b]$ is a connected component of $\sigma(A)$, then $||(A - \lambda I)^{-1}|| \leqq Kd_\lambda^{-2}$ in a neighbourhood of $[a, b]$ ($a = b$ permitted).*

Proof. We will begin by proving that $||(A - \lambda I)^{-1}|| \leqq K|\mathrm{Im}\,\lambda|^{-2}$ for all $|\lambda| \leqq R$. For any operator B, it is true that $||(B - wI)^{-1}|| \leqq 1/\mathrm{dist}[w, W(B)]$. For $|t| \leqq \pi/4$ and $|t - \pi| \leqq \pi/4$,

$$||(A^2 - r^2e^{i2t}I)^{-1}|| \leqq 1/r^2|\sin 2t| \leqq 1/|r \sin t|^2,$$

since $W(A^2)$ is real. For $|t \pm \pi/2| \leqq \pi/4$,

$$||(A^2 - r^2e^{i2t}I)^{-1}|| \leqq 1/r^2 \leqq 1/|r \sin t|^2,$$

since $W(A^2)$ is positive. Thus,

$$||(A - re^{it}I)^{-1}|| \leqq ||A + re^{it}I||\,||A^2 - r^2e^{i2t}I)^{-1}|| \leqq K'|r \sin t|^2$$

for $0 \leqq t \leqq 2\pi$ and $r \leqq R$.

From here on, the proof is similar to one by Bartle (**1**, Theorem 1). We will present a direct proof, rather than pointing out the changes that must be made in his theorem.

Let (α, β) be an open interval in $\rho(A)$, the resolvent set of A, with $\alpha, \beta \in \sigma(A)$. Set $f(z) = (z - \alpha_1)^2(z - \beta_1)^2(A - zI)^{-1}$, where $\alpha < \alpha_1 < \beta_1 < \beta$. Then, $|f(z)|$ is bounded by K on the rectangle $\mathrm{Re}\, z = \alpha_1$, $\mathrm{Re}\, z = \beta_1$,

https://doi.org/10.4153/CJM-1969-058-4 Published online by Cambridge University Press

$\operatorname{Im} z = \pm 1$; and this constant K does not depend on α_1, β_1 or the particular open set (α, β) selected ($|\alpha|$ and $|\beta|$ are bounded, of course). Then, for λ in the rectangle, it follows that $|f(\lambda)| = ||(\lambda - \alpha_1)^2(\lambda - \beta_1)^2(A - \lambda I)^{-1}|| \leqq K$. Letting α_1 approach α and β_1 approach β yields

$$||(A - \lambda I)^{-1}|| \leqq K|(\lambda - \alpha)^{-2}(\lambda - \beta)^{-2}| \leqq K d_\lambda^{-4}$$

for λ in the rectangle. The two end points of $\sigma(A)$ may be handled in a similar way. Combining all this completes the first half of the theorem.

The case $[a, b] \subset \sigma(A)$ is proved in a similar manner; however, one uses the function $f(z) = (z - \alpha_1)^2(A - zI)^{-1}$ and a semi-circular contour at the end points.

The last theorem permits us to conclude that A has a C^5 functional calculus in the sense of Tillmann (**16**), Maeda (**13**, Remark 1.3) or Kantorovitz (**9**, Theorem 1). If the underlying Banach space is weakly complete, Theorem 3 yields a more detailed spectral structure. Moreover, the resolvent $R_\lambda(A)$ would have first-order growth on sets bounded away from the origin in this case.

THEOREM 6. *Let A be adjoint abelian, and let w be an isolated point of $\sigma(A)$. By the Riesz-Dunford theory, there exist projections P and Q such that $P + Q = I$, and*

(1) *$APX \subset PX$; $AQX \subset QX$,*

(2) *$\sigma(A|PX) = \{w\}$, $\sigma(A|QX) = \sigma(A)\backslash\{w\}$.*

Then $Ax_1 = wx_1$ for $x_1 \in PX$; and $[x_2, x_1^*] = 0$, for $x_2 \in QX$.

Proof. By the Riesz-Dunford theory, $A = wI + N$ on PX, where N is quasi-nilpotent. Since the resolvent of A has second-order growth near w, $N^2 = 0$. Thus, $A^2y = (w^2I + 2N)y$, for $y \in PX$, and hence $(A^2 - zI)^{-1}y = (w^2 - z)^{-1}y + 2(w^2 - z)^{-2}Ny$. However, A^2 is hermitian, and consequently $||(A^2 - zI)^{-1}|| \leqq |\operatorname{Im} z|^{-1}$, which implies that $Ny = 0$. Thus, $A = wI$ on PX, which completes the first part of the proof.

