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The Classical N-body Problem in the Context of Curved Space

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Abstract. We provide the differential equations that generalize the Newtonian N-body problem of celestial mechanics to spaces of constant Gaussian curvature κ, for all $\kappa \in \mathbb{R}$. In previous studies, the equations of motion made sense only for $\kappa \neq 0$. The system derived here does more than just include the Euclidean case in the limit $\kappa \to 0$; it recovers the classical equations for $\kappa = 0$. This new expression of the laws of motion allows the study of the N-body problem in the context of constant curvature spaces and thus offers a natural generalization of the Newtonian equations that includes the classical case. We end the paper with remarks about the bifurcations of the first integrals.

1 Introduction

The idea that geometry and physics are intimately related made its way into human thought during the early part of the 19th century due to the discovery of hyperbolic geometry. After that, the Euclidean nature of physical space could not be taken for granted anymore. Gauss measured the angles of a triangle formed by three mountain peaks near Göttingen, Germany, apparently hoping to learn whether the universe has positive or negative curvature, but the inevitable observational errors rendered his results inconclusive [6,20,21]. In the 1830s, Bolyai and Lobachevsky took these investigations further. They independently addressed the connection between geometry and physics by seeking a natural extension of the gravitational law from Euclidean to hyperbolic space [2, 24]. Their idea led to the study of the Kepler problem and the 2-body problem in spaces of nonzero constant Gaussian curvature $\kappa \neq 0$, two fundamental problems that are not equivalent, unlike in Euclidean space [26]. A detailed history of the results obtained in this direction since Bolyai and Lobachevsky, as well as the reasons why their approach provides a natural way of extending gravitation to spaces of constant Gaussian curvature (a crucial aspect we also briefly address in Section 2), can be found in [6, 8, 9].

Some recent studies [4–17,25] introduced a suitable framework for generalizing the equations of motion suggested by Bolyai and Lobachevsky to $N \geq 2$ bodies. Like the curved Kepler problem and the curved 2-body problem, our equations made sense in spaces of constant Gaussian curvature $\kappa \neq 0$, *i.e.*, on 3-spheres of radius $R = \kappa^{-1/2}$ embedded in $\mathbb{R}^4$, for $\kappa > 0$, and on hyperbolic 3-spheres of imaginary radius $iR = \kappa^{-1/2}$ embedded in the Minkowski space $\mathbb{R}^{3,1}$, for $\kappa < 0$. But whether written in extrinsic or intrinsic coordinates, these equations contain undetermined expressions for $\kappa = 0$, although we can recover the classical Newtonian system when $\kappa \to 0$. So a study of

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the flat case in the context of curved space, including some understanding of the bifurcations and the stability of solutions when the parameter κ is varied through 0, is impossible to perform in that setting.

In this paper we derive some equations of motion that overcome the difficulties mentioned above. Using a coordinate system in $\mathbb{R}^4$ having the origin at the north pole of the 3-spheres (the only point that is common to all the manifolds involved), we prove that the N-body problem in spaces of constant Gaussian curvature $\kappa \in \mathbb{R}$ can be written as

$$\ddot{\mathbf{r}}_i = \sum_{j=1, j\neq i}^{N} \frac{m_j\Big[\mathbf{r}_j - \big(1 - \frac{\kappa r_{ij}^2}{2}\big)\mathbf{r}_i + \frac{r_{ij}^2 \mathbf{R}}{2}\Big]}{r_{ij}^3\big(1 - \frac{\kappa r_{ij}^2}{4}\big)^{3/2}} - (\dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_i)(\kappa \mathbf{r}_i + \mathbf{R}), \quad i = 1, \dots, N, \tag{1.1}$$

where $m_1, \dots, m_N > 0$ represent the masses. The dot $\cdot$ denotes the standard inner product of signature $(+,+,+,+)$ for $\kappa \geq 0$, but the Lorentz inner product of signature $(+,+,+,-)$ for $\kappa < 0$. The vectors $\mathbf{R}$ and $\mathbf{r}_i$ are given by

$$\mathbf{R} = (0, 0, 0, \sigma|\kappa|^{1/2}), \quad \mathbf{r}_i = (x_i, y_i, z_i, \omega_i), \quad i = 1, \dots, N,$$

σ is the signum function, *i.e.*, $\sigma = +1$ for $\kappa \geq 0$ and $\sigma = -1$ for $\kappa < 0$, and

$$r_{ij} := \big[(x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2 + \sigma(\omega_i - \omega_j)^2\big]^{1/2}$$

is the Euclidean distance for $\kappa \geq 0$ and the Minkowski distance for $\kappa < 0$.

Notice that the distances r_{ij} vary smoothly with κ. In particular, the values of the coordinates ω_i, $i = 1, \dots, N$, and consequently the values of the expressions

$$(\omega_i - \omega_j)^2, \quad i, j \in \{1, \dots, N\}, i \neq j,$$

become small when κ gets close to 0, to vanish at $\kappa = 0$ since $\omega_i = 0$, $i = 1, \dots, N$, on the three-dimensional Euclidean manifold.

For $\kappa \neq 0$, the initial conditions must be taken such that the bodies are restricted to 3-spheres for $\kappa > 0$ and hyperbolic 3-spheres for $\kappa < 0$. For $\kappa = 0$ and $\mathbf{r}_i = (x_i, y_i, z_i, 0)$, $i = 1, \dots, N$, we recover the Newtonian equations,

$$\ddot{\mathbf{r}}_i = \sum_{j=1, j\neq i}^{N} \frac{m_j(\mathbf{r}_j - \mathbf{r}_i)}{r_{ij}^3}, \quad i = 1, \dots, N.$$

To make (1.1) analytic for all values of the parameter, we can introduce the substitution $\delta = \sigma|\kappa|^{1/2}$. This slight modification of the equations of motion will be helpful in future studies of the bifurcations of solutions when the new parameter passes through the value $\delta = 0$.

