# ON THE CLIFFORD COLLINEATION, TRANSFORM AND SIMILARITY GROUPS. II.

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## 1. Preliminaries

The present paper deals with the Clifford groups in the case $p = 2$. For the most part, it runs parallel to the previous paper I ([1]) on the case $p > 2$, and a number of proofs are therefore either given in outline or omitted. A general introduction to both papers is given in I, § 1.

1.1 *Index to notation.*

General notation: see I, § 2.1.

Symplectic groups: see I, § 2.2.

Orthogonal groups: see II, § 1.2.

§ 1.2: $q(\alpha)$, $q_1(\alpha)$, $q_2(\alpha)$, $Q(\alpha, \beta)$, $f(\alpha, \beta)$, $O(q)$, $O^+(q)$, $O_1(2m, 2)$, $O_2(2m, 2)$.

§ 2.1: $V$, $v_\lambda$, $W^\alpha$, $\phi(\alpha, \beta)$, $CG$, $\mathscr{CG}$, $CT$.

§ 3 (introduction): $W_i$.

§ 3.1: real, semi-real, $CG_i$, $\mathscr{CG}_i$, $CT_i$, $\mathfrak{R}(q)$, $w^\alpha$.

§ 3.3: $CT_i^+$, $\mathscr{CT}_i^+$, $\mathscr{CT}_i$.

1.2 *Orthogonal groups.* We set down for reference some properties of orthogonal groups over $GF(2)$. Let

$$q(\alpha) = \sum_{1 \leq i \leq j \leq 2m} a_{ij}\alpha_i\alpha_j \tag{1.2.1}$$

be a quadratic form on $\mathscr{V}_{2m}^{(2)}$. The skew form

$$Q(\alpha, \beta) = q(\alpha + \beta) + q(\alpha) + q(\beta) = \sum_{i<j} a_{ij}(\alpha_i\beta_j + \alpha_j\beta_i) \tag{1.2.2}$$

is called the *polar form* of $q$. $q$ is said to be *non-defective* when $Q$ is non-degenerate. We assume that this is the case.

The matrices $T$ such that $q(\alpha T') \equiv q(\alpha)$ form the *orthogonal group* $O(q)$ of $q$. $O(q)$ is clearly a subgroup of $Sp(Q)$. A necessary and sufficient condition that $T \in O(q)$ is that the rows $t_1, \cdots, t_{2m}$ of $T'$ satisfy the conditions:

$$q(t_i) = a_{ii} \ (i = 1, \cdots, 2m), \ Q(t_i, t_j) = a_{ij} \ (1 \leqq i < j \leqq 2m).$$

It is known that $q$ is equivalent to one, and only one of the two *canonical* quadratic forms

$$
\begin{aligned}
q_1(\alpha) &= \sum_{i=1}^{m} \alpha_i \alpha_{m+i}, \\
q_2(\alpha) &= q_1(\alpha) + \alpha_m^2 + \alpha_{2m}^2.
\end{aligned}
\tag{1.2.3}
$$

Hence $O(q)$ is isomorphic to one of the two groups $O(q_1)$, $O(q_2)$. The standard notations for these groups are $O_1(2m, 2)$, $O_2(2m, 2)$. Notice that the polar form of each of $q_1, q_2$ is the canonical skew form

$$f(\alpha, \beta) = \sum_{i=1}^{m} (\alpha_i \beta_{m+i} + \alpha_{m+i} \beta_i),$$

so that $O_1(2m, 2)$ and $O_2(2m, 2)$ are subgroups of $Sp(f) = Sp(2m, 2)$.

A vector $\boldsymbol{a}$ in $\mathscr{V}_{2m}$ is called isotropic or non-isotropic (with respect to $q$) according as $q(\boldsymbol{a}) = 0$ or 1. The transvections in $O(q)$ are the linear transformations

$$\alpha \to \alpha + Q(\alpha, \boldsymbol{a})\boldsymbol{a}, \tag{1.2.4}$$

where $\boldsymbol{a}$ is non-isotropic.

The *Dickson invariant* $D(T)$ is a certain function of the elements $T$ of $O(q)$ with values in $GF(2)$; we omit the precise definition (cf. Dieudonné [3], p. 62 et seq.). The $T$ such that $D(T) = 0$ form a subgroup of $O(q)$ of index 2, which we denote by $O^+(q)$. If $T$ is an orthogonal transvection, then $D(T) = 1$, so that $O(q) = O^+(q) + TO^+(q)$.

(1.2.5) $O_i^+(2m, 2)$ *is the commutator group of* $O_i(2m, 2)$, *except when* $m = 2$ *and* $i = 1$. $O_1(4, 2)'$ *is a subgroup of* $O_1^+(4, 2)$ *of index* 2.

(1.2.6) *Structure Theorem. If* $m = i = 2$, *or if* $m \geqq 3$, $O_i^+(2m, 2)$ *is a non-cyclic simple group.*

The orthogonal groups of low dimension have the following structures. $O_1(2, 2)$ has order 2. $O_1(4, 2)$ has a subgroup of index 2 (not $O_1^+(4, 2)$), which is isomorphic to the direct product $S_3 \times S_3$. $O_2(2, 2)$ $(= Sp(2, 2))$, $O_2(4, 2)$, $O_1(6, 2)$ are isomorphic to $S_3$, $S_5$, $S_8$ respectively. $O_2(6, 2)$ is the group of order 51840 associated with the lines on a cubic surface, $O_2^+(6, 2)$ its simple subgroup of order 25920.

(1.2.7) *Cahit Arf's Theorem. If* $\alpha_1, \cdots, \alpha_s$ and $\beta_1, \cdots, \beta_s$ *are two sets of linearly independent vectors in* $\mathscr{V}_{2m}$ *such that*

$$q(\alpha_i) = q(\beta_i), \qquad Q(\alpha_i, \alpha_j) = Q(\beta_i, \beta_j) \qquad (i, j = 1, \cdots, s)$$

*there exists an element* $T$ *of* $O(q)$ *such that*

$$\alpha_i T' = \beta_i \qquad (i = 1, \cdots, s).$$

(1.2.8) *Corollary. If* $\boldsymbol{a}$, $\boldsymbol{b}$ *are non-zero vectors such that* $q(\boldsymbol{a}) = q(\boldsymbol{b})$, *there exists an element* $T$ *of* $O(q)$ *such that* $\boldsymbol{a}T' = \boldsymbol{b}$.

(1.2.9) $O_i^+(2m, 2)$ *is an irreducible group except when* $m = i = 1$.

## 2. The Complex Clifford Groups

2.1 *CG*, $\mathscr{C}\mathscr{G}$, *CT*. Let $V$ be a $2^m$-dimensional complex vector space with basis elements $v_{\boldsymbol{\lambda}}$ $(\boldsymbol{\lambda} \in \mathscr{V}_m)$. Writing the elements $\boldsymbol{\alpha}, \boldsymbol{\beta}, \cdots$ of $\mathscr{V}_{2m}$ as pairs of elements of $\mathscr{V}_m$:

$$\boldsymbol{\alpha} = (\boldsymbol{a}_1, \boldsymbol{a}_2), \qquad \boldsymbol{\beta} = (\boldsymbol{b}_1, \boldsymbol{b}_2), \cdots$$

we define

$$W^{\boldsymbol{\alpha}} v_{\boldsymbol{\lambda}} = (-1)^{\boldsymbol{a}_1 \cdot (\boldsymbol{\lambda} + \boldsymbol{a}_2)} v_{\boldsymbol{\lambda} + \boldsymbol{a}_2}. \tag{2.1.1}$$

Write

$$\phi(\boldsymbol{\alpha}, \boldsymbol{\beta}) = \boldsymbol{a}_2 \cdot \boldsymbol{b}_1. \tag{2.1.2}$$

Then $\phi$ is related to the canonical skew form $f$ by:

$$f(\boldsymbol{\alpha}, \boldsymbol{\beta}) = \phi(\boldsymbol{\alpha}, \boldsymbol{\beta}) + \phi(\boldsymbol{\beta}, \boldsymbol{\alpha}),$$

and therefore $f$ is the polar form of the quadratic form $\phi(\boldsymbol{\alpha}, \boldsymbol{\alpha})$.