For $x_1 \in PX$ and $x_2 \in QX$, let $f(z) = [(zI - A)^{-1}x_2, x_1^*]$. Clearly, $f(z)$ is analytic for $z \in \rho(A) \cup \{w\}$, and hence in a neighbourhood of w. However, for $|z| > ||A||$, we have

$$f(z) = \sum_{n=0}^{\infty} z^{-(n+1)}[A^n x_2, x_1^*] = \sum_{n=0}^{\infty} z^{-(n+1)}[x_2, (A^n x_1)^*]$$
$$= \sum_{n=0}^{\infty} z^{-(n+1)} w^n [x_2, x_1^*] = [x_2, x_1^*](z - w)^{-1} \quad (w \text{ is real}).$$

The last expression is analytic for $z \neq w$, and is an analytic extension of $f(z)$. Hence, $f(z)$ is entire and vanishes at infinity. Thus, $f(z) = 0$, and therefore $[x_2, x_1^*] = 0$.

COROLLARY 7. *If A is adjoint abelian, and $\sigma(A) = \{c\}$, then $A = cI$.*

The second part of the theorem is a restricted version of the following well-known result. If A is a self-adjoint operator on a Hilbert space, then $\sigma(A, x_1) \cap \sigma(A, x_2) = \emptyset$ implies that $(x_1, x_2) = 0$.

V. An example. Let X be the two-dimensional Euclidean space over the complex numbers, with the L^p metric. Thus, we write $x = (x_1, x_2)$ for $x \in X$. If $||x|| = [|x_1|^p + |x_2|^p]^{1/p} = 1$, then $x^* = (\bar{x}_1|x_1|^{p-2}, \bar{x}_2|x_2|^{p-2})$. Let

$$A = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}$$

with respect to the basis indicated above. It is easy to check that A is adjoint abelian. However, $[Ax, x^*] = x_2\bar{x}_1|x_1|^{p-2} + x_1\bar{x}_2|x_2|^{p-2}$. If we choose x_1 real and x_2 imaginary, with $|x_1| \neq |x_2|$, it follows that $[Ax, x^*]$ is not real. This shows first that the numerical range of an adjoint abelian operator need not be real (as it is in Hilbert space), and second that not every adjoint abelian operator is hermitian.

This operator also shows that adjoint abelian operators are not invariant under translation by real scalars. Indeed, if

$$A = \begin{vmatrix} a & 1 \\ 1 & a \end{vmatrix},$$

for a real and $||x|| = 1$, then

$$A^*\phi x = (a\bar{x}_1|x_1|^{p-2} + \bar{x}_2|x_2|^{p-2}, \bar{x}_1|x_1|^{p-2} + a\bar{x}_2|x_2|^{p-2}).$$

However, $\phi Ax = ||Ax||^{2-p}((a\bar{x}_1 + \bar{x}_2)|ax_1 + x_2|^{p-2}, \; (\bar{x}_1 + a\bar{x}_2)|x_1 + ax_2|^{p-2})$, and these are not equal in general.

VI. We may define operators on Banach space analogous to the unitary operators on Hilbert space by means of another compatibility relation between the duality map of X to X^* and the natural map of $\mathscr{B}(X)$ to $\mathscr{B}(X^*)$.

Definition. Let U be an invertible operator on a Banach space X. Let ϕ be a duality map of X to X^*. Then U is iso-abelian if $\phi U = U^{*-1}\phi$, or equivalently, $(Ux)^* = U^{*-1}x^*$, for all $x \in X$. Note that if X is a Hilbert space, then the iso-abelian operators are precisely the unitary ones.

LEMMA 6. *Let U be iso-abelian on X. Then $(U^{-1}x)^* = U^*x^*$ and $||Ux|| = ||U^{-1}x|| = ||x||$, for $x \in X$.*

Proof. Note that $[x, (U^{-1}y)^*] = [U^{-1}Ux, (U^{-1}y)^*] = [Ux, y^*] = [x, U^*y^*]$, and thus $(U^{-1}y)^*$ and U^*y^* must be the same linear functional. Since $||Ux||^2 = [Ux, (Ux)^*] = [x, x^*] = ||x||^2$, the proof is complete.

From this lemma, it is clear that iso-abelian operators have their spectrum on the unit circle, and $||R_z(U)|| \leqq | \, |z| - 1|^{-1}$ for $||z|| \neq 1$. The analogues to Lemmas 1, 4, and Theorem 6 carry over without difficulty, and we will omit statements and proofs.

However, the "spectral theorem" does not seem to generalize. Consider $X = L^p[0, 2\pi]$, $p \neq 2$. Then for $f(\theta) \in X$, $||f||_p = 1, f^* = \overline{f(\theta)}|f(\theta)|^{p-2}$. Define $Uf(\theta) = f(\theta + \alpha)$ for $f \in X$, where α is irrational. Then U is not spectral or scalar (see **5**), but it is a routine matter to show that U is iso-abelian. What is needed is an analogue to Lemma 3; however, this does not hold in L^p in general.

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