The rest of this paper is organized as follows. We first introduce the equations of motion in extrinsic coordinates and explain why they fall short of our goal (Section 2). Then we derive the north pole equations in the hope that they will help us solve our problem (Section 3). Unfortunately they do not, but they get us a step closer towards finding a solution. We also derive the equations of motion in intrinsic coordinates (Section 4) and explain why they also fail to address our concerns. Then we prove that all these equations can be extended to (1.1), the only framework we have found so far that offers a unified picture for all $\kappa \in \mathbb{R}$ (Section 5). We end our paper with a discussion of the bifurcations encountered by the integrals of motion when the new parameter $\delta = \sigma|\kappa|^{1/2}$ passes through the value $\delta = 0$ (Section 6).

2 Equations of Motion in Extrinsic Coordinates

In this section we present the equations of motion of the curved N-body problem in extrinsic coordinates and explain how the flat case is obtained in the limit when $\kappa \to 0$. But before getting into the details, we would like to mention why the approach of Bolyai and Lobachevsky is the natural way to extend gravitation to spaces of nonzero constant Gaussian curvature.

The reason for introducing this extension is purely mathematical. There is no unique way of generalizing the classical equations of motion in order to recover them when the curved ambient space becomes flat. So the potential we want to use should satisfy the same basic properties the Newtonian potential does in its most basic setting, the Kepler problem, a particular case when one body moves around a fixed attracting centre.

Two fundamental properties characterize the Newtonian potential of the Kepler problem: it is a harmonic function in three dimensions (but not in two dimensions), *i.e.*, it satisfies Laplace's equation, and it generates a central field in which all bounded orbits are closed, a result proved by Joseph Louis Bertrand in 1873 [1]. In the early years of the twentieth century, Heinrich Liebmann proved that these properties are also satisfied by the Kepler problem in spaces of constant curvature, thus offering strong arguments for this mathematical generalization of the gravitational force [22, 23].

Let us further present our approach to the gravitational extension first suggested by Bolyai and Lobachevski. Take $N \geq 2$ point masses, $m_1, \dots, m_N > 0$, moving on the 3-sphere (of constant Gaussian curvature $\kappa > 0$),

$$\mathbb{S}^3_\kappa := \{(x, y, z, w) \mid x^2 + y^2 + z^2 + w^2 = \kappa^{-1}, \kappa > 0\},$$

viewed as embedded in $\mathbb{R}^4$, or on the hyperbolic 3-sphere (of constant Gaussian curvature $\kappa < 0$), $\mathbb{H}^3_\kappa := \{(x, y, z, w) \mid x^2 + y^2 + z^2 - w^2 = \kappa^{-1}, w > 0,\ \kappa < 0\}$, viewed as embedded in the Minkowski space $\mathbb{R}^{3,1}$. We consider these spaces in the framework of classical mechanics; so, unlike in special or general relativity, the Minkowski space mentioned above has four spatial components instead of one temporal and three spatial dimensions. Consequently the notation $\mathbb{R}^{3,1}$ we adopt here rather expresses the signature of the inner product defined below instead of the nature of the components.

The coordinates of the point mass m_i are given by the components of the vector $\mathbf{q}_i = (x_i, y_i, z_i, w_i)$, and they satisfy the constraints $x_i^2 + y_i^2 + z_i^2 + \sigma w_i^2 = \kappa^{-1}$, for $i = 1, \dots, N$, where σ is the signum function

$$\sigma := \begin{cases} +1 & \text{for } \kappa \geq 0 \\ -1 & \text{for } \kappa < 0. \end{cases}$$

We define the inner product of the vectors $\mathbf{q}_i$ and $\mathbf{q}_j$ by the formula

$$\mathbf{q}_i \cdot \mathbf{q}_j := x_i x_j + y_i y_j + z_i z_j + \sigma w_i w_j.$$

This is the standard inner product in $\mathbb{R}^4$, of signature $(+,+,+,+)$, for $\kappa \geq 0$, but the Lorentz inner product in the Minkowski space $\mathbb{R}^{3,1}$, of signature $(+,+,+,-)$, for $\kappa < 0$.

Let us consider the notations

$$q^{ij} := \mathbf{q}_i \cdot \mathbf{q}_j, \quad i, j \in \{1, 2, \dots, N\}, i \neq j,$$
$$q_i^2 := \mathbf{q}_i \cdot \mathbf{q}_i, \quad i = 1, \dots, N,$$

and define the distance between the point masses m_i and m_j as

$$d_\kappa(\mathbf{q}_i, \mathbf{q}_j) := \begin{cases} \kappa^{-1/2} \cos^{-1}(\kappa \mathbf{q}_i \cdot \mathbf{q}_j) & \kappa > 0, \\ |\mathbf{q}_i - \mathbf{q}_j| & \kappa = 0, \\ (-\kappa)^{-1/2} \cosh^{-1}(\kappa \mathbf{q}_i \cdot \mathbf{q}_j) & \kappa < 0, \end{cases}$$

which in $\mathbb{S}^3_\kappa$ and $\mathbb{H}^3_\kappa$ represents the arc distance and implies that the force between bodies acts along geodesics (see [6, 8, 9]).

As shown in [6, 8, 14], the cotangent force function,

$$U_\kappa(\mathbf{q}) = \begin{cases} \sum_{1 \leq i < j \leq N} m_i m_j \cot(d_\kappa(\mathbf{q}_i, \mathbf{q}_j)) & \kappa > 0, \\ \sum_{1 \leq i < j \leq N} m_i m_j \coth(d_\kappa(\mathbf{q}_i, \mathbf{q}_j)) & \kappa < 0, \end{cases}$$

which extends the classical Newtonian force function to $\mathbb{S}^3_\kappa$, and $\mathbb{H}^3_\kappa$ for $\kappa \neq 0$ in the direction suggested by Bolyai and Lobachevski, can be put into the form

$$U_\kappa(\mathbf{q}) = \sum_{1 \leq i < j \leq N} \frac{m_i m_j |\kappa|^{1/2} \kappa q^{ij}}{|(\kappa q_i^2)(\kappa q_j^2) - (\kappa q^{ij})^2|^{1/2}}, \tag{2.1}$$

where $\mathbf{q} := (\mathbf{q}_1, \dots, \mathbf{q}_N)$ is the configuration of the particle system. But U_κ is a homogeneous function of degree 0, so Euler's relationship,

$$\mathbf{q}_i \cdot \nabla_{\mathbf{q}_i} U_\kappa(\mathbf{q}) = 0, \quad i = 1, \dots, N,$$

is satisfied [19] where the gradient operator is defined as

$$\nabla_{\mathbf{q}_i} = \Big(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i}, \frac{\partial}{\partial z_i}, \sigma \frac{\partial}{\partial w_i} \Big), \quad i = 1, \dots, N.$$

