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SOME RESULTS IN A CORRELATED RANDOM WALK

BY
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1. **Introduction.** In connection with a statistical problem concerning the Galton-test Cśaki and Vincze [1] gave for an equivalent Bernoullian symmetric random walk the joint distribution of g and k, denoting respectively the number of positive steps and the number of times the particle crosses the origin, given that it returns there on the last step. In the present paper the corresponding results are obtained for an unsymmetrically correlated random walk in a compact form in terms of the hypergeometric function ${}_2F_1$. The event of a return to the starting position has been investigated in some detail.

A particle moves along a straight line a unit distance during every interval τ. During the first interval τ, the particle moves to the right with probability ρ_1 and to the left with probability $\rho_2 = 1-\rho_1$. Thereafter during each interval τ, its movements are governed by the transition probability matrix

$$\begin{array}{c} \\ \text{right} \\ \text{left} \end{array} \begin{array}{c} \begin{array}{cc} \text{right} & \text{left} \end{array} \\ \left[\begin{array}{cc} p_1 & q_1 \\ q_2 & p_2 \end{array}\right] \end{array}; \qquad p_1+q_1 = 1 = p_2+q_2.$$

It can be proved by induction that $P(x_k = 1)$, the probability that the particle moves to the right during the kth step is

$$q_2[1-(\rho_2-q_1\rho_1/q_2)(p_1-q_2)^{k-1}]/(q_1+q_2).$$

It follows that, for large k,

$$P(x_k = 1) = 1-P(x_k = -1) = q_2/(q_1+q_2),$$

indicating asymptotic stable phase of the walk except in the trivial case $|p_1-q_2| = 1$.

The coefficient of correlation between two consecutive steps when $\rho_1 = \rho_2$ is

$$\rho = \delta/(\delta_1+\delta_2-\delta_1\delta_2)^{1/2},$$

where

$$(1.1) \quad \delta = p_1-q_2,\ \delta_1 = [(p_1p_2)^{1/2}-(q_1q_2)^{1/2}]^2,\ \delta_2 = [(p_1p_2)^{1/2}+(q_1q_2)^{1/2}]^2,$$

so that $\delta^2 = \delta_1\delta_2$.

The square roots are as usual taken with the +ve sign. As will be seen in the sequel, δ_1 and δ_2 are useful in abbreviating a number of expressions.

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2. **Derivation of telegraph equation.** Defining $P(x, t)$ as the probability that a particle, performing a symmetric correlated random walk along a straight line reaches x from 0 at a time t, Goldstein [2] obtains under specialized conditions a second order differential equation in $P(x, t)$ known as the telegraph equation. The corresponding equation is derived below for an unsymmetric correlated random walk on taking Δx as the length of a step and Δt as the time between two consecutive steps. If $A(x, t)$ and $B(x, t)$ are the respective probabilities that the particle arrives at x from the left or from the right, then the following relations hold:

$$P(x, t) = A(x, t)+B(x, t)$$

(2.1)
$$\begin{aligned} A(x, t+\Delta t) &= p_1 A(x-\Delta x, t)+q_2 B(x-\Delta x, t) \\ &= p_1 P(x-\Delta x, t)-\delta B(x-\Delta, t), \end{aligned}$$

(2.2)
$$\begin{aligned} B(x, t+\Delta t) &= p_2 B(x+\Delta x, t)+q_1 A(x+\Delta x, t) \\ &= p_2 P(x+\Delta x, t)-\delta A(x+\Delta x, t). \end{aligned}$$

Here the probabilities $P(x, t)$, $A(x, t)$, and $B(x, t)$ may be regarded as the limit of probabilities concerning the discrete process.

