THE TIME TRANSFORMATION TDB–TDT: AN ANALYTICAL FORMULA AND RELATED PROBLEM OF CONVENTION.

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Abstract. An analytical formula for the time transformation TDB–TDT valid over a few thousand years around J2000 has been computed with an accuracy at the 1 ns level. The coefficients for a formula accurate at the 100 ns level are provided here. The numerical and analytical procedures to compute this transformation are discussed. We note that these procedures cannot comply with recommendation 5 of the 1976 IAU meeting. Furthermore, these procedures yield different units for the corresponding TDB time scales. We also note that this transformation is independent of the two PPN parameters γ and β and of the 3 most commonly used coordinate systems (isotropic, standard-Schwarzschild, Painlevé) at least at the 1 ns level.

Introduction

Recommendation 5 of the 1976 IAU meeting in Grenoble (see Winkler and Van Flandern, 1977) states that:

the time scales for equations of motions referred to the barycentre of the solar system be such that there be only periodic variations between these time scales and that for the apparent geocentric ephemerides.

It should be noted that reference is made to the time scale associated with the geocentric ephemerides and not to a topocentric time scale. The complete formulation for transforming from a barycentric coordinate time to the time of an observatory located on the surface of the Earth is in Guinot (1986). We shall restrict our transformation to the time scales of the IAU recommendation.

A new impetus for the need of an accurate time transformation between the observed time of an event in Temps Dynamique Terrestre (TDT) and the corresponding coordinate time in Temps Dynamique Barycentrique (TDB) is given by the analysis of timing data of the millisecond pulsar PSR1937+214. At present, the precision of these data acquired at Arecibo is approximately 0.3 microsecond and is expected to improve to 0.1 microsecond soon. Hence, the physical model required to analyse these observations must include a time transformation TDB–TDT which is precise at the 0.01 microsecond level (one tenth of the expected observation error). We have finally set the requirement that our formula be accurate to the 1 nanosecond level for future applications.

There are two possible procedures to calculate this time transformation. The first procedure is numerical. The linear trend is computed by averaging a numerically integrated "time ephemeris" over less than a century and is then substracted from the

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"time ephemeris" itself. The resulting tabulated values provide a time transformation TDB−TDT which matches only to some extent the IAU recommendation. The second procedure is based on analytical theories for the motions of the planets and Moon. The planetary theory VSOP82 (Bretagnon, 1982) and the lunar theory ELP2000 (Chapront-Touzé and Chapront, 1983) developed at the Bureau des Longitudes are used to calculate a more accurate analytical formula for this transformation.

An Analytical Formula for the Time Transformation TDB−TDT

The Parametrised Post Newtonian (PPN) metric given by equation (4) in Brumberg (1986) describes the space-time properties of the solar system. This metric includes the contributions of all the planets and their mutual interactions. It also includes the two physical PPN parameters γ and β and the integer ν (with possible values 0, 1, 2) for selecting one of the three most commonly used coordinate systems (isotropic, standard Schwarzschild, Painlevé) in celestial mechanics.

This complete metric was used to derive the differential relation between TDT and TDB. The resulting expression is rather voluminous. However, only the terms larger than 10^{-11} in this expression must be kept to have a formula providing a 1 nanosecond accuracy after integration. The final differential expression is:

$$\frac{d\,TDT}{d\,TDB} = 1 - \left(\sum_i \frac{m_i}{\rho_i} + \frac{1}{2}\left(\frac{v}{c}\right)^2\right) \tag{1}$$

where $m_i = \frac{GM_i}{c^2}$, M_i mass of planet i

$\vec{\rho}_i = \vec{r} - \vec{r}_i$, $\vec{r}_i$ barycentric position of planet i

$\vec{r}$ and v stand for the position and velocity of the Earth , respectively.

It is interesting to note that differential equation (1) does not anymore depend explicitly either on the two physical PPN parameters γ and β or on the integer ν selecting the coordinates. Hence, at the 1 ns level, the time transformation TDB−TDT is independent of the 3 systems of coordinates considered by Brumberg (1986).

Equation (1) is given by Thomas (1975) and by Moyer (1981) who provides an analytical solution accurate to 20 μs. Hirayama and Kinoshita (1986) provide a solution which is more precise (a few μs). Various typographic errors in the solution as it is printed need to be corrected: the sign of the phase of the $2E$ term should be changed; the argument $4V - 8E + 3M$ should read $4E - 8M + 3J$; the term $0.468 \sin(E - U + 180.000)$ should read $0.468 \sin(E - N + 180.000)$.