Using the variational method of constrained Lagrangian dynamics (see [6, 8, 14]), it can be shown that the equations of motion are given by the system of differential equations

$$m_i \ddot{\mathbf{q}}_i = \nabla_{\mathbf{q}_i} U_\kappa(\mathbf{q}) - \kappa m_i (\dot{\mathbf{q}}_i \cdot \dot{\mathbf{q}}_i) \mathbf{q}_i, \quad i = 1, \dots, N, \tag{2.2}$$

where $\kappa \neq 0$ and

$$\nabla_{\mathbf{q}_i} U_\kappa(\mathbf{q}) = \sum_{j=1, j \neq i}^{N} \frac{m_i m_j |\kappa|^{3/2} \kappa q_j^2 [(\kappa q_i^2) \mathbf{q}_j - (\kappa q^{ij}) \mathbf{q}_i]}{|(\kappa q_i^2)(\kappa q_j^2) - (\kappa q^{ij})^2|^{3/2}}, \quad i = 1, \dots, N.$$

To keep the bodies on the respective manifolds, it is enough to assume that at the initial time $t = 0$, the position vectors and the velocities satisfy the constraints

$$\kappa q_i^2 = 1, \quad \mathbf{q}_i \cdot \dot{\mathbf{q}}_i = 0, \quad i = 1, \dots, N,$$

conditions that hold for all the time t for which the solution is defined.

Using the constraints $\kappa q_i^2 = 1$, $i = 1, \dots, N$, we can write the gradient of the force function U_κ on the manifolds of curvature $\kappa \neq 0$ as

$$\nabla_{\mathbf{q}_i} U_\kappa(\mathbf{q}) = \sum_{j=1, j\neq i}^{N} \frac{m_i m_j |\kappa|^{3/2}[\mathbf{q}_j - (\kappa q^{ij})\mathbf{q}_i]}{\left|1 - (\kappa q^{ij})^2\right|^{3/2}}, \quad i = 1, \dots, N.$$

Notice that as the curvature of the manifolds $\mathbb{S}_\kappa^3$ and $\mathbb{H}_\kappa^3$ nears 0, the distance between m_i and m_j approaches the Euclidean distance, as it is obvious from geometrical considerations (see Figure 1). But this is not at all obvious from the formula defining the distance, since $|\mathbf{q}_i|, |\mathbf{q}_j| \to \infty$ as $\kappa \to 0$. So although it is geometrically clear that

$$\lim_{\kappa\to 0} U_\kappa(\mathbf{q}) \to U_0(\mathbf{q}) := \sum_{1\le i<j\le N} \frac{m_i m_j}{|\mathbf{q}_i - \mathbf{q}_j|},$$

i.e., U_κ tends to the Newtonian force function when $\kappa \to 0$, this fact becomes less obvious when trying to use (2.1). A similar problem appears when attempting to prove that

$$\lim_{\kappa\to 0} \nabla_{\mathbf{q}_i} U_\kappa(\mathbf{q}) = \sum_{j=1, j\neq i}^{N} \frac{m_i m_j(\mathbf{q}_j - \mathbf{q}_i)}{|\mathbf{q}_j - \mathbf{q}_i|^3},$$

i.e., that the equations of the curved problem ($\kappa \neq 0$) tend to the Newtonian equations when $\kappa \to 0$. But again, the above geometric considerations about the distance support the validity of this conclusion.

A similar technical difficulty shows up when substituting $\kappa = 0$ into (2.2), an operation that leads to undetermined expressions on the right-hand side of the equations of motion. Although from the geometrical and dynamical point of view we can conclude that the equations of the curved problem tend in the limit to the Newtonian equations, (2.2) does not include both the curved and the flat case, since the lengths of the position vectors tend to infinity when $\kappa \to 0$.

It is natural to suspect that the reason for this failure stays with the fact that the origin of the co-ordinate system is at the centre of the spheres, so the radii of the spheres become infinite as $\kappa \to 0$. We could therefore shift the origin of the coordinate system to the north pole of the 3-spheres, namely to the point $(0, 0, 0, |\kappa|^{-1/2})$, a move that would keep the values of the coordinates finite when $\kappa \to 0$. But as we will show in the next section, this approach alone does not fare better either.

3 The North Pole Equations

In this section we attempt to include the case $\kappa = 0$ in the equations of motion by shifting the origin of the coordinate system to the North-Pole of the 3-spheres (see Figure 1). For this purpose we consider the change of variables

$$\omega_i = w_i - |\kappa|^{-1/2}, \quad i = 1, \dots, N,$$

which leaves the coordinates x_i, y_i, z_i, $i = 1, \dots, N$, unchanged. If

$$\overline{q}^{ij} := x_i x_j + y_i y_j + z_i z_j + \sigma \omega_i \omega_j,$$

we have that $\kappa q^{ij} = \kappa \overline{q}^{ij} + |\kappa|^{1/2}(\omega_i + \omega_j) + 1$, for $i, j \in \{1, \dots, N\}$, $i \neq j$.

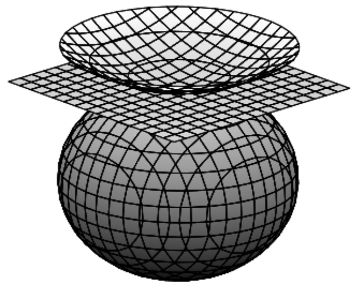

Figure 1: A two-dimensional representation of the continuous transition from $\mathbb{S}^3_\kappa$, up, and from $\mathbb{H}^3_\kappa$, down, to $\mathbb{R}^3$. The only common point of these manifolds is the north pole of the 3-spheres.