We have

$$W^{\boldsymbol{\alpha}} W^{\boldsymbol{\beta}} = (-1)^{\phi(\boldsymbol{\alpha}, \boldsymbol{\beta})} W^{\boldsymbol{\alpha} + \boldsymbol{\beta}}, \tag{2.1.3}$$

$$[W^{\boldsymbol{\alpha}}, W^{\boldsymbol{\beta}}] = (-1)^{f(\boldsymbol{\alpha}, \boldsymbol{\beta})} I, \tag{2.1.4}$$

$$(W^{\boldsymbol{\alpha}})^2 = (-1)^{\phi(\boldsymbol{\alpha}, \boldsymbol{\alpha})} I. \tag{2.1.5}$$

Thus, the order of $W^{\boldsymbol{\alpha}} (\boldsymbol{\alpha} \neq \boldsymbol{0})$ is 2 or 4 according as $\phi(\boldsymbol{\alpha}, \boldsymbol{\alpha}) = 0$ or 1.

*Definition.* $CG(2^m)$ is the group formed by the linear transformations $\lambda W^{\boldsymbol{\alpha}}$ $(\lambda \in C^*, \boldsymbol{\alpha} \in \mathscr{V}_{2m})$, $\mathscr{C}\mathscr{G}(2^m)$ the subgroup formed by the linear transformations $i^k W^{\boldsymbol{\alpha}}$ $(k = 0, 1, 2, 3; \boldsymbol{\alpha} \in \mathscr{V}_{2m})$ and $CT(2^m)$ the normalizer of $CG(2^m)$ (or $\mathscr{C}\mathscr{G}(2^m)$) in $GL(V)$.

As before, $\mathscr{C}\mathscr{G}$ is a fully invariant subgroup of $CG$: $X \in CG$ satisfies $X^4 = I$ if, and only if, $X \in \mathscr{C}\mathscr{G}$. (Notice that the subgroup formed by the $\pm W^{\boldsymbol{\alpha}}$ is *not* fully invariant.) $\mathscr{C}\mathscr{G}$ has order $2^{2m+2}$ and exponent 4, and its commutator group is $\{-I\}$.

Let $\mathfrak{H}(F)$ denote the abstract group with generators $w_1, \cdots, w_{2m}, v$, and defining relations

$$w_i^2 = v^4 = [w_i, v] = 1, \quad [w_i, w_j] = v^{2F_{ij}}, \quad (i, j = 1, \cdots, 2m), \tag{2.1.6}$$

where

$$F(\boldsymbol{\alpha}, \boldsymbol{\beta}) = \sum_{i,j} F_{ij} \alpha_i \beta_j$$

is a non-degenerate skew form on $\mathscr{V}_{2m}$. Then $\mathscr{C}\mathscr{G} \cong \mathfrak{H}(F)$. To prove this, choose vectors $\alpha_1, \cdots, \alpha_{2m}$ as in the corresponding proof in I, § 3.1. Let $k_1, \cdots, k_{2m}$ be integers such that $k_i \equiv \phi(\alpha_i, \alpha_i) \pmod 2$. Then $i^{k_1}W^{\alpha_1}, \cdots, i^{k_{2m}}W^{\alpha_{2m}}$, $iI$ satisfy the defining relations for $\mathfrak{H}(F)$ and generate $\mathscr{C}\mathscr{G}$, so that $\mathscr{C}\mathscr{G} \cong \mathfrak{H}(F)$ as required.

2.2 Theorems 1 and 2 of I carry over verbatim to the present case. Theorem 3 becomes:

THEOREM 3′. *Every automorphism $\psi$ of $CG$ which leaves the scalars fixed is a similarity over $R_0(i)$.*

2.3 *The structure of $CT$.* The results of I, § 3.3 carry over to the present case, except that (in general) $\mathfrak{A}$ is not isomorphic to $ASp$.

THEOREM 4. *Every automorphism of $CG$ which leaves the scalars fixed has the form*

$$\psi(\lambda W^{\alpha}) = \lambda i^{g(\alpha)} W^{\alpha T'}, \tag{2.3.1}$$

*where $T \in Sp(f)$. Conversely, if $T \in Sp(f)$ there exist $g(\alpha)$ such that* (2.3.1) *is an automorphism.*

PROOF. Write $W_i = W^{\varepsilon_i}$ $(i = 1, \cdots, 2m)$ where $\varepsilon_i$ is the $i$-th unit vector in $\mathscr{V}_{2m}$. Then $W_1, \cdots, W_{2m}, iI$ satisfy the defining relations (2.1.6) for $\mathfrak{H}(f)$. Suppose now that $\psi$ is an automorphism of $CG$ which leaves the scalars fixed and let

$$\psi(W_i) = \lambda_i W^{t_i} \qquad (i = 1, \cdots, 2m). \tag{2.3.2}$$

Let $T$ be the transpose of the matrix with rows $t_1, \cdots, t_{2m}$. By (2.1.5),

$$\left.\begin{aligned} \lambda_j &= i^{g_j} \qquad\qquad\quad, \\ g_j &\equiv \phi(t_j, t_j) \pmod 2, \end{aligned}\right\} \qquad (j = 1, \cdots, 2m) \tag{2.3.3}$$

and therefore, by (2.1.3), $\psi$ has the form (2.3.1). By (2.1.4), $T \in Sp$.

Conversely, consider the equations (2.3.2) when (2.3.3) holds and $T \in Sp$. By (2.1.5) and (2.3.3),

$$(\psi(W_i))^2 = I;$$

and since $f(\varepsilon_i, \varepsilon_j) = f(t_i, t_j)$, (2.1.4) shows that

$$[W_i, W_j] = [\psi(W_i), \psi(W_j)].$$

It follows from the defining relations (2.1.6) that (2.3.2) determines a unique endomorphism of $CG$ of the form (2.3.1). But since $T$ is non-singular, (2.3.1) is clearly a one-to-one mapping of $CG$ onto itself and therefore an automorphism. This completes the proof of the theorem.

COROLLARY. *Let $g$ be an arbitrary function, $T$ an arbitrary matrix. Then* (2.3.1) *is an automorphism of $CG$ if, and only if,*

$$g(\alpha + \beta) - g(\alpha) - g(\beta) \equiv 2(\phi(\alpha T', \beta T') + \phi(\alpha, \beta)) \pmod 4 \qquad (\alpha, \beta \in \mathscr{V}_{2m}). \tag{2.3.4}$$

PROOF. The proof of the theorem shows that (2.3.1) is an automorphism if, and only if, it has the homomorphic property

$$\psi(W^{\alpha}W^{\beta}) = \psi(W^{\alpha})\psi(W^{\beta}).$$

A direct calculation shows that this is equivalent to (2.3.4).

THEOREM 5. *In the notation of I, theorem* 4,

$$PCT(2^m) \cong \mathfrak{A}(2^m)$$

$$CT(2^m)/CG(2^m) \cong \mathfrak{A}(2^m)/\mathfrak{J}(2^m) \cong Sp(2m, 2).$$

The isomorphism $\mathfrak{A}/\mathfrak{J} \cong Sp$ is an easy consequence of theorem 4. The rest of the proof is not essentially different from that of I, theorem 5.

2.4 *The commutator group of CT.*

THEOREM 6. *The commutator group $CT'$ has the following properties*:

(i) *$CT'$ is unitary;*
(ii) *$CT'$ is a group over $R_0(i)$;*
(iii) *$CT'$ is finite, and has scalar subgroup $\{iI\}$ except when $m = 1$;*
(iv) *$P(CT') = PCT$ except when $m = 1,2$.*

PROOF. The method of I, theorem 7, proves (i), (ii), (iv) and shows that the scalar subgroup $S$ of $CT'$ satisfies $\{-I\} \subseteq S \subseteq \{iI\}$. It remains to prove only that $iI \in CT'$ when $m > 1$.

By I, (2.2.5), we can choose $S, T \in Sp$ such that

$$(\varepsilon_1 + \varepsilon_{m+1})S' = \varepsilon_2, \qquad (\varepsilon_1 + \varepsilon_2)T' = \varepsilon_{m+1}.$$

Since $m \geqq 2$, $\phi(u, u) = 1$ when $u = \varepsilon_1 + \varepsilon_{m+1}$, 0 when $u = \varepsilon_2, \varepsilon_{m+1}$, $\varepsilon_1 + \varepsilon_2$. Therefore, by the proof of theorem 4, we can choose $X, Y \in CT$ such that

$$XW^{\varepsilon_1+\varepsilon_{m+1}}X^{-1} = iW^{\varepsilon_2}, \qquad YW^{\varepsilon_1+\varepsilon_2}Y^{-1} = W^{\varepsilon_{m+1}}.$$

Then

$$[X, W^{\varepsilon_1+\varepsilon_{m+1}}] = i[Y, W^{\varepsilon_1+\varepsilon_2}] \qquad (= -iW^{\varepsilon_1+\varepsilon_2+\varepsilon_{m+1}}),$$

so that $iI \in CT'$ as required.