The analytical theories VSOP82 and ELP2000 for the motions of the planets and Moon have integration constants which are adjusted on the Jet Propulsion Laboratory ephemeris DE200 (Newhall, Standish and Williams, 1983). These analytical theories provide ρ_i and v as periodic time series. Using these in (1) and keeping all the periodic terms larger than 10^{-11}, the equation is integrated and the resulting formula is in the form :

$$
\begin{aligned}
TDB = TDT + C_0T^2 + D_0T^3 + \ldots + \\
&+ \sum_i A_i \sin(\omega_{ai}T + \phi_{ai}) + \\
&+ T\sum_i B_i \sin(\omega_{bi}T + \phi_{bi}) + \\
&+ T^2\sum_i C_i \sin(\omega_{ci}T + \phi_{ci}) + \\
&+ T^3\sum_i D_i \sin(\omega_{di}T + \phi_{di})
\end{aligned} \tag{2}
$$

where T is in thousands of years from J2000.0 and TDB will be given in microseconds. The coefficients are given in Table I.
We have compared this formula with the one given by Hirayama and Kinoshita (1986) and found a good agreement at the $0.04\mu s$ level for the terms present in the two solutions. The uncertainty mentioned above (a few μs) for Hirayama's and Kinoshita's formula arises from all the neglected terms; the largest of which has an amplitude of 0.376 μs.

As pointed out in Fukushima *et al.* (1986) and in Hellings (1986), the use of the TDB unit as defined by the IAU implies that the space coordinates be multiplied by a constant factor so that, in particular, the speed of light remain constant. This factor partly depends on the linear term of the integrated equation which is dropped in formula (2) to conform to the IAU recommendation. It has the value 467.308935202 $s/10^3y$

The Convention Problem

The IAU convention, as stated in the introduction, implies that the theory of motions of the planets and Moon is constructed with purely periodic terms. However, such a theory which is valid over several millions of years is not precise locally in time. Secular variation theories are, on the contrary, more precise by construction because long periodic terms of perihelie and nodes (with periods between 45 000 and 2 000 000 years) are expanded as time polynomials . Integrating (1) analytically with such a theory leads to an expression of the time transformation TDB−TDT in the form of a time polynomial, of periodic terms and of mixed terms of the form $t^n \sin \omega t$ as in (2). To conform to the IAU recommendation, the linear term of the time polynomial is dropped. However, the higher order terms in the polynomial and the mixed terms subsist. These terms do not arise from the particular analytical representation used but are actually present in the transformation. For example, they are partly generated by the long-term variations in the eccentricity of the Earth's orbit. Hence, the IAU recommendation cannot be formally complied with.

At present, the timing data of the fast pulsar PSR1937+214 are analysed using numerical transformations (Backer and Hellings (1986), Davis *et al.* (1985)). These transformations are computed over specific spans of time (60 and 100 years ending at J2000.) emphasized in the description of the physical model. This precise definition is necessary since such numerical transformations depend slightly on the span chosen. As already mentioned, the "time ephemeris" is averaged over a few decades though it includes terms with periods longer than a few centuries or even a few millenia. Locally, these long period terms are analogous to a slope in time. To conform to the IAU recommendation, this linear term is dropped and thence long period terms are practically ignored in numerical transformations. On the contrary, long period terms are kept in the analytical transformation. The

index i	A_i (μs)	ω_{ai} (rd/10^3y)	ϕ_{ai} (rd)
1	1656.6894	6283.0758494	6.2400497
2	22.4175	5753.3848843	4.2969771
3	13.8399	12566.1516988	6.1968995
4	4.7701	529.6909651	0.4444038
5	4.6767	6069.7767539	4.0211937
6	2.2566	213.2990954	5.5431320
7	1.7307	-3.5231591	5.0189615
8	1.5555	77713.7714679	5.1984671
9	1.2768	7860.4193937	5.9888233
10	1.1934	5223.6939192	3.6498063
11	1.1153	3930.2096968	1.4227456
12	0.7942	11506.7697686	2.3223126
13	0.6003	1577.3435443	2.6782570
14	0.4968	6208.2942508	5.6967011
15	0.4863	5884.9268358	0.5199988
16	0.4686	6244.9428137	5.8663983
17	0.4484	26.2983277	3.6116882
18	0.4353	-398.1490136	4.3493415
19	0.4324	74.7815986	2.4358996
20	0.3755	5507.5532411	4.1034739
21	0.2431	-775.5226083	3.6519195
22	0.2307	5856.4776585	4.7740285
23	0.2037	12036.4607337	4.3339850
24	0.1734	18849.2275481	6.1537378
25	0.1591	10977.0788035	1.8900771
26	0.1440	-796.2980272	5.9574876
27	0.1379	11790.6290905	1.1359361
28	0.1200	38.1330356	4.5515858
29	0.1190	5486.7778222	1.9145317
30	0.1161	1059.3819302	0.8734863
31	0.1019	-5573.1428016	5.9845038
32	0.0984	2544.3144043	0.0927835
33	0.0802	206.1855484	2.0953827
34	0.0796	4694.0029541	2.9492402
35	0.0750	2942.4634179	4.9809276
36	0.0626	20.7754189	2.6543767
37	0.0644	5746.2713373	1.2804037
38	0.0638	5760.4984313	4.1680021
39	0.0588	426.5981909	4.8396652
40	0.0571	-0.9804182	0.9252472
41	0.0541	17260.1546529	3.4110896
42	0.0482	155.4203227	2.2517971
43	0.0480	2146.1653907	1.4958314
44	0.0427	632.7837393	5.7206226