Then the equations of motion (2.2) take the form

$$\ddot{x}_i = \sum_{1\le i<j\le N} \frac{m_j|\kappa|^{3/2}[x_j - (\kappa\overline{q}^{ij} + |\kappa|^{1/2}(\omega_i+\omega_j)+1)x_i]}{|1-[\kappa\overline{q}^{ij} + |\kappa|^{1/2}(\omega_i+\omega_j)+1]^2|^{3/2}} - \kappa(\dot{\bar{\mathbf{q}}}_i\cdot\dot{\bar{\mathbf{q}}}_i)x_i$$

$$\ddot{y}_i = \sum_{1\le i<j\le N} \frac{m_j|\kappa|^{3/2}[y_j - (\kappa\overline{q}^{ij} + |\kappa|^{1/2}(\omega_i+\omega_j)+1)y_i]}{|1-[\kappa\overline{q}^{ij} + |\kappa|^{1/2}(\omega_i+\omega_j)+1]^2|^{3/2}} - \kappa(\dot{\bar{\mathbf{q}}}_i\cdot\dot{\bar{\mathbf{q}}}_i)y_i$$

$$\ddot{z}_i = \sum_{1\le i<j\le N} \frac{m_j|\kappa|^{3/2}[z_j - (\kappa\overline{q}^{ij} + |\kappa|^{1/2}(\omega_i+\omega_j)+1)z_i]}{|1-[\kappa\overline{q}^{ij} + |\kappa|^{1/2}(\omega_i+\omega_j)+1]^2|^{3/2}} - \kappa(\dot{\bar{\mathbf{q}}}_i\cdot\dot{\bar{\mathbf{q}}}_i)z_i$$

$$\ddot{\omega}_i = \sum_{1\le i<j\le N} \frac{m_j|\kappa|^{3/2}\{\omega_j + |\kappa|^{-1/2} - [\kappa\overline{q}^{ij} + |\kappa|^{1/2}(\omega_i+\omega_j)+1](\omega_i+|\kappa|^{-1/2})\}}{|1-[\kappa\overline{q}^{ij} + |\kappa|^{1/2}(\omega_i+\omega_j)+1]^2|^{3/2}} - \kappa(\dot{\bar{\mathbf{q}}}_i\cdot\dot{\bar{\mathbf{q}}}_i)(\omega_i+|\kappa|^{-1/2}),$$

for $i = 1,\dots,N$, where $\dot{\bar{\mathbf{q}}}_i = (\dot{x}_i,\dot{y}_i,\dot{z}_i,\dot{\omega}_i)$ and $\dot{\bar{\mathbf{q}}}_i\cdot\dot{\bar{\mathbf{q}}}_i = \dot{x}_i^2+\dot{y}_i^2+\dot{z}_i^2+\sigma\dot{\omega}_i^2$, for $i = 1,\dots,N$.

As in the previous section, the equations of motion are undetermined when $\kappa = 0$, although we know from the above geometrical considerations that they tend to the Newtonian equations as $\kappa \to 0$. This fact suggests that the extrinsic coordinates might not be good enough for solving our problem, so let us see if the use of intrinsic coordinates allows us to include the case $\kappa = 0$ into the equations of motion.

4 Equations of Motion in Intrinsic Coordinates

In this section we introduce the equations of motion in intrinsic coordinates in a unified context. For $\kappa < 0$ and $\kappa > 0$, these equations were separately derived and studied in [16, 25]. These papers, however, treat only the two-dimensional case, which is enough to justify our point.

So we assume in this section that the bodies move on the 2-spheres $\mathbb{S}^2_\kappa$ or the hyperbolic 2-spheres $\mathbb{H}^2_\kappa$, which we can write together as

$$\mathbb{M}^2_\kappa = \{(\mathfrak{x},\mathfrak{y},\mathfrak{z}) \mid \mathfrak{x}^2+\mathfrak{y}^2+\sigma\mathfrak{z}^2 = \kappa^{-1}, \kappa\ne 0,\ \text{with } \mathfrak{z}>0 \text{ for } \kappa<0\}.$$

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In this new setting, the force function (2.1) has the form

$$\mathfrak{U}_\kappa(\mathbf{p}) = \sum_{1\le i<j\le N} \frac{m_i m_j |\kappa|^{1/2}\kappa p^{ij}}{|(\kappa p_i^2)(\kappa p_j^2) - (\kappa p^{ij})^2|^{1/2}}, \tag{4.1}$$

where $\mathbf{p}_i = (\mathfrak{x}_i, \mathfrak{y}_i, \mathfrak{z}_i)$, $i = 1, \dots, N$,

$p^{ij} := \mathbf{p}_i \cdot \mathbf{p}_j = \mathfrak{x}_i\mathfrak{x}_j + \mathfrak{y}_i\mathfrak{y}_j + \sigma\mathfrak{z}_i\mathfrak{z}_j$, $p_i^2 := \mathbf{p}_i \cdot \mathbf{p}_i = \mathfrak{x}_i^2 + \mathfrak{y}_i^2 + \sigma\mathfrak{z}_i$, $i, j \in \{1, 2, \dots, N\}$,

and $\mathbf{p} = (\mathbf{p}_1, \dots, \mathbf{p}_N)$ is the configuration of the system. Then the equations of motion are given by

$$m_i\ddot{\mathbf{p}}_i = \nabla_{\mathbf{p}_i}\mathfrak{U}_\kappa(\mathbf{p}) - \kappa m_i(\dot{\mathbf{p}}_i \cdot \dot{\mathbf{p}}_i)\mathbf{p}_i, \quad i = 1, \dots, N, \tag{4.2}$$

where $\kappa \neq 0$,

$$\nabla_{\mathbf{p}_i}\mathfrak{U}_\kappa(\mathbf{p}) = \sum_{\substack{j=1\\ j\neq i}}^{N} \frac{m_i m_j |\kappa|^{3/2}\kappa p_j^2[(\kappa p_i^2)\mathbf{p}_j - (\kappa p^{ij})\mathbf{p}_i]}{|(\kappa p_i^2)(\kappa p_j^2) - (\kappa p^{ij})^2|^{3/2}}, \quad i = 1, \dots, N,$$

and the coordinates satisfy the constraints $\kappa p_i^2 = 1$, $\mathbf{p}_i \cdot \dot{\mathbf{p}}_i = 0$, for $i = 1, \dots, N$.