## 3. The Real and Semi-Real Clifford Groups

The "real" Clifford theory of the present section arises out of the fact that the linear transformations $W^{\alpha}$ are real. Since the "real" and "complex" theories are very similar, we shall omit those proofs which do not differ essentially from the corresponding ones in section 2. The main point of divergence in the "real" theory is that the symplectic group $Sp(2m, 2)$ is replaced by the orthogonal group $O_1(2m, 2)$. The "semi-real" Clifford theory, which we develop simultaneously with the "real" theory, is an analogous generalization in which $Sp(2m, 2)$ is replaced by $O_2(2m, 2)$.

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*Notation.* $W_i = W^{\varepsilon_i}$, where $\varepsilon_i$ is the $i$-th unit vector in $\mathscr{V}_{2m}$.

3.1 *The groups* $CG_i$, $\mathscr{C}\mathscr{G}_i$, $CT_i$. Our previous definition of a real linear transformation $X$ on $V$ (I § 3.1) can be restated as follows: $X$ is real if it commutes with the semi-linear transformation

(3.1.1) $$\wp_1 : \sum \xi_{\lambda} v_{\lambda} \to \sum \bar{\xi}_{\lambda} v_{\lambda}.$$

$X$ is called semi-real if it commutes with the semi-linear transformation

(3.1.2) $$\wp_2 = \wp_1 W_m W_{2m} : \sum \xi_{\lambda} v_{\lambda} \to \sum (-1)^{\lambda_m+1} \bar{\xi}_{\lambda} v_{\lambda+e_m}.$$

We remark that a scalar $\lambda I$ is semi-real if, and only if, it is real. The real (semi-real) elements of a linear group $G$ form a subgroup which we call the real (semi-real) subgroup of $G$.

(3.1.3) *Definition.* The real Clifford groups $CG_1(2^m)$, $\mathscr{C}\mathscr{G}_1(2^m)$, $CT_1(2^m)$, and semi-real Clifford groups $CG_2(2^m)$ $\mathscr{C}\mathscr{G}_2(2^m)$, $CT_2(2^m)$ are the real and semi-real subgroups of $CG(2^m)$, $\mathscr{C}\mathscr{G}(2^m)$, $CT(2^m)$ respectively.

Thus, $\mathscr{C}\mathscr{G}_1(2^m)$ is generated by the elements $W_i$ $(i = 1, \cdots, 2m)$, $\mathscr{C}\mathscr{G}_2(2^m)$ by the elements $W_i$, $W_{m+i}$, $(i = 1, \cdots, m-1)$ and $iW_m$, $iW_{2m}\cdot \mathscr{C}\mathscr{G}_i$ is a fully invariant subgroup of $CG_i$ and has scalar subgroup $\{-I\}$.

Let $q(\alpha)$ be the non-defective quadratic form (2.3.1) on $\mathscr{V}_{2m}$. Let $\mathfrak{K}(q)$ denote the abstract group with generators $w_1, w_2, \cdots, w_{2m}, w$ and defining relations

(3.1.4) $$\begin{aligned} w^2 = [w, w_i] = 1, \quad w_i^2 = w^{a_{ii}}, \quad & (1 \leqq i \leqq 2m) \\ [w_i, w_j] = w^{a_{ij}} \quad & (1 \leqq i < j \leqq 2m). \end{aligned}$$

Then

(3.1.5) $$\mathscr{C}\mathscr{G}_i \cong \mathfrak{K}(q_i) \qquad (i = 1, 2),$$

where $q_i$, $q_2$ are the canonical quadratic forms (2.3.3).

Write
$$w^{\alpha} = w_1^{\alpha_1} \cdots w_{2m}^{\alpha_{2m}} \qquad (\alpha = (\alpha_1, \cdots, \alpha_{2m}) \in \mathscr{V}_{2m}),$$

where $w_i^{\alpha_i}$ is interpreted as $1$, $w_i$ according as $\alpha_i = 0, 1 \in GF(2)$. Then

(3.1.6) $$\begin{aligned} [w^{\alpha}, w^{\beta}] &= w^{Q(\alpha, \beta)}, \\ (w^{\alpha})^2 &= w^{q(\alpha)}, \end{aligned}$$

where $Q(\alpha, \beta)$ is the polar form (1.2.2) of $q(\alpha)$.

If $q(\alpha)$ and $q'(\alpha) = \sum_{i \leqq j} a'_{ij} \alpha_i \alpha_j$ are non-defective quadratic forms on $\mathscr{V}_{2m}$, then either $q$ is equivalent to $q'$, or $q$ is equivalent to one of the forms $q_1$, $q_2$ and $q'$ to the other. In the first case, there exists a basis $\boldsymbol{t}_1, \cdots, \boldsymbol{t}_{2m}$ of $\mathscr{V}_{2m}$ such that

$$q(\boldsymbol{t}_i) = a'_{ii}, \qquad Q(\boldsymbol{t}_i, \boldsymbol{t}_j) = a'_{ij} \qquad (i < j);$$

by (3.1.6), $w^{t_1}, w^{t_2}, \cdots, w^{t_{2m}}, w$ satisfy the defining relations for $\mathfrak{K}(q')$ and generate $\mathfrak{K}(q)$, so that $\mathfrak{K}(q') \cong \mathfrak{K}(q)$. In the second case, $\mathfrak{K}(q') \not\cong \mathfrak{K}(q)$, for the equations $q_1(\alpha) = 0$ and $q_2(\alpha) = 0$ have $2^{m-1}(2^m + 1)$ and $2^{m-1}(2^m - 1)$ solutions respectively, and therefore, by (3.1.6), $\mathfrak{K}(q_1)$ and $\mathfrak{K}(q_2)$ have different numbers of elements of order 2.

By the same kind of argument, it is easy to prove that the equations

$$\psi(w_i) = w^{\kappa_i} w^{t_i} \qquad (i = 1, \cdots, 2m) \tag{3.1.7}$$

determine an automorphism of $\mathfrak{K}(q)$ if, and only if,

$$T \in O(q), \tag{3.1.8}$$

where $T$ is the transpose of the matrix with rows $t_1, \cdots, t_{2m}$. Such an automorphism has the form

$$\psi(w^{\alpha}) = w^{g(\alpha)} w^{\alpha T'} \quad (g(\alpha) \in GF(2)). \tag{3.1.9}$$

Conversely, (3.1.9) represents an automorphism of $\mathfrak{K}(q)$ if, and only if,

$$g(\alpha + \beta) = g(\alpha) + g(\beta) + \chi(\alpha T', \beta T') + \chi(\alpha, \beta) \quad (\alpha, \beta \in \mathscr{V}_{2m}), \tag{3.1.10}$$

where

$$\chi(\alpha, \beta) = \sum_{i \leq j} a_{ij} \beta_i \alpha_j. \tag{3.1.11}$$

We remark that

$$\chi(\alpha, \alpha) = q(\alpha), \qquad \chi(\alpha, \beta) + \chi(\beta, \alpha) = Q(\alpha, \beta).$$

3.2 *The structure of* $CT_i$.

THEOREM 3*. *Every automorphism $\psi$ of $CG_1$ ($CG_2$) which leaves the scalars fixed has the form $\psi(X) = Y^{-1}XY$, where $Y$ is a linear transformation over $R_0$ (a semi-real linear transformation over $R_0(i)$).*

PROOF. (Semi-real case) If $Z$ is a linear transformation on $V$, we write $\tilde{Z} = \wp_2^{-1} Z \wp_2 = W_{2m}^{-1} W_m^{-1} \bar{Z} W_m W_{2m}$. By theorem 3′, $\psi(X) = Y^{-1}XY$ ($X \in CG_2$), where $Y$ is a linear transformation over $R_0(i)$. Since $X = \tilde{X}$ and $Y^{-1}XY = \tilde{Y}^{-1}\tilde{X}\tilde{Y}$, $\tilde{Y}Y^{-1}$ commutes with $X$. Therefore, by theorem 1, $\tilde{Y}Y^{-1}$ is a scalar $\lambda I$. Applying $\sim$ to the equation $\tilde{Y} = \lambda Y$, we have $Y = \bar{\lambda}\tilde{Y} = \bar{\lambda}\lambda Y$, so that $\bar{\lambda}\lambda = 1$. Hence $\lambda$ has the form $\bar{\mu}\mu^{-1}$, where $\mu \in R_0(i)$ (we may take $\mu = i$ when $\lambda = -1$, $\mu = (1 + \lambda)^{-1}$ when $\lambda \neq -1$). Then $\widetilde{(\mu^{-1}Y)} = \mu^{-1}Y$ is semi-real, which gives the theorem.