From (2.1) and (2.2)

$$\begin{aligned} A(x+\Delta x, t)+B(x-\Delta x, t) &= p_1 A(x, t-\Delta t)+q_2 B(x, t-\Delta t) \\ &\quad +p_2 B(x, t-\Delta t)+q_1 A(x, t-\Delta t) \\ &= A(x, t-\Delta t)+B(x, t-\Delta t) = P(x, t-\Delta t). \end{aligned}$$

Hence, by addition of (2.1) and (2.2)

$$P(x, t+\Delta t) = p_1 P(x-\Delta x, t)+p_2 P(x+\Delta x, t)-\delta P(x, t-\Delta t).$$

Expanding this by Taylor's theorem, neglecting terms of a higher order than $(\Delta t)^2$ and using the norming

$$p_1+p_2 = 2(1-\Delta t/2C), \qquad p_1-p_2 = 2D\Delta x,$$

we obtain

(2.3)
$$\frac{\partial^2 P}{\partial t^2}+\frac{1}{C}\frac{\partial P}{\partial t} = v^2\left(\frac{\partial^2 P}{\partial x^2}-2D\frac{\partial P}{\partial x}\right),$$

where $v=\Delta x/\Delta t$ is the velocity of the particle and C and D are constants.

This equation with $D=0$ (i.e. $p_1=p_2$) is the Telegraph equation without leakage. Equation (2.3) can easily be solved by standard methods.

3. **First passage to** r. Let $(a_{r,n}; b_{r,n})\equiv$conditional probability of a particle reaching r for the first time on the nth step (i.e. at time $\tau_0+n\tau$) given that it arrived on the origin at time τ_0 from (left; right).

Supposing that the first passage through 1 occurs at the kth step ($k=1, 2, \ldots$) and also supposing that it reaches 0 from the right at time τ_0, the probability of the first passage through 2 at the nth step is

(3.1) $$b_{2,n} = \sum_k b_{1,k}\, a_{1,n-k}.$$
Similarly $$a_{-2,n} = \sum_k a_{-1,k}\, b_{-1,n-k}.$$

Defining $a_{r,0} \equiv 0 \equiv b_{r,0}$ for $r > 0$, the relations (3.1) are true for $n \geq 0$ and the limits for k can be taken as 0 to n.

Multiply (3.1) by t^n, sum over all n and denote the probability generating functions (PGF) by corresponding capital letters; then

(3.2) $$\begin{aligned} B_2(t) &= B_1(t)A_1(t); \\ A_{-2}(t) &= B_{-1}(t)A_{-1}(t). \end{aligned}$$

The PGF for the first passage through $+1$ and -1 are further seen to be connected by the relations

(3.3) $$\begin{aligned} A_1(t) &= p_1t + q_1tB_2(t), \\ B_1(t) &= q_2t + p_2tB_2(t), \\ A_{-1}(t) &= q_1t + p_1tA_{-2}(t), \\ B_{-1}(t) &= p_2t + q_2tA_{-2}(t). \end{aligned}$$

From (3.2) and (3.3), we obtain

(3.4) $$\begin{aligned} p_2A_1(t) = p_1B_{-1}(t) &= [1 + \delta t^2 - \{(1-\delta_1 t^2)(1-\delta_2 t^2)\}^{1/2}]/2t, \\ q_1B_1(t) = q_2A_{-1}(t) &= p_2A_1(t) - t\delta. \end{aligned}$$

From (3.3) and (3.4)

(3.5) $$A_1(t) = p_1t/[1 - q_1tB_1(t)];$$

(3.6) $$A_1(t)B_1(t) = [(1-\delta_1t^2)^{1/2} - (1-\delta_2t^2)^{1/2}]^2/4q_1p_2t^2.$$

(i) *First return to the origin.* Define $(p_n^{(1)}; q_n^{(1)}) \equiv$ probability of a particle returning for the first time (on the nth step) to the starting position, given that the first step is to the (right; left).