Table I : Coefficients of equation (2).

index i	A_i (μs)	ω_{ai} (rd/10^3y)	ϕ_{ai} (rd)
45	0.0426	161000.6857375	1.2708377
46	0.0424	6275.9623024	2.8695872
47	0.0421	-7.1135470	3.5707209
48	0.0408	12352.8526033	3.9814932
49	0.0405	15720.8387873	2.5466120
50	0.0370	3154.6870886	5.0717851
51	0.0366	5088.6288086	3.3246566
52	0.0365	801.8209360	6.2487864
53	0.0349	522.5774181	5.2100747
54	0.0335	6062.6632069	4.1452250
55	0.0335	9437.7629379	2.4047140
56	0.0324	8827.3902537	5.5414605
57	0.0324	6076.8903009	0.7495680
58	0.0302	7084.8967854	3.3896043
59	0.0299	12139.5535079	1.7701727
60	0.0293	-71430.6956185	4.1831763
61	0.0279	-6286.5990085	5.0737086
62	0.0272	6279.5526903	5.0450074
63	0.0252	1748.0163771	2.9018643
64	0.0248	-1194.4470408	1.0870978
65	0.0226	6133.5126522	3.3080189
66	0.0225	10447.3878384	1.4607311
67	0.0217	14143.4952430	5.9526579
68	0.0209	8429.2412401	0.6522829
69	0.0203	419.4846439	3.7354887
70	0.0178	73.2971259	3.4759751
71	0.0177	6812.7668145	3.1861180
72	0.0162	10213.2855462	1.3311023
73	0.0160	-2352.8661526	6.1453853
74	0.0159	-220.4126424	4.0052889
75	0.0151	19651.0484841	3.9694831
76	0.0147	1349.8673635	4.3089139
77	0.0143	16730.4636878	3.0160582
78	0.0142	17789.8456180	2.1045498
79	0.0137	-536.8045121	5.9716728
80	0.0125	103.0927742	1.7374759
81	0.0123	3.5903879	1.7853927
82	0.0124	4690.4797950	4.7340616
83	0.0119	5643.1785631	5.4893206
84	0.0119	8031.0922265	2.0533868
85	0.0117	-4705.7323051	2.6541366
86	0.0116	5120.6011450	4.8639255
87	0.0108	553.5693363	0.8427244
88	0.0104	951.7183499	5.7177869

Table I (cont.) : Coefficients of equation (2).