THEOREM 4*. *Every automorphism of $CG_i$ which leaves the scalars fixed has the form*

$$\psi(\lambda W^{\alpha}) = \lambda i^{g(\alpha)} W^{\alpha T'}, \tag{3.2.1}$$

*where $T \in O(q_i)$. Conversely, if $T \in O(q_i)$, there exist functions $g(\alpha)$ such that (3.2.1) is an automorphism of $CG_i$ ($i = 1, 2$).*

THEOREM 5*. *Let $\mathfrak{A}_i(2^m)$ denote the group of automorphisms of $CG_i(2^m)$ which leave the scalars fixed, $\mathfrak{J}_i(2^m)$ the group of inner automorphisms of $CG_i(2^m)$. Then*

$$PCT_i(2^m) \cong \mathfrak{A}_i(2^m),$$
$$CT_i(2^m)/CG_i(2^m) \cong \mathfrak{A}_i(2^m)/\mathfrak{J}_i(2^m) \cong O_i(2m, 2).$$

$(i = 1, 2)$.

3.3 *The commutator group of* $CT_i$. If $X \in CT_i$ then

$$XW^{\alpha}X^{-1} = i^k W^{\alpha T'},$$

where $T \in O(q_i)$. The $X$ such that $T \in O^+(q_i)$ form a subgroup $CT_i^+$ of $CT_i$ of index 2.

THEOREM 6*. *The commutator group $CT_i'$ has the following properties:*

(i) $CT_i'$ *is unitary;*
(ii) $CT_1'$ *is a group over* $R_0$, $CT_2'$ *a semi-real group over* $R_0(i)$;
(iii) $CT_i'$ *is finite, and has scalar subgroup* $\{-I\}$.
(iv) $P(CT_i') = P(CT_i)$ *except when* $i = 1$ *and* $m = 1, 2$.

This theorem can be proved by the same methods as before ((i), (ii), (iii) are almost immediate consequences of theorem 6). The exceptions in (iv) are due to the facts that $O_1(2, 2)$ is a reducible group and that $O_1(4, 2)' \neq O_1^+(4, 2)$.

$CT_i'$ stands in much the same relation to $CT_i^+$ as $\mathscr{CG}$ to $CG$, and we shall therefore write

$$\mathscr{CT}_i^+ = CT_i,$$

the exceptional cases $i = 1$, $m = 1, 2$ being excluded. We shall now prove that the existence of a group $\mathscr{CT}_i$ which stands in a similar relation to $CT_i$.

Consider the transvection

(3.3.1) $\quad (\alpha_1, \cdots, \alpha_m, \alpha_{m+1}, \cdots, \alpha_{2m})T_m' = (\alpha_1, \cdots, \alpha_{2m}, \alpha_{m+1}, \cdots, \alpha_m)$

on $\mathscr{V}_{2m}$. Since $T_m \in O(q_1) \cap O(q_2)$, we have

$$O(q_i) = O^+(q_i) + T_m O^+(q_i) \qquad (i = 1, 2).$$

Consider now the linear transformation

(3.3.2) $\quad X_m v_{(\lambda_1, \cdots, \lambda_{m-1}, \lambda_m)} = v_{(\lambda_1, \cdots, \lambda_{m-1}, 0)} + (-1)^{\lambda_m+1} v_{(\lambda_1, \cdots, \lambda_{m-1}, 1)}$

on $V(2^m)$. Since

$$X_m W^{\alpha} X_m^{-1} = (-1)^{\alpha_m(1+\alpha_{2m})} W^{\alpha T'_m},$$

and

$$X_m^2 = 2W_m W_{2m},$$

we have

$$X_m \in CT_1 \cap CT_2$$

and therefore

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$$CT_i = CT_i^+ + X_m CT_i^+ \qquad (i = 1, 2).$$

We define

$$(3.3.3) \qquad \mathscr{C}\mathscr{T}_i = \mathscr{C}\mathscr{T}_i^+ + (2^{-\frac{1}{2}}X_m)\mathscr{C}\mathscr{T}_i^+ \qquad (i = 1, 2),$$

the exceptional cases being always excluded.

It is clear that $\mathscr{C}\mathscr{T}_i$ has the properties:

(i) the scalar subgroup of $\mathscr{C}\mathscr{T}_i$ is $\{-I\}$;
(ii) $P\mathscr{C}\mathscr{T}_i = PCT_i$.

We prove finally that $\mathscr{C}\mathscr{T}_i$ is the only subgroup of $CT_i$ with these properties. In fact, suppose that $H \subset CT_i$ satisfies (i) and (ii). By (ii), $H' = CT_i'$ and therefore $H = \mathscr{C}\mathscr{T}_i^+ + Y\mathscr{C}\mathscr{T}_i^+$, where $Y = \lambda 2^{-\frac{1}{2}}X_m$. Since $Y^2 = \lambda^2 I$, we have, by (i), $\lambda^2 = \pm 1$. Since $\lambda I$ is real or semi-real, $\lambda$ is real and so either 1 or $-1$. Hence $H = \mathscr{C}\mathscr{T}_i$ as required.

## 4. The Projective Clifford Transform Groups

In the present section we prove that the projective Clifford transform groups are *not* isomorphic to the corresponding affine classical groups. It is sufficient to prove that $PCG$ is not complemented in $PCT$ (or in the real and semi-real cases, that $PCG$ ($= PCG_i$) is not complemented in $PCT_i$). Certain cases of low dimension are exceptional and these are considered in §§ 4.2, 4.3.

### 4.1 *The general case.*

THEOREM 7. (a) *If $m \geqq 2$, $PCG(2^m)$ is not complemented in $PCT(2^m)$.* (b) *If $m \geqq 3$, $PCG(2^m)$ is not complemented in $PCT_i(2^m)$ $(i = 1, 2)$.* (c) *If $m = 3$ and $i = 2$, or if $m \geqq 4$ and $i = 1, 2$, $PCG(2^m)$ is not complemented in $PCT_i^+(2^m)$*

PROOF. (a) Suppose that the result were false. Then $\mathfrak{A}(2^m)$ would have a subgroup $H = \{\psi_T | T \in Sp\}$, where $\psi_T$ has the form

$$\psi_T(W^{\alpha}) = i^{g_T(\alpha)} W^{\alpha T}.$$

Since $\psi_S \psi_T = \psi_{ST}$,

$$(4.1.1) \qquad g_{ST}(\alpha) = g_T(\alpha) + g_S(\alpha T') \quad (\alpha \in \mathscr{V}_{2m};\ S, T \in Sp),$$

and by (2.3.4),

$$(4.1.2) \quad g_T(\alpha + \beta) = g_T(\alpha) + g_T(\beta) + 2(\phi(\alpha T', \alpha T') + \phi(\alpha, \beta)) \qquad (\alpha, \beta \in \mathscr{V}_{2m};\ T \in Sp).$$

(Equality in (4.1.1), (4.1.2) means equality modulo 4.)

Since $m \geqq 2$, we can choose linearly independent vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$, such that $f(\boldsymbol{a}, \boldsymbol{b}) = 1$, $f(\boldsymbol{a}, \boldsymbol{c}) = 0$. Let $T$ denote the symplectic transvection

$\alpha T' = \alpha + f(\alpha, \boldsymbol{a})\boldsymbol{a}$ and write $t(\alpha) = g_T(\alpha)$. By (4.1.2), $t(\boldsymbol{a}) = t(\boldsymbol{b}) + t(\boldsymbol{a} + \boldsymbol{b}) + 2$, and by (4.1.1) with $S = T$, $0 = g_{T^2}(\boldsymbol{b}) = t(\boldsymbol{b}) + t(\boldsymbol{a} + \boldsymbol{b})$. Hence

$$t(\boldsymbol{a}) = 2. \tag{4.1.3}$$

By I, (2.2.5), we can choose $S \epsilon Sp$ such that $\boldsymbol{a}S' = \boldsymbol{a}$, $\boldsymbol{c}S' = \boldsymbol{a} + \boldsymbol{c}$. By (4.1.1) and since $ST = TS$, we have

$$t(\boldsymbol{c}) + g_S(\boldsymbol{c}T') = g_S(\boldsymbol{c}) + t(\boldsymbol{c}S'),$$

and therefore, since $\boldsymbol{c}T' = \boldsymbol{c}$ and $\boldsymbol{c}S' = \boldsymbol{a} + \boldsymbol{c}$,

$$t(\boldsymbol{c}) = t(\boldsymbol{a} + \boldsymbol{c}).$$

Since, by (4.1.2), $t(\boldsymbol{a} + \boldsymbol{c}) = t(\boldsymbol{a}) + t(\boldsymbol{c})$, we deduce that $t(\boldsymbol{a}) = 0$. This contradiction to (4.1.3) proves our result.