Transferring the origin to the position reached by the particle in the first step, we get

(3.7) $$P^{(1)}(t) \equiv \sum_{n=1}^{\infty} p_n^{(1)}t^n = tA_{-1}(t) = q_1tB_1(t)/q_2,$$
and $$Q^{(1)}(t) \equiv \sum_{n=1}^{\infty} q_n^{(1)}t^n = tB_1(t).$$

From (3.4),

$$q_1B_1(1) = \tfrac{1}{2}\{q_2 + q_1 - |q_2 - q_1|\}.$$

It therefore follows that

$$P^{(1)}(1) = \begin{cases} 1 & \text{if } q_1 > q_2 \\ q_1/q_2 & \text{if } q_2 > q_1, \end{cases}$$

and
$$Q^{(1)}(1) = \begin{cases} q_2/q_1 & \text{if } q_1 > q_2 \\ 1 & \text{if } q_2 > q_1. \end{cases}$$

To interpret these results in physical terms, a large number α of noninteracting particles should be supposed to have started from the origin, $\alpha\rho_1$ to the right and $\alpha\rho_2$ to the left. In case $q_1 > q_2$, all the $\alpha\rho_1$ particles and a fraction q_2/q_1 of the $\alpha\rho_2$ particles return sooner or later to the origin, i.e. a fraction $\rho_2(1-q_2/q_1)$ of the particles on an average never returns to the origin. Similarly, in case $q_2 > q_1$, a fraction $\rho_1(1-q_1/q_2)$ of the particles on an average will not return to the origin.

The PGF of a first return to the starting position is then

$$\begin{aligned} F(t) &= \rho_1 P^{(1)}(t) + \rho_2 Q^{(1)}(t) \\ &= (\rho_1 q_1 + \rho_2 q_2)[1-\delta t^2 - \{(1-\delta_1 t^2)(1-\delta_2 t^2)\}^{1/2}]/2q_1 q_2, \end{aligned}$$

and the coefficient of t^{2n} in its expansion shows that the probability of a first return to the starting position at the $(2n)$th step is

$$(3.8) \qquad f_{0,2n} = \frac{(\rho_1 q_1 + \rho_2 q_2)}{2q_1 q_2} \frac{(-)^{n+1}\delta_2^n \Gamma(\frac{3}{2})}{\Gamma(\frac{3}{2}-n)\Gamma(n+1)} \, {}_2F_1\left(-\tfrac{1}{2}; -n; \tfrac{3}{2}-n; \frac{\delta_1}{\delta_2}\right)$$

in generalization of a result obtained by Seth [3]. The function ${}_2F_1(a; b; c; x)$ is the well known Gauss function defined by

$${}_2F_1(a; b; c; x) = \sum_{r=0}^{\infty} \frac{(a)_r (b)_r}{(c)_r} \frac{x^r}{x!},$$

where $(a)_r = a(a+1)\ldots(a+r-1)$, $r = 1, 2, 3, \ldots$; $(a)_0 = 1$, the series being absolutely convergent whenever $|x| < 1$ and when $x = 1$, provided that $\text{Re}\,(c-a-b) > 0$.

The probability that the accumulated number of positive and negative steps (i.e. to the right and to the left) will ever equalize (i.e. the particle would return to the origin) is

$$\begin{aligned} F(1) &= (\rho_1 q_1 + \rho_2 q_2)[q_1 + q_2 - |q_2 - q_1|]/2q_1 q_2 \\ &= \begin{cases} \rho_2 + q_1\rho_1/q_2 & \text{for } q_2 > q_1, \\ \rho_1 + q_2\rho_2/q_1 & \text{for } q_1 < q_2, \end{cases} \end{aligned}$$

indicating that the return to the starting position for $q_1 \neq q_2$ is not a persistent event.

While (3.8) gives the recurrence time distribution for the first return, that for subsequent returns is obtained by substituting $\rho_1 = q_2$.

Thus return to the origin becomes an undelayed recurrent event if $\rho_1 = q_2$ so that $\rho_2 = q_2$.