index i	A_i (μs)	ω_{ai} (rd/10^3y)	ϕ_{ai} (rd)
89	0.0104	5863.5912055	1.9138804
90	0.0101	283.8593219	1.9421795
index i	**B_i (μs/10^3y)**	**ω_{bi} (rd/10^3y)**	**ϕ_{bi} (rd)**
1	102.1574	6283.0758494	4.2490312
2	1.7068	12566.1516988	4.2059040
3	0.2697	213.2990954	3.4002911
4	0.2659	529.6909651	5.8360513
5	0.2158	-3.5231591	0.0349384
6	0.0780	5223.6939192	4.6703356
7	0.0547	1577.3435443	4.5348996
8	0.0593	26.2983277	1.0873123
9	0.0344	-398.1490136	5.9800691
10	0.0321	18849.2275481	4.1629120
11	0.0336	5507.5532411	5.9801641
12	0.0292	5856.4776585	0.6238510
13	0.0277	155.4203227	3.7453675
14	0.0252	5746.2713373	2.9803823
15	0.0230	-796.2980272	1.1743887
16	0.0250	5760.4984313	2.4679632
17	0.0218	206.1855484	3.8547865
18	0.0179	-775.5226083	1.0918412
19	0.0138	426.5981909	2.6998356
20	0.0133	6062.6632069	5.8459339
21	0.0118	12036.4607337	2.2928350
22	0.0129	6076.8903009	5.3335561
23	0.0122	1059.3819302	6.2228683
24	0.0106	-7.1135470	5.1924310
25	0.0101	4694.0029541	4.0451363
26	0.0101	522.5774181	0.7493158
index i	**C_i (μs/10^6y)**	**ω_{ci} (rd/10^3y)**	**ϕ_{ci} (rd)**
1	4.3230	6283.0758494	2.6428936
2	0.1226	12566.1516988	2.4381357
3	0.1648	0.0000000	4.7123890
4	0.0195	213.2990954	1.6421878
5	0.0169	529.6909651	4.5109594
6	0.0131	-3.5231591	1.3410365
index i	**D_i (μs/10^9y)**	**ω_{di} (rd/10^3y)**	**ϕ_{di} (rd)**
1	0.1434	6283.0758494	1.1314526

Table I (end.) : Coefficients of equation (2).

difference is small but nevertheless significant as it corresponds fondamentally to a change of unit between the TDB time scales generated.

As an example, the period of the 7^{th} periodic term in formula (2), with coefficient

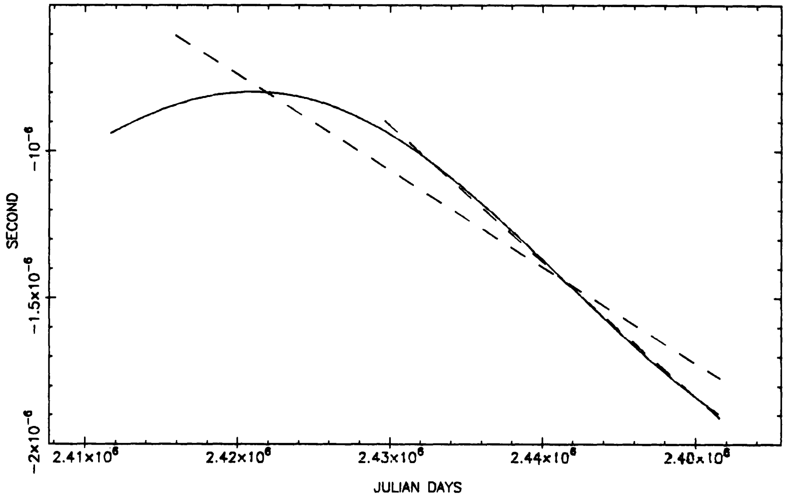

Fig. 1: the continuous curve represents the sum of all the terms with periods longer than 100 years in the time transformation TDB−TDT. The two dashed straight lines represent the averages of this sum over 60 and 100 years ending at J2000.

A_7, is 1800 years and it can be expanded locally around J2000 as $a + bt + ct^2 + \ldots$, where $a = -16500 \times 10^{-10}s$, $b = -5.831 \times 10^{-17}ss^{-1}$ and $c = 1.0283 \times 10^{-26}ss^{-2}$. In the case of the fast pulsar PSR1937+214, dropping b in the numerical procedure is equivalent to a relative change of the TDB unit of -5.831×10^{-17}. Therefore, the measured period of PSR1937+214 ($\approx 1.56ms$) would differ by $\approx 10^{-19}s$ if compared to its value determined with an analytical formula. At present, the period of PSR1937+214 is measured at the 10^{-16}s level. In reality, the total effect arises from many long period terms and we have plotted in figure 1 the time function summing all the terms with periods longer than 100 years and fitted 2 slopes to this function over the 2 time intervals used by Backer and Hellings (1986) and Davis *et al.* (1985). The slopes differ by $16 \times 10^{-17}s/s$ when the time span is changed for these two numerical transformations.

Conclusion

The integration of a "time ephemeris" over a time interval spanning just a few decades implies that the resulting numerical time transformations TDB−TDT are only valid for that particular time interval. This difficulty vanishes when one uses an analytical expression of this time transformation even if one considers an interval as long as a few thousand years around J2000. The expression (2) above provides a time transformation TDB−TDT accurate at the 100 ns level; however we have computed a complete expression accurate at the 1 ns level which is available on request.

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