(b) We omit the proof, which closely follows that of (c).

(c) In the present proof we assume that $m \geqq 4$; the outstanding case $m = 3$, $i = 2$, will be dealt with in § 4.3. For a suitable choice of the quadratic form $q$, $\mathscr{C}\mathscr{G}_i$ is isomorphic to the group $\mathfrak{R}(q)$ given by (3.1.4). It is convenient to express the proof in terms of $\mathfrak{R}(q)$, and we shall use the notations (2.3.1), (2.3.2), (3.1.11).

Suppose that our result were false. Then the group of automorphisms of $\mathfrak{R}(q)$ would have a subgroup $H = \{\psi_T | T \epsilon O^+(q)\}$, where $\psi_T$ has the form

$$\psi_T(w^{\alpha}) = w^{h_T(\alpha)} w^{\alpha T'} \quad (h_T(\alpha) \epsilon GF(2)).$$

As in case (a),

$$h_{ST}(\alpha) = h_T(\alpha) + h_S(\alpha T'), \tag{4.1.4}$$

$$h_T(\alpha + \beta) = h_T(\alpha) + h_T(\beta) + \chi(\alpha T', \beta T') + \chi(\alpha, \beta). \tag{4.1.5}$$

Since $m \geqq 3$, we can choose linearly independent vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ such that $q(\boldsymbol{a}) = q(\boldsymbol{b}) = 1$, $Q(\boldsymbol{a}, \boldsymbol{b}) = Q(\boldsymbol{b}, \boldsymbol{c}) = Q(\boldsymbol{c}, \boldsymbol{a}) = 0$. Let $T$ denote the product of the (commuting) orthogonal transvections with "centres" $\boldsymbol{a}, \boldsymbol{b}$:

$$\alpha T' = \alpha + Q(\alpha, \boldsymbol{a})\boldsymbol{a} + Q(\alpha, \boldsymbol{b})\boldsymbol{b}.$$

Writing $t(\alpha) = h_T(\alpha)$, we have as in case (a)

$$t(\boldsymbol{a}) = 1. \tag{4.1.6}$$

Since $m \geqq 4$, we can choose a vector $\boldsymbol{d}$ such that $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}$ are linearly independent and $Q(\boldsymbol{a}, \boldsymbol{d}) = Q(\boldsymbol{b}, \boldsymbol{d}) = 0$, $Q(\boldsymbol{c}, \boldsymbol{d}) = q(\boldsymbol{c})$. Then $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{d}$ and $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} + \boldsymbol{d}$ are linearly independent sets such that

$$Q(\boldsymbol{a}, \boldsymbol{d}) = Q(\boldsymbol{a}, \boldsymbol{c} + \boldsymbol{d}), \qquad Q(\boldsymbol{b}, \boldsymbol{d}) = Q(\boldsymbol{b}, \boldsymbol{c} + \boldsymbol{d}),$$
$$q(\boldsymbol{c} + \boldsymbol{d}) \ (= q(\boldsymbol{c}) + Q(\boldsymbol{c}, \boldsymbol{d}) + q(\boldsymbol{d})) = q(\boldsymbol{d}).$$

Therefore, by (2.3.7) we can choose $S \epsilon O(q)$ such that $\boldsymbol{a}S' = \boldsymbol{a}$, $\boldsymbol{b}S' = \boldsymbol{b}$,

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$dS' = c + d$. Since the transvection with centre $a$ leaves $a, b, d$ fixed, we can suppose that $S \in O^+(q)$. As in part (a), we deduce that $t(d) = t(c+d)$ and thence that $t(c) = 0$. Similarly, $t(a+c) = 0$ and therefore, by (4.1.5), $t(a) = t(c) + t(a+c) = 0$. This contradiction to (4.1.6) completes the proof.

4.2 *The exceptional cases*. When considering the groups $CT_i$ we shall replace $\mathscr{C}\mathscr{G}_i$ by the isomorphic abstract group $\mathfrak{K}(q)$, where $q = q(\alpha)$ is a conveniently chosen quadratic form equivalent to $q_i(\alpha)$. $\mathfrak{A}$ denotes the group of automorphisms, $\mathfrak{J}$ the group of inner automorphisms, of $\mathfrak{K}(q)$. $Q(\alpha, \beta)$ is the polar form of $q(\alpha)$. $T_a$ denotes the orthogonal transvection with centre $\alpha$:

$$\alpha T'_a = \alpha + Q(\alpha, a)a \quad (q(a) \neq 0).$$

Notice that

$$T_a T_b = T_b T_a \qquad \text{if } Q(a, b) = 0,$$
$$T_a T_b = T_{a+b} T_a \qquad \text{if } Q(a, b) = 1.$$

We require several preliminary results.

LEMMA 1. *The generators $V_1, \cdots, V_{n-2}$ and relations*

$$V_i^3 \doteq (V_i V_j)^2 = 1 \qquad (i \neq j) \tag{4.2.1}$$

*define the alternating group $A_n$*. (Coxeter and Moser [2], p. 67).

Suppose now that $a, b, c$, are linearly independent vectors such that

$$q(a) = q(b) = q(c) = Q(a, b) = Q(b, c) = Q(c, a) = 1. \tag{4.2.2}$$

Consider the elements

$$S = T_c T_a, \qquad T = T_b T_a$$

of $O^+(q)$. Then $ST = T_c T_{a+b}$ and

$$S^3 = T^3 = (ST)^2 = I,$$

so that $\{S, T\} \cong A_4$.

LEMMA 2. *Let*

$$\left.\begin{aligned} \psi_S(w^\alpha) &= w^{g(\alpha)} w^{\alpha S'}, \\ \psi_T(w^\alpha) &= w^{h(\alpha)} w^{\alpha T'}, \end{aligned}\right\} (g(\alpha), h(\alpha) \in GF(2))$$

*be automorphisms of $\mathfrak{K}(q)$ corresponding to $S, T$. Then $\{\psi_S, \psi_T\} \cong A_4$ if, and only if,*

$$g(\alpha) = 0 \text{ when } Q(\alpha, a) = Q(\alpha, c) = 0; \tag{4.2.3}$$

$$h(\alpha) = 0 \text{ when } Q(\alpha, a) = Q(\alpha, b) = 0; \tag{4.2.4}$$

$$g(a + c) + \chi(c, a) = h(a + b) + \chi(b, a), \tag{4.2.5}$$

*where $\chi(\alpha, \beta)$ is the bilinear form* (3.1.11).

PROOF. By direct computation, we find that $\psi_S^3 = \psi_T^3 = (\psi_S \psi_T)^2 = 1$ if, and only if,

$$g(\alpha) + g(\alpha S') + g(\alpha S'^2) \equiv 0; \tag{4.2.6}$$

$$h(\alpha) + h(\alpha T') + h(\alpha T'^2) \equiv 0; \tag{4.2.7}$$

$$k(\alpha) + k(\alpha T' S') \equiv 0, \tag{4.2.8}$$

where $k(\alpha) \equiv h(\alpha) + g(\alpha T')$. It is in fact sufficient that each of these conditions should hold for a set of vectors which span $\mathscr{V}_{2m}$. For then each of $\psi_S^3$, $\psi_T^3$ and $(\psi_S \psi_T)^2$ leaves a set of generators of $\mathfrak{R}(q)$ fixed and so is the identity.