To obtain the equations of motion in intrinsic coordinates, we further introduce new geometric models, both for the 2-spheres and the hyperbolic 2-spheres. For this, we use the stereographic projection, which takes the points of coordinates $(\mathfrak{x}, \mathfrak{y}, \mathfrak{z}) \in \mathbb{M}_\kappa^2$ to the points of coordinates (u, v) of the plane $\mathfrak{z} = 0$ through the bijective transformation

$$u = \frac{\mathfrak{x}}{1 - \sigma|\kappa|^{1/2}\mathfrak{z}}, \quad v = \frac{\mathfrak{y}}{1 - \sigma|\kappa|^{1/2}\mathfrak{z}}.$$

The inverse of the stereographic projection takes the points of coordinates (u, v) of the plane $\mathfrak{z} = 0$ to the points $(\mathfrak{x}, \mathfrak{y}, \mathfrak{z}) \in \mathbb{M}_\kappa^2$ through the formulae

$$\mathfrak{x} = \frac{2u}{1 + \kappa(u^2 + v^2)}, \quad \mathfrak{y} = \frac{2v}{1 + \kappa(u^2 + v^2)}, \quad \mathfrak{z} = \frac{\kappa(u^2 + v^2) - 1}{|\kappa|^{3/2}(u^2 + v^2) + \sigma|\kappa|^{1/2}}. \tag{4.3}$$

From the geometric point of view, the correspondence between a point of $\mathbb{M}_\kappa^2$ and a point of the plane $\mathfrak{z} = 0$ is made via a straight line through the point $(0, 0, |\kappa|^{-1/2})$ for $\kappa > 0$ and $(0, 0, -|\kappa|^{-1/2})$ for $\kappa < 0$. In the former case, the projection of $\mathbb{S}_\kappa^2$ is $\mathbb{R}^2$, but with a different metric than the Euclidean one. We denote this plane by $\mathbb{P}_\kappa^2$. In the latter case the projection of $\mathbb{H}_\kappa^2$ is the Poincaré disk $\mathbb{D}_\kappa^2$ of radius $(-\kappa)^{-1/2}$, with the corresponding hyperbolic metric. Let $\mathbb{B}_\kappa^2$ denote either of $\mathbb{P}_\kappa^2$ and $\mathbb{D}_\kappa^2$. With this notation we say that the stereographic projection of $\mathbb{M}_\kappa^2$ that preserves the geometric structure is $\mathbb{B}_\kappa^2$.

The metric of $\mathbb{B}_\kappa^2$ in coordinates (u, v) is given by

$$ds^2 = \frac{4}{[1 + \kappa(u^2 + v^2)]^2}(du^2 + dv^2).$$

This metric can be obtained by substituting (4.3) into $ds^2 = d\mathfrak{x}^2 + d\mathfrak{y}^2 + \sigma d\mathfrak{z}^2$, which defines the metric in $\mathbb{R}^3$, for $\sigma = 1$, and in the Minkowski space $\mathbb{M}^{2,1}$, for $\sigma = -1$

(see [18]). In other words, we can say that the metric in $\mathbb{B}^2_\kappa$ is given by the matrix $G = (g_{ij})_{i,j=1,2}$ with

$$g_{11} = g_{22} = \frac{4}{[1+\kappa(u^2+v^2)]^2}, \quad g_{12} = g_{21} = 0.$$

The inverse of G is $G^{-1} = (g^{ij})_{i,j=1,2}$ with

$$g^{11} = g^{22} = \frac{[1+\kappa(u^2+v^2)]^2}{4}, \quad g^{12} = g^{21} = 0.$$

Assume that the stereographic projection maps the points $\mathbf{q}_i$ and $\mathbf{q}_j$ from $\mathbb{M}^2_\kappa$ to the points $\mathbf{w}_i = (u_i, v_i)$ and $\mathbf{w}_j = (u_j, v_j)$ of $\mathbb{B}^2_\kappa$, respectively. Then, using (4.3), we obtain that

$$\mathbf{q}_i \cdot \mathbf{q}_j = \frac{4\kappa \mathbf{w}_i \cdot \mathbf{w}_j + (\kappa|\mathbf{w}_i|^2 - 1)(\kappa|\mathbf{w}_j|^2 - 1)}{\kappa(\kappa|\mathbf{w}_i|^2 + 1)(\kappa|\mathbf{w}_j|^2 + 1)},$$

where $\mathbf{w}_i \cdot \mathbf{w}_j = u_i u_j + v_i v_j$, so $|\mathbf{w}_i|^2 = u_i^2 + v_i^2$.

To simplify the computations, we introduce the complex coordinates $(z, \bar{z})$ with the help of the transformation $z = u + iv$, $\bar{z} = u - iv$. Then the metric of $\mathbb{B}^2_\kappa$ is given by $ds^2 = \frac{4}{(1+\kappa|z|^2)^2}\, dz\, d\bar{z}$, where $\frac{4}{(1+\kappa|z|^2)^2}$ is the conformal factor.