The GF of u_n, the probability of a return to the starting position, on taking $\rho_1 q_1 + \rho_2 q_2 = 2q_1 q_2$ and also $u_0 \equiv 1$ is

$$\begin{aligned} u(t) \equiv \sum_{n=0}^{\infty} u_n t^n &= 1/[1-F(t)] \\ &= \{[(1-\delta_1 t^2)(1-\delta_2 t^2)]^{1/2} - \delta t^2\} \sum_{i=0}^{\infty} (\delta_1 + \delta_2)^i t^{2i}, \end{aligned}$$

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so that

$$u_n = \frac{(-)^n\sqrt{\pi}}{2}\sum_{i=0}^{n}\frac{(-)^i(\delta_1+\delta_2)^i\delta_2^{n-i}}{\Gamma(n+1-i)\Gamma(\frac{3}{2}+i-n)}$$
$$\times {}_2F_1\left(-\tfrac{1}{2};\, i-n;\, \tfrac{3}{2}+i-n;\, \frac{\delta_1}{\delta_2}\right)-\delta(\delta_1+\delta_2)^{n-1}.$$

(ii) *Return without crossing the origin.* Let $(p_n; q_n)$ denote the probability with which a particle, which has taken the first step to the (right; left), returns to the starting position at the nth step without having in the meantime crossed it.

Now if in a path of the type p_n, the first step is removed and instead of -1 is attached at the nth step, there results a path of the type $a_{-2,n}$ giving

$$p_2 p_n = \alpha_{-2,n},$$

so that

$$P(t) \equiv \sum_{n=1}^{\infty} p_n t^n = A_{-2}(t)/p_2 = q_1 A_1(t) B_1(t)/q_2 p_1 \tag{3.9}$$
$$= [(1-\delta_1 t^2)^{1/2}-(1-\delta_2 t^2)^{1/2}]^2/4t^2 p_1 p_2 q_2.$$

Similarly

$$Q(t) \equiv \sum_{n=1}^{\infty} q_n t^n = q_2 P(t)/q_1.$$

4. **PGF for i crosses of the origin and g steps on the right.** Let $(p_{g,n}^{(i)}; q_{g,n}^{(i)}) \equiv$ probability of a particle returning to the starting position on the nth step after crossing it i times and spending g steps on the right of it, given that the first step is to be (right; left).

Clearly for $n<g+i$

$$p_{g,n}^{(i)} = 0 = q_{g,n}^{(i)}.$$

We write

$$p_{n,n}^{(0)} = p_n \quad \text{and} \quad q_{0,n}^{(0)} = q_n,$$

p_n and q_n being defined in (iii) of §2.

Defining

$$p_{0,0}^{(0)} \equiv q_{0,0}^{(0)} \equiv 0,$$

and the PGF

$$P_n^{(i)}(s) \equiv \sum_{g=0}^{n} p_{g,n}^{(i)} s^g; \qquad P^{(i)}(s,t) \equiv \sum_{n=0}^{\infty} P_n^{(i)}(s) t^n,$$

and similarly for

$$Q_n^{(i)}(s) \quad \text{and} \quad Q^{(i)}(s,t),$$

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we have

$$P_n^{(0)}(s) = p_n s^n; \qquad Q_n^{(0)}(s) = q_n,$$

and

$$\text{(4.1)} \quad \begin{aligned} P^{(0)}(s, t) &\equiv \sum_{n=0}^{\infty} p_n(st)^n = P(st), \\ Q^{(0)}(s, t) &\equiv \sum_{n=0}^{\infty} q_n t^n = Q(t); \end{aligned}$$

the actual expressions can be obtained by using (3.9) and (1.1).

Considering the two contingencies that after the first return to the origin on the rth step, say, that the particle goes again to the right or crosses the origin, we have

$$p_{g,n}^{(i)} = \sum_r a_{-1,r-1}(q_2 p_{g-r,n-r}^{(i)} + p_2 q_{g-r,n-r}^{(i-1)}),$$

and

$$q_{g,n}^{(i)} = \sum_r b_{1,r-1}(q_1 q_{g,n-r}^{(i)} + p_1 p_{g,n-r}^{(i-1)}).$$