Suppose first that (4.2.6)—(4.2.8) hold. With $\alpha$ as in (4.2.3), we have $\alpha = \alpha S' = \alpha S'^2$, so that, by (4.2.6), $3g(\alpha) = 0$; hence (4.2.3) holds. (4.2.4) is proved similarly. To prove (4.2.5), we choose $\beta$ such that $Q(\beta, \mathbf{a}) = Q(\beta, \mathbf{b}) = Q(\beta, \mathbf{c}) = 1$. By (4.2.8), we have

$$h(\beta) + h(\beta + \mathbf{c}) = g(\beta + \mathbf{a}) + g(\beta + \mathbf{c}). \tag{4.2.9}$$

By (4.2.4), $h(\mathbf{a} + \mathbf{b} + \mathbf{c}) = 0$. Adding $h(\mathbf{a} + \mathbf{b} + \mathbf{c})$ to the left hand side of (4.2.9) and then simplifying both sides with the help of (3.1.11), we get the required formula (4.2.5).

Conversely, let the conditions of the lemma hold. When $\alpha = \mathbf{a}$ or $\mathbf{c}$, (4.2.6) becomes $g(\mathbf{a}) + g(\mathbf{c}) + g(\mathbf{a} + \mathbf{c}) = 0$, which is a consequence of (3.1.11). When $Q(\alpha, \mathbf{a}) = Q(\alpha, \mathbf{c}) = 0$, (4.2.6) follows from (4.2.3). Similarly, (4.2.7) holds when $\alpha = \mathbf{a}$ or $\mathbf{b}$ and when $Q(\alpha, \mathbf{a}) = Q(\alpha, \mathbf{b}) = 0$. (4.2.8) obviously holds when $\alpha T' S' = \alpha$, i.e. when $Q(\alpha, \mathbf{c}) = Q(\alpha, \mathbf{a} + \mathbf{b}) = 0$. When $Q(\alpha, \mathbf{a}) = Q(\alpha, \mathbf{b}) = Q(\alpha, \mathbf{c}) = 1$, (4.2.8) holds by the considerations of the previous paragraph. When $\alpha = \mathbf{a}$, (4.2.8) becomes $h(\mathbf{a}) + h(\mathbf{b} + \mathbf{c}) = g(\mathbf{c}) + g(\mathbf{a} + \mathbf{b})$; by (3.1.11), this reduces to $h(\mathbf{a} + \mathbf{b} + \mathbf{c}) = g(\mathbf{a} + \mathbf{b} + \mathbf{c})$, of which both sides are zero by (4.2.3) and (4.2.4). We have now verified that each of (4.2.6), (4.2.7) and (4.2.8) holds for a set of vectors which span $\mathscr{V}_{2m}$, so that $\psi_S^3 = \psi_T^3 = (\psi_S \psi_T)^2 = 1$ as required.

LEMMA 3. *Suppose that $\mathfrak{R}(q)$ is identified with $\mathscr{C}\mathscr{G}_i$ (according to some fixed isomorphism) and that $X_S$, $X_T$ are elements of $CT_i$ such that*

$$X_S w^{\alpha} X_S^{-1} = \psi_S(w^{\alpha}),$$
$$X_T w^{\alpha} X_T^{-1} = \psi_T(w^{\alpha}).$$

*Then* $\{X_S, X_T\} \not\cong A_4$.

PROOF. $w^{\mathbf{a}}, w^{\mathbf{b}}, w^{\mathbf{c}}$ are elements $\pm W^{\mathbf{u}}, \pm W^{\mathbf{v}}, \pm W^{\mathbf{w}}$ of $CG$ such that $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are linearly independent and $f(\mathbf{u}, \mathbf{v}) = f(\mathbf{v}, \mathbf{w}) = f(\mathbf{w}, \mathbf{u}) = 1$. By I, (2.2.5), we may suppose (after transforming by a suitable element of $CT$) that $\mathbf{u} = \varepsilon_m$, $\mathbf{v} = \varepsilon_{2m}$, $\mathbf{w} = \varepsilon_{m-1} + \varepsilon_m + \varepsilon_{2m}$. We may clearly also suppose that $\{\psi_S, \psi_T\} \cong A_4$. Then, by lemma 2 ((4.2.3) and (4.2.4)), $X_S$ and $X_T$

commute with each element of $CG$ which commutes with $W^u$, $W^v$, $W^w$—in particular, with each element $W_i = W^{\varepsilon_i} (i = 1, \cdots, m-1)$.

Let $M_0^{(i)}$, $M_1^{(i)}$ be the eigenspaces of $W_i$ corresponding to the eigenvalues $1, -1$ respectively. Each of the $2^{m-1}$ intersections

$$M_{i_1,\cdots,i_{m-1}} = M_{i_1}^{(1)} \cap M_{i_2}^{(2)} \cap \cdots \cap M_{i_{m-1}}^{(m-1)}$$

is invariant under the group $G = \{X_S, X_T\}$, and since

$$M_{i_1,\cdots,i_{m-1}} = \{v_{(i_1,\cdots,i_{m-1},0)}, v_{(i_1,\cdots,i_{m-1},1)}\},$$

$V$ is the direct sum

$$V = \sum_{i_1,\cdots,i_{m-1}} M_{i_1,\cdots,i_{m-1}}.$$

It follows that each irreducible component of $G$ has degree $\leqq 2$. On the other hand, every faithful representation of $A_4$ has an irreducible component of degree 3. Hence $G \not\cong A_4$.

5.4 *The exceptional cases* (*cont.*). We now consider the exceptional groups in turn.

(i) $CT_2(2)$. $\mathscr{C}\mathscr{G}_2$ is a quaternion group of order 8 and $O_2(2, 2) \cong S_3$. It is well known that $\mathfrak{A} \cong S_4$ and that $\mathfrak{J}$ (which corresponds to the normal subgroup of $S_4$ of order 4) is complemented in $\mathfrak{A}$. It can be shown that $CG$ is complemented in $C^*(CT_2)$ (though $\mathscr{C}\mathscr{G}_2$ is not complemented in $\mathscr{C}\mathscr{T}_2$).

(ii) $CT(2)$, $CT_1(2)$. These groups do not require separate consideration for $PCT_1 \subset PCT = PCT_2$.

(iii) $CT_1(4)$. Since $\mathscr{C}\mathscr{G}_1(4)$ is isomorphic to the direct product of two copies of $\mathscr{C}\mathscr{G}_2(2)$ with the central elements identified, it is clear from case (i) that $\mathscr{C}\mathscr{G}_1(4)$ has a group $H$ of outer automorphisms of order $2(3!)^2 = 72$. Since the order of $O_1(4, 2)$ is 72, $H$ is a complement of $\mathfrak{J}$ in $\mathfrak{A}$. It can be shown that $CG$ is complemented in $C^*(CT_1)$.

In the cases which follow, we shall write $ijk\cdots$ for $\varepsilon_i + \varepsilon_j + \varepsilon_k + \cdots$ and correspondingly $T_{ij\cdots}$ for $T_{\varepsilon_i+\varepsilon_j+\cdots}$ (where the $\varepsilon$'s are the unit vectors in $\mathscr{V}_{2m}$).

(iv) $CT_2(4)$. In this case, $O_2(4, 2) \cong S_5$, $O_2(4, 2) \cong A_5$. We may take $q(\alpha) = q_2(\alpha) = \alpha_1\alpha_3 + \alpha_2^2 + \alpha_2\alpha_4 + \alpha_4^2$.

The linear transformations

$$V_1 = T_4 T_2, \qquad V_2 = T_{124} T_2, \qquad V_3 = T_{234} T_2,$$

belong to $O^+(q)$ and satisfy the defining relations (4.2.1) for $A_5$; for, if $\alpha$, $\beta$ are any two of the vectors 2, 4, 124, 234, then $q(\alpha) = Q(\alpha, \beta) = 1$. Let

$$\psi_i : w^\alpha \to w^{g_i(\alpha)} w^{\alpha V'_i} \qquad (i = 1, 2, 3)$$

be corresponding automorphisms of $\mathfrak{R}(q)$, and write $g_{ij} = g_i(\varepsilon_j)$. By lemma 2, the relations

$$\psi_i^3 = (\psi_i\psi_j)^2 = 1 \qquad (i \neq j)$$

impose 8 independent linear conditions on the 12 available parameters $g_{ij}$. Hence there are $2^{12-8} = 2^4$ complements of $PCG$ in $PCT_2^+$.