Some long but straightforward computations show that, for $\kappa \neq 0$, these changes of variables applied to the position vectors bring the force function U_κ given by (4.1) to the form

$$W_\kappa(\mathbf{z}, \bar{\mathbf{z}}) = \sum_{1 \le i < j \le N} \frac{|\kappa|^{1/2} m_i m_j B_{ij}}{|A_{ij}^2 - B_{ij}^2|^{1/2}},$$

where $\mathbf{z} = (z_1, \dots, z_N)$, $\bar{\mathbf{z}} = (\bar{z}_1, \dots, \bar{z}_N)$, and z_i is the coordinate of the body of mass m_i, $i = 1, \dots, N$,

$$B_{ij} := B(z_i, z_j, \bar{z}_i, \bar{z}_j) := 2\kappa^{-1}(z_i \bar{z}_j + z_j \bar{z}_i) + (|z_i|^2 - \kappa^{-1})(|z_j|^2 - \kappa^{-1}),$$
$$A_{ij} := A(z_i, z_j, \bar{z}_i, \bar{z}_j) := (|z_i|^2 + \kappa^{-1})(|z_j|^2 + \kappa^{-1}), \quad i, j \in \{1, \dots, N\},\ i \neq j.$$

The equations of motion (4.2) take the form

$$m_i \ddot{z}_i = \frac{(\kappa|z_i|^2 + 1)^2}{2} \frac{\partial W_\kappa}{\partial \bar{z}_i}(\mathbf{z}, \bar{\mathbf{z}}) + \frac{2|\kappa| m_i \bar{z}_i \dot{z}_i^2}{\kappa|z_i|^2 + 1}, \quad i = 1, \dots, N, \tag{4.4}$$

where

$$\frac{\partial W_\kappa}{\partial \bar{z}_i}(\mathbf{z}, \bar{\mathbf{z}}) = \sum_{j=1, j \neq i}^{N} \frac{2 m_i m_j E_{ij}}{|\kappa|^{11/2}[\sigma(A_{ij}^2 - B_{ij}^2)]^{3/2}},$$

$$E_{ij} := E(z_i, z_j, \bar{z}_i, \bar{z}_j) := 2(\kappa|z_i|^2 + 1)(\kappa|z_j|^2 + 1)^2 (z_j - z_i)(\kappa z_i \bar{z}_j + 1).$$

For $\kappa = 0$, (4.4) is undetermined. By looking just at these equations, it is also far from obvious that the Newtonian equations are recovered when $\kappa \to 0$, but this property is satisfied because equations (4.4) and (2.2) are equivalent, a result proved in [16, 25].

Since the equations of motion written in intrinsic coordinates do not solve our problem either, let us move to another attempt to find a solution. The idea is to combine the use of extrinsic coordinates given by a frame centred at the north pole of the 3-spheres with different distances than the geodesic ones, namely the Euclidean distance for $\kappa \geq 0$ and the Minkowski distance for $\kappa < 0$.

5 Extension to the Flat Case

In this section we provide a form of the equations of motion that extends from $\kappa \neq 0$ to $\kappa = 0$, thus solving the problem we posed at the beginning of this paper. Given the position vectors $\mathbf{q}_i = (x_i, y_i, z_i, w_i)$ for the body m_i and $\mathbf{q}_j = (x_j, y_j, z_j, w_j)$ for the body m_j, $i, j \in \{1, \dots, N\}$, $i \neq j$, let us introduce the notation

$$q_{ij} := \left[(x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2 + \sigma(w_i - w_j)^2\right]^{1/2}.$$

For $\kappa \geq 0$, q_{ij} is the Euclidean distance between m_i and m_j in $\mathbb{R}^4$. But for $\kappa < 0$, q_{ij} is not a distance in the usual mathematical sense of the word. Although the quantities

$$(x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2 - (w_i - w_j)^2$$

are always non-negative, such that the expressions q_{ij} are positive for distinct point masses m_i and m_j, with positions given by the vectors $\mathbf{q}_i$ and $\mathbf{q}_j$, it is not generally true that $q_{ik} \leq q_{ij} + q_{jk}$, so this "distance" does not satisfy the triangle inequality. Nevertheless, the "Minkowski distance" misnomer has been employed in the mathematics and physics literature for more than a century now.

Using the fact that $2q^{ij} = q_i^2 + q_j^2 - q_{ij}^2$, which follows from a straightforward computation, the force function U_κ given by (2.1) can be written in the ambient space as

$$V_\kappa(\mathbf{q}) = \sum_{1 \leq i < j \leq N} \frac{m_i m_j (\kappa q_i^2 + \kappa q_j^2 - \kappa q_{ij}^2)}{[2(\kappa q_i^2 + \kappa q_j^2) q_{ij}^2 - \kappa (q_i^2 - q_j^2)^2 - \kappa q_{ij}^4]^{1/2}}. \tag{5.1}$$

On the manifolds of constant curvature κ, the force function V_κ becomes

$$V_\kappa(\mathbf{q}) = \sum_{1 \leq i < j \leq N} \frac{m_i m_j (2 - \kappa q_{ij}^2)}{q_{ij}(4 - \kappa q_{ij}^2)^{1/2}},$$

which is the same as

$$V_\kappa(\mathbf{q}) = \sum_{1 \leq i < j \leq N} \frac{m_i m_j \left(1 - \frac{\kappa q_{ij}^2}{2}\right)}{q_{ij}\left(1 - \frac{\kappa q_{ij}^2}{4}\right)^{1/2}}. \tag{5.2}$$

The dependence of V_κ on $\mathbf{q}$ is obvious from the definition of the extrinsic mutual distances q_{ij}. We prefer to emphasize the dependence on $\mathbf{q}$ instead of the dependence on q_{ij} alone, because the equations of motion involve $\nabla_{\mathbf{q}_i} V_\kappa$. But whereas the formula of U_κ in (2.1) cannot be extended to the flat case, the right-hand side of (5.2) makes immediate sense for $\kappa = 0$. Since V_κ depends only on the mutual distances, which are finite, we recover for $\kappa = 0$ the classical Newtonian force function of the Euclidean space, $V_0(\mathbf{q}) = \sum_{1 \leq i < j \leq N} \frac{m_i m_j}{q_{ij}}$.

Let us now see how the equations of motion (2.2) get transformed. Straightforward computations show that we can put them into the form

$$\ddot{\mathbf{q}}_i = \sum_{j=1, j\neq i}^{N} \frac{m_j\left[\mathbf{q}_j - \left(1 - \frac{\kappa q_{ij}^2}{2}\right)\mathbf{q}_i\right]}{q_{ij}^3\left(1 - \frac{\kappa q_{ij}^2}{4}\right)^{3/2}} - \kappa(\dot{\mathbf{q}}_i \cdot \dot{\mathbf{q}}_i)\mathbf{q}_i, \quad i = 1, \ldots, N. \tag{5.3}$$

For $\kappa \neq 0$, the $2N$ initial conditions at $t = 0$, $\kappa q_i^2 = 1$, $\kappa \mathbf{q}_i \cdot \dot{\mathbf{q}}_i = 0$, for $i = 1, \ldots, N$, must be satisfied to keep the bodies on the manifolds $\mathbb{S}_\kappa^3$ or $\mathbb{H}_\kappa^3$.