Multiplying these by $s^g t^n$ and summing over

$$g = 0, 1, 2, \ldots, n \quad \text{and} \quad n = 0, 1, \ldots, \infty$$

we get

$$P^{(i)}(s, t) = p_2 P(st) Q^{(i-1)}(s, t),$$

and

$$Q^{(i)}(s, t) = p_1 Q(t) P^{(i-1)}(s, t),$$

whence

$$\text{(4.2)} \left\{ \begin{aligned} & P^{(i)}(s, t) = p_1 p_2 P(st) Q(t) P^{(i-2)}(s, t), \\ \text{and} \quad & Q^{(i)}(s, t) = p_1 p_2 P(st) Q(t) Q^{(i-2)}(s, t). \end{aligned} \right.$$

These, on using (4.1) show that for an even number $2k$ of crosses, the corresponding PGF are

$$\text{(4.3)} \quad P^{(2k)}(s, t) = [p_1 p_2 q_2 P(st) P(t)/q_1]^k P(st),$$

and

$$\text{(4.4)} \quad Q^{(2k)}(s, t) = q_2[p_1 p_2 q_2 P(st) P(t)/q_1]^k P(t)/q_1.$$

Using (4.1) and (4.2) the corresponding PGF for an odd number $(2k-1)$ of crosses are given by

$$\text{(4.5)} \quad p_1 P^{(2k-1)}(s, t) = p_2 Q^{(2k-1)}(s, t) = [p_1 p_2 q_2 P(st) P(t)/q_1]^k.$$

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Writing out the actual expressions by using (3.9) and (1.1) and expanding by the binomial theorem we obtain the probabilities

$$\sqrt{(p_1p_2q_2/q_1)}p^{(2k)}_{2g,2n} = \delta(k+1, g)\delta(k, n-g);$$
$$\sqrt{(p_1p_2q_1/q_2)}q^{(2k)}_{2g,2n} = \delta(k, g)\delta(k+1, n-g);$$

and

$$p_1p^{(2k-1)}_{2g,2n} = p_2q^{(2k-1)}_{2g,2n} = \delta(k, g)\delta(k, n-g)$$

where

$$\delta(k, g) = \frac{(-\delta_2)^{k+g}}{(16p_1p_2q_1q_2)^{k/2}} \sum_{i=0}^{2k} \frac{\binom{2k}{i}(-)^i\Gamma(k+1-\frac{1}{2}i)}{\Gamma(g+k+1)\Gamma(1-g-\frac{1}{2}i)} \times {}_2F_1\left(-\tfrac{1}{2}i;\ -g-k;\ 1-g-\tfrac{1}{2}i;\ \frac{\delta_1}{\delta_2}\right). \tag{4.6}$$

The substitution $s=1$ in (4.3) gives the PGF for a particle which has taken the first step to the right, returning to its starting position on the $2n$th step after crossing it $2k$ times, without regard to the number of steps on the right; and the coefficient of t^{2n} in the expansion of $p^{(2k)}(1, t)$ gives $p^{(2k)}_{2n}$. These operations in (4.3), (4.4) and (4.5) then give

$$\sqrt{(p_1p_2q_2/q_1)}p^{(2k)}_{2n} = \sqrt{(p_1p_2q_1/q_2)}q^{(2k)}_{2n} = \delta(2k+1, n),$$
$$p_1p^{(2k-1)}_{2n} = p_2q^{(2k-1)}_{2n} = \delta(2k, n).$$

For a Bernoullian symmetric random walk (i.e. $p_1=p_2=\rho_1=\frac{1}{2}$), a use of $\delta_1=0$ and $\delta_2=1$ in (4.3), (4.4) and (4.5) verifies the following results due to Cśaki and Vincze [1]:

$$2^{2n}\, z^{(2k)}_{2g,2n} = \frac{k+1}{g}\binom{2g}{g-k-1}\frac{k}{(n-g)}\binom{2n-2g}{n-g-k}+\frac{k}{g}\binom{2g}{g-k}\frac{k+1}{(n-g)}\binom{2n-2g}{n-g-k-1}$$

and

$$2^{2n}\, z^{(2k-1)}_{2g,2n} = 2\frac{k}{g}\binom{2g}{g-k}\frac{k}{(n-g)}\binom{2n-2g}{n-g-k}.$$

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