Let $G$ be one of these complements, $N$ the normalizer of $G$ in $PCT_2$. Since $O^+(q)$ acts irreducibly on $\mathscr{V}_4$, $N \cap PCG = 1$. On the other hand, since the total number of complements is less than the index, $2^5$, of $G$ in $PCT_2$, $G$ is a proper subgroup of $N$. It follows that $N$ is a complement of $PCG$ in $PCT_2$. Clearly, there are $2^4$ complements of $PCG$ in $PCT_2$ and any two are conjugate in $PCT_2$. $PCT_2$ is isomorphic to the group of orthogonal affine transformations $\boldsymbol{\alpha} \to \boldsymbol{\alpha}T' + \boldsymbol{t}$ ($T \in O_2(4, 2)$, $\boldsymbol{t} \in \mathscr{V}_4$). By lemma 3, $CG$ is not complemented in $C^*(CT_2^+)$.
(v) $CT(4)'$. In this case, $Sp(4, 2) \cong S_6$, $Sp(4, 2)' \cong A_6$. Although $PCG$ is not complemented in $PCT$ (theorem 7), we shall prove that it *is complemented in* $P(CT')$.

Since any two of the vectors 1, 3, 123, 134, 1234 satisfy $f(\boldsymbol{\alpha}, \boldsymbol{\beta}) = 1$, the elements

$$V_1 = T_3T_1, \qquad V_2 = T_{123}T_1, \qquad V_3 = T_{134}T_1, \qquad V_4 = T_{1234}T_1$$

of $Sp'$ satisfy the defining relations (4.2.1) for $A_6$. (For the present case $T_{\boldsymbol{a}}$ denotes the symplectic transvection $\boldsymbol{\alpha}T'_{\boldsymbol{a}} = \boldsymbol{\alpha} + f(\boldsymbol{\alpha}, \boldsymbol{a})\boldsymbol{a}$.)

Let

$$\psi_i : W^{\boldsymbol{\alpha}} \to i^{g_i(\boldsymbol{\alpha})} W^{\boldsymbol{\alpha}V'_i} \qquad (i = 1, 2, 3, 4)$$

be corresponding automorphisms of $\mathscr{C}\mathscr{G}$. Let $k_{ij}$ be fixed integers such that $k_{ij} \equiv \phi(\varepsilon_j V'_i, \varepsilon_j V'_i)$ (mod 2). By (2.3.4), $g_i(\varepsilon_j)$ has the form $k_{ij} + 2g_{ij}$. By a slight modification of lemma 2, the relations

$$\psi_i^3 = (\psi_i\psi_j)^2 = 1 \qquad (i \neq j)$$

impose 11 independent linear conditions on the 16 available parameters $g_{ij}$, so that there are $2^5$ complements of $PCG$ in $P(CT')$. Hence $P(CT')$ is isomorphic to the group of affine transformations $\boldsymbol{\alpha} \to \boldsymbol{\alpha}T' + \boldsymbol{t}$ ($T \in Sp'(4, 2)$, $\boldsymbol{t} \in \mathscr{V}_4$). By the argument of the previous case (and since $PCG$ is not complemented in $PCT$) any two complements of $PCG$ in $P(CT')$ are conjugate in $PCT$. By the argument of lemma 3, $CG$ is not complemented in $CT'$.
(vi) $CT_1(8)$. In this case, $O_1(6, 2) \cong S_8$, $O_1^+(6, 2) \cong A_8$. We may take $q(\boldsymbol{\alpha}) = q_1(\boldsymbol{\alpha}) = \alpha_1\alpha_4 + \alpha_2\alpha_5 + \alpha_3\alpha_6$.

If $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ are any two of the vectors 14, 125, 1236, 1356, 346, 2345, 2456, then $q(\boldsymbol{\alpha}) = Q(\boldsymbol{\alpha}, \boldsymbol{\beta}) = 1$. Arguing as in case (iv), we conclude that there are $2^7$ complements of $PCG$ in $PCT_1^+$ and that any two are conjugate in $PCT_1$. $PCT_1^+$ is isomorphic to the group of affine transformations $\boldsymbol{\alpha} \to \boldsymbol{\alpha}T' + \boldsymbol{t}$ ($T \in O_1^+(6, 2)$, $\boldsymbol{t} \in \mathscr{V}_6$). $CG$ is not complemented in $C^*(CT_1^+)$.
(vii) $CT_2(8)$. In this case, $O_2(6, 2)$ is the cubic surface group of order

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51840, $O_2^+(6, 2)$ its simple subgroup of order 25920. We may take $q(\alpha) = \sum_{i=1}^{3} (\alpha_{2i-1}^2 + \alpha_{2i-1}\alpha_{2i} + \alpha_{2i}^2)$. It is required to prove that $PCG$ is not complemented in $PCT_2^+$ (see theorem 7, part (c)).

The orthogonal transvections

$$R_1 = T_{12}, \quad R_2 = T_1, \quad R_3 = T_{123456}, \quad R_4 = T_5, \quad R_5 = T_{56}, \quad R_6 = T_3,$$

satisfy the defining relations for the cubic surface group given in Coxeter and Moser [2], p. 122. Therefore the elements

$$P_0 = T_{123456} T_3 = T_{12456} T_{123456},$$
$$P_i = T_i T_{123456} \qquad (i = 1, 2, 5, 6),$$

generate $O^+(q)$. Suppose now, contrary to theorem 7, that $PCG$ is complemented in $PCT_2^+$. Since

$$P_0^3 = P_i^3 = (P_0 P_i)^2 = I \qquad (i = 1, 2, 5, 6),$$

it is possible to choose automorphisms

$$\psi_i : w^{\alpha} \to w^{g_i(\alpha)} w^{\alpha P'_i} \qquad (i = 0, 1, 2, 5, 6)$$

so that

$$\psi_0^3 = \psi_i^3 = (\psi_0 \psi_i)^2 = 1 \qquad (i = 1, 2, 5, 6).$$

By lemma 2, these relations impose 24 independent linear conditions on the 30 available parameters $g_i(\varepsilon_j)$, and therefore there are at most $2^6$ complements of $PCG$ in $PCT_2^+$. Let $G$ be one of these complements, $N$ the normalizer of $G$ in $PCT_2$. Since the index, $2^7$, of $G$ in $PCT_2$ exceeds the total number of complements, $G$ is a proper subgroup of $N$. On the other hand, since $O^+(q)$ acts irreducibly on $\mathscr{V}_6$, $N \cap PCG = 1$. It follows that $N$ is a complement of $PCG$ in $PCT_2$, contrary to theorem 7, part (b). This contradiction proves our result.

## 5. The Elements of the Clifford Transform Groups

We shall determine an element $X$ of $CT$ which induces the automorphism (2.3.1) of $CG$. If

$$Xv_{\lambda} = \sum_{\mu} \xi_{\lambda,\mu} v_{\mu} \qquad (\lambda \in \mathscr{V}_m), \tag{5.1}$$

then

$$XW^{\alpha} = i^{g(\alpha)} W^{\alpha T'} X, \tag{5.2}$$

so that

$$\xi_{\lambda+a_2,\mu+A_2} = i^{g(\alpha)+2(A_1\cdot(\mu+A_2)+a_1\cdot(\lambda+a_2))} \xi_{\lambda,\mu} \tag{5.3}$$

$$(\lambda, \mu \in \mathscr{V}_m; \quad (a_1, a_2) \in \mathscr{V}_{2m}, \qquad (a_1, a_2)T' = (A_1, A_2)).$$

Let $\mathscr{V}_T$ denote the subspace of $\mathscr{V}_{2m}$ formed by all vectors $(a_2, A_2)$, where

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$(\boldsymbol{a}_1, \boldsymbol{a}_2)$ runs over $\mathscr{V}_{2m}$, By the argument of I. § 4.1, the solution of (5.3) is unique apart from a scalar multiplier. The form of (5.3) now shows that there is a fixed coset

$$\mathscr{W}_{T,g} = (\boldsymbol{\lambda}_0, \boldsymbol{\mu}_0) + \mathscr{V}_T \tag{5.4}$$

of $\mathscr{V}_T$ such that $\xi_{\boldsymbol{\lambda},\boldsymbol{\mu}} = 0$ unless $(\boldsymbol{\lambda}, \boldsymbol{\mu}) \in \mathscr{W}_{T,g}$. Therefore, apart from an arbitrary scalar multiplier, we have

$$\begin{aligned} \xi_{\boldsymbol{\lambda}_0+\boldsymbol{a}_2, \boldsymbol{\mu}_0+\boldsymbol{A}_2} &= i^{g((\boldsymbol{a}_1, \boldsymbol{a}_2))+2(\boldsymbol{A}_1\cdot(\boldsymbol{\mu}_0+\boldsymbol{A}_2)-\boldsymbol{a}_1\cdot(\boldsymbol{\lambda}_0+\boldsymbol{a}_2))} \\ \xi_{\boldsymbol{\lambda},\boldsymbol{\mu}} &= 0 \textit{ when } (\boldsymbol{\lambda}, \boldsymbol{\mu}) \notin \mathscr{W}_{T,g}. \end{aligned} \tag{5.5}$$

Our argument shows that the first line of (5.5) depends only on $(\boldsymbol{a}_2, \boldsymbol{A}_2)$, and not directly on $(\boldsymbol{a}_1, \boldsymbol{a}_2)$. It is easy to verify this independently.