Since the origin of the coordinate system lies at the centre of the 3-spheres, when $\kappa \to 0$, we have that $|\mathbf{q}_i| \to \infty$. So for $\kappa = 0$ the equations are still undetermined. To overcome this last difficulty we can now make use of the idea introduced in Section 3, namely shift the origin of the coordinate system to the North-Poles $(0, 0, 0, |\kappa|^{-1/2})$ of the 3-spheres. For this consider again the transformations

$$\omega_i = w_i - |\kappa|^{-1/2}, \quad i = 1, \ldots, N, \tag{5.4}$$

which leave the variables x_i, y_i, z_i, $i = 1, \ldots, N$, unchanged, and make the notations

$$\mathbf{R} = (0, 0, 0, \sigma|\kappa|^{1/2}), \quad \mathbf{r}_i = (x_i, y_i, z_i, \omega_i), \text{ for } i = 1, \ldots, N, \quad \mathbf{r} = (\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N),$$

$$r_{ij} := \left[(x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2 + \sigma(\omega_i - \omega_j)^2\right]^{1/2}.$$

By noticing that $r_{ij} = q_{ij}$, we can see that the potential is practically unchanged, *i.e.*,

$$V_\kappa(\mathbf{r}) = \sum_{1\le i<j\le N} \frac{m_i m_j\left(1 - \frac{\kappa r_{ij}^2}{2}\right)}{r_{ij}\left(1 - \frac{\kappa r_{ij}^2}{4}\right)^{1/2}},$$

and that the equations of motion become

$$\ddot{\mathbf{r}}_i = \sum_{j=1, j\neq i}^{N} \frac{m_j\left[\mathbf{r}_j - \left(1 - \frac{\kappa r_{ij}^2}{2}\right)\mathbf{r}_i + \frac{r_{ij}^2\mathbf{R}}{2}\right]}{r_{ij}^3\left(1 - \frac{\kappa r_{ij}^2}{4}\right)^{3/2}} - (\dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_i)(\kappa\mathbf{r}_i + \mathbf{R}), \quad i = 1, \ldots, N. \tag{5.5}$$

At $t = 0$, the initial conditions must have the $2N$ constraints

$$\kappa r_i^2 + 2|\kappa|^{1/2}\omega_i = 0, \quad \kappa\mathbf{r}_i \cdot \dot{\mathbf{r}}_i + |\kappa|^{1/2}\dot{\omega}_i = 0, \quad i = 1, \ldots, N.$$

Due to the invariance of $\mathbb{S}_\kappa^3$ and $\mathbb{H}_\kappa^3$ relative to the equations of motion, these conditions are satisfied for all t. They are also identically satisfied for $\kappa = 0$. System (5.5) still bears a small inconvenience: it is not analytic for all $\kappa \in \mathbb{R}$. In any study of bifurcations of solutions relative to some given parameter, it is desirable that the system be analytic in that parameter. To endow our system with this property, we introduce the new parameter $\delta = \sigma|\kappa|^{1/2}$, which implies that $\kappa = \sigma\delta^2$. Then equations (5.5) become

$$\ddot{\mathbf{r}}_i = \sum_{j=1, j\neq i}^{N} \frac{m_j\left[\mathbf{r}_j - \left(1 - \frac{\sigma\delta^2 r_{ij}^2}{2}\right)\mathbf{r}_i + \frac{r_{ij}^2\mathbf{Q}}{2}\right]}{r_{ij}^3\left(1 - \frac{\sigma\delta^2 r_{ij}^2}{4}\right)^{3/2}} - (\dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_i)(\sigma\delta^2\mathbf{r}_i + \mathbf{Q}), \quad i = 1, \ldots, N, \tag{5.6}$$

where $\mathbf{Q} = (0, 0, 0, \delta)$. The constraints are then given by

$$\sigma\delta^2 r_i^2 + 2|\delta|\omega_i = 0, \quad \sigma\delta^2\mathbf{r}_i \cdot \dot{\mathbf{r}}_i + |\delta|\dot{\omega}_i = 0, \quad i = 1, \ldots, N. \tag{5.7}$$

System (5.6) is obviously analytic in the parameter δ. The fact that the constraints do not share this property at 0 is of no consequence in the study of bifurcations of solutions when passing through $\delta = 0$.

On components, system (5.6) can be written as

$$
\begin{aligned}
\ddot{x}_i &= \sum_{j=1, j\neq i}^{N} \frac{m_j\Big[x_j - \Big(1 - \frac{\sigma\delta^2 r_{ij}^2}{2}\Big)x_i\Big]}{r_{ij}^3\Big(1 - \frac{\sigma\delta^2 r_{ij}^2}{4}\Big)^{3/2}} - \sigma\delta^2(\dot{\mathbf{r}}_i\cdot\dot{\mathbf{r}}_i)x_i,\\
\ddot{y}_i &= \sum_{j=1, j\neq i}^{N} \frac{m_j\Big[y_j - \Big(1 - \frac{\sigma\delta^2 r_{ij}^2}{2}\Big)y_i\Big]}{r_{ij}^3\Big(1 - \frac{\sigma\delta^2 r_{ij}^2}{4}\Big)^{3/2}} - \sigma\delta^2(\dot{\mathbf{r}}_i\cdot\dot{\mathbf{r}}_i)y_i,\\
\ddot{z}_i &= \sum_{j=1, j\neq i}^{N} \frac{m_j\Big[z_j - \Big(1 - \frac{\sigma\delta^2 r_{ij}^2}{2}\Big)z_i\Big]}{r_{ij}^3\Big(1 - \frac{\sigma\delta^2 r_{ij}^2}{4}\Big)^{3/2}} - \sigma\delta^2(\dot{\mathbf{r}}_i\cdot\dot{\mathbf{r}}_i)z_i,\\
\ddot{\omega}_i &= \sum_{j=1, j\neq i}^{N} \frac{m_j\Big[\omega_j - \Big(1 - \frac{\sigma\delta^2 r_{ij}^2}{2}\Big)\omega_i + \frac{\delta r_{ij}^2}{2}\Big]}{r_{ij}^3\Big(1 - \frac{\sigma\delta^2 r_{ij}^2}{4}\Big)^{3/2}} - (\dot{\mathbf{r}}_i\cdot\dot{\mathbf{r}}_i)[\sigma\delta^2\omega_i + \delta],
\end{aligned}
$$