It remains to determine the coset $\mathscr{W}_{T,g}$. Let $\mathscr{U}_T$ denote the subspace of $\mathscr{V}_{2m}$ formed by the vectors $(\mathbf{a}, \mathbf{0})$ such that $(\boldsymbol{a}, \mathbf{0})T'$ has the form $(\boldsymbol{A}, \mathbf{0})$. Then, by (2.3.4),

$$g(\alpha) = 2h(\alpha) \qquad (\alpha \in \mathscr{U}_T), \tag{5.6}$$

where $h(\alpha)$ is a certain linear form on $\mathscr{U}_T$. Since the first line of (5.5) is unaltered when we replace $(\boldsymbol{a}_1, \boldsymbol{a}_2)$ by $(\boldsymbol{a}_1 + \boldsymbol{a}, \boldsymbol{a}_2)$, it follows that each element $(\boldsymbol{\lambda}_0, \boldsymbol{\mu}_0)$ of $\mathscr{W}_{T,g}$ is a solution of the system of linear equations

$$\boldsymbol{a} \cdot \boldsymbol{\lambda}_0 - \boldsymbol{A} \cdot \boldsymbol{\mu}_0 = h(\beta) \qquad (\beta = (\boldsymbol{a}, \mathbf{0}) \in \mathscr{U}_T). \tag{5.7}$$

It is clear that the rank of the system (5.7) is the dimension of $\mathscr{U}_T$. On the other hand, since $\mathscr{U}_T$ is the kernel of the linear mapping $(\boldsymbol{a}_1, \boldsymbol{a}_2) \to (\boldsymbol{a}_2, \boldsymbol{A}_2)$ of $\mathscr{V}_{2m}$ onto $\mathscr{V}_T$, we have $\dim \mathscr{V}_T + \dim \mathscr{U}_T = 2m$. *Therefore* $\mathscr{W}_{T,g}$ *is precisely the set of solutions of* (5.7).

Arguing as in I § 4.1, we see that the matrix $(\xi_{\boldsymbol{\lambda},\boldsymbol{\mu}})$ has $2^{d_T}$ elements in each row and column, where $m + d_T$ is the dimension of $\mathscr{V}_T$. Hence, by theorem 2, if $X$ is the linear transformation (5.5) we have

$$X^* X = 2^{d_T} I. \tag{5.8}$$

We now determine the elements of $CT'$, $\mathscr{C}\mathscr{T}_1$ and $\mathscr{C}\mathscr{T}_2$.

(i) $CT(2^m)'$. *We prove that, if* $T \in Sp(2m, 2)'$, *then*

$$(1 + i)^{-d_T} X \in CT'. \tag{5.9}$$

Since the scalar subgroup of $CT'$ is $\{iI\}$, it is sufficient to prove that if $\lambda^{-1}X \in CT'$ then $\lambda$ has the form $i^s(1 + i)^{d_T}$. Now $\det \lambda^{-1}X = 1$, and by theorem 6, $\lambda \in R_0(i)$. Thus $\xi = i^k\lambda^{2m}$, where $\xi = \det X$. By (5.8), $\xi\bar{\xi} = 2^{d_T 2^m}$. From these two formulae it follows that $\lambda$ is an algebraic integer of $R_0(i)$ such that $\lambda\bar{\lambda} = 2^{d_T}$. Since the principal ideal (2) of $R_0(i)$ is the square of the prime ideal $((1 + i))$, and since the units of $R_0(i)$ are the powers of $i$, $\lambda$ has the required form $i^s(1 + i)^{d_T}$.

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(ii) $\mathscr{C}\mathscr{T}_1(2^m)$. *We prove that, if* $T \in O_1(2m, 2)$ $(m \geqq 3)$, *then*

$$2^{-\frac{1}{2}d_T} X \in \mathscr{C}\mathscr{T}_1. \tag{5.10}$$

In fact, let $\lambda^{-1}X \in \mathscr{C}\mathscr{T}_1$; it is required to prove that $\lambda = \pm 2^{\frac{1}{2}d_T}$. Since $T \in O_1(2m, 2)$, $\phi(\alpha T', \alpha T') + \phi(\alpha, \alpha) \equiv 0$. Hence, by (2.3.4), $g(\alpha) \equiv 0 \pmod 2$ for all $\alpha$. By (5.5), $X$ is real. Since $\lambda^{-1}X$ is real, $\lambda$ is also real. By § 3.3, $\det \lambda^{-1}X = \pm 1$. Arguing as in the previous case, we get $\lambda = \pm 2^{\frac{1}{2}d_T}$ as required.

Comparing (5.10) with theorem 6* and (3.3.3), we get the (well known) result that $T \in O_1^+(2m, 2)$ if, and only if, $d_T$ is even.

(iii) $\mathscr{C}\mathscr{T}_2(2^m)$. Let $\varepsilon$ denote the primitive eighth root of unity $2^{-\frac{1}{2}}(1+i)$. *We prove that, if* $T \in O_2(2m, 2)$, *then there exist vectors* $\boldsymbol{s}, \boldsymbol{t}$ *such that*

$$(\boldsymbol{s}, \boldsymbol{e}_m)T' = (\boldsymbol{t}, \boldsymbol{e}_m), \tag{5.11}$$

*and that*

$$\varepsilon^k 2^{-\frac{1}{2}d_T} X \in \mathscr{C}\mathscr{T}_2, \tag{5.12}$$

*where*

$$k \equiv g\big((\boldsymbol{s}, \boldsymbol{e}_m)\big) + 2\big((\boldsymbol{s}+\boldsymbol{e}_m)\cdot \boldsymbol{\lambda}_0 + (\boldsymbol{t}+\boldsymbol{e}_m)\cdot \boldsymbol{\mu}_0\big) \pmod 4. \tag{5.13}$$

In fact, let $\lambda^{-1}X \in \mathscr{C}\mathscr{T}_2$. Since $\lambda^{-1}X$ commutes with the semi-linear transformation (3.1.2), we have

$$\xi_{\boldsymbol{\lambda}+\boldsymbol{e}_m, \boldsymbol{\mu}+\boldsymbol{e}_m} = (-1)^{\boldsymbol{e}_m\cdot(\boldsymbol{\lambda}+\boldsymbol{\mu})}\lambda\bar{\lambda}^{-1}\bar{\xi}_{\boldsymbol{\lambda},\boldsymbol{\mu}}, \tag{5.14}$$

and therefore, by (5.5), $(\boldsymbol{e}_m, \boldsymbol{e}_m) \in \mathscr{V}_T$. Hence it is possible to choose $\boldsymbol{s}, \boldsymbol{t}$ so that (5.11) holds.

We remark that, by (2.3.4) with $\boldsymbol{\alpha} = \boldsymbol{\beta} = (\boldsymbol{s}, \boldsymbol{e}_m)$,

$$g\big((\boldsymbol{s}, \boldsymbol{e}_m)\big) \equiv \boldsymbol{e}_m \cdot (\boldsymbol{s}+\boldsymbol{t}) \pmod 2. \tag{5.15}$$

By (5.15) and (5.14) with $(\boldsymbol{\lambda}, \boldsymbol{\mu}) = (\boldsymbol{\lambda}_0, \boldsymbol{\mu}_0)$, we have $\lambda\bar{\lambda}^{-1} = i^{-k}$, where $k$ satisfies (5.13). Hence $\lambda = \varepsilon^{-k}\mu$, where $\mu$ is real. Arguing as in the previous case, we now deduce that $\mu = \pm 2^{\frac{1}{2}d_T}$, which proves (5.12).

Comparing (5.12) with theorem 6 and (3.3.3), and using (5.15), we deduce that $T \in O_2^+(2m, 2)$ if, and only if, $d_T + \boldsymbol{e}_m \cdot (\boldsymbol{s}+\boldsymbol{t})$ is even.

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