for $i = 1,\ldots,N$, with the $2N$ constraints

$$
\begin{aligned}
\sigma\delta^2(x_i^2 + y_i^2 + z_i^2) + \delta^2\omega_i^2 + 2|\delta|\omega_i &= 0,\\
\sigma\delta^2(x_i\dot{x}_i + y_i\dot{y}_i + z_i\dot{z}_i) + \delta^2\omega_i\dot{\omega}_i + |\delta|\dot{\omega}_i &= 0, \quad i = 1,\ldots,N.
\end{aligned}
$$

We can now assume that, when δ varies and the bodies are fixed on their respective manifolds, only the direction (but not the length) of the position vectors changes. Then for $\delta = 0$, the values of $|\mathbf{r}_i|$ are finite, so we recover Newton's equations in the Euclidean case,

$$
\ddot{\mathbf{r}}_i = \sum_{j=1, j\neq i}^{N} \frac{m_j(\mathbf{r}_j - \mathbf{r}_i)}{r_{ij}^3}, \quad i = 1,\ldots,N, \tag{5.8}
$$

where, since $\delta = 0$ and $\omega_i = 0$, $i = 1,\ldots,N$, the position vectors, $\mathbf{r}_i = (x_i, y_i, z_i, 0)$, for $i = 1,\ldots,N$, are free of constraints. Notice that, for consistency, we consider that the motion in $\mathbb{R}^3$ takes place in a hyperplane of $\mathbb{R}^4$, *i.e.*, in a space of zero Gaussian curvature with position vectors $\mathbf{r}_i = (x_i, y_i, z_i, 0)$ and velocities $\dot{\mathbf{r}}_i = (\dot{x}_i, \dot{y}_i, \dot{z}_i, 0)$, so the coordinates and the velocities can be assumed to have the $2N$ constraints, $\omega_i = \dot{\omega}_i = 0$, $i = 1,\ldots,N$, the same number as the constraints (5.7) that occur for $\delta \neq 0$. Consequently the dimension of the phase space of system (5.8) is $6N$, a conclusion that can be drawn either because there are no constraints in $\mathbb{R}^3$ or since there are $8N$ coordinate and velocity components bound by $2N$ constraints in $\mathbb{R}^4$.

The equations of motion (5.5) are apparently less natural than the other equations of motion presented in this paper, because they use the Euclidean distance in $\mathbb{R}^4$ and the Minkowski distance in $\mathbb{R}^{3,1}$ instead of the standard geodesic distance between bodies. But for any given parameter δ, the Euclidean or the Minkowski distance uniquely determines the geodesic distance, so there is no room for confusion. Moreover, (5.6) is very convenient when we regard the classical Newtonian approach as the flat case of the more general problem that describes the gravitational motion of point masses in spaces of constant curvature. We emphasize that, unlike (2.2), for which it is far from obvious what happens when $\kappa \to 0$, (5.6) brings forth the equations of motion of the curved N-body problem for any $\delta \in \mathbb{R}$.

System (5.5) thus opens the way towards the study of the classical N-body problem in the larger context of spaces of constant curvature. In particular it allows us to understand the dynamical behaviour of solutions near $\delta = 0$. This is an important physical problem, since we actually still do not know whether physical space is flat or curved, although it is now widely agreed that, should the curvature be nonzero, its absolute value must be very small. Although this is more of a cosmological problem, which refers to very large distances and not to those traditionally encountered in celestial mechanics, it is still an interesting mathematical problem to regard the equations describing the gravitational motion of N bodies from the point of view of curved space.

6 The Integrals of Motion

In this last section we complete our paper with a study of the bifurcations that occur for the integrals of motion when the parameter passes through the value $\delta = 0$. The results we obtain here show that the classical case appears to be quite special in the context of curved space in the sense that it is the exception rather than the rule. The only integral of motion that encounters no bifurcations is the integral of energy, which exists for all $\delta \in \mathbb{R}$, whereas all the other integrals change in number.

It has been known since 1887 that the equations that describe the three-dimensional Newtonian N-body problem have ten linearly independent integrals of motion that are algebraic functions relative to position vectors and momenta and that there are no other such integrals of this kind [3]. Today these aspects are better understood from the perspective of Noether's theorem, and they have been discussed in [6] relative to generators of isometry groups. There is one integral of energy, three integrals of the centre of mass, three integrals of the linear momentum, and three integrals of the total angular momentum. As we previously proved, for nonzero curvature there is one integral of energy and six integrals of the angular momentum, but no integrals of the centre of mass and of the linear momentum [7, 8, 14]. We will further show how these bifurcations occur in (5.6) when the parameter δ passes through 0.

6.1 The Integrals of the Centre of Mass and the Linear Momentum

The typical way to obtain the integrals of the linear momentum is to sum up $m_i\ddot{\mathbf{r}}_i$ in (5.6) from $i = 1$ to $i = N$, notice that the obtained expression is 0, and then integrate this identity. The integrals of the centre of mass follow after another integration. More precisely, we have that

$$\sum_{i=1}^{N} m_i\ddot{\mathbf{r}}_i = \sum_{i=1}^{N}\sum_{j=1, j\neq i}^{N} \frac{m_i m_j\Big[\mathbf{r}_j - \Big(1 - \frac{\sigma\delta^2 r_{ij}^2}{2}\Big)\mathbf{r}_i + \frac{r_{ij}^2\mathbf{Q}}{2}\Big]}{r_{ij}^3\Big(1 - \frac{\sigma\delta^2 r_{ij}^2}{4}\Big)^{3/2}} - \sum_{i=1}^{N} m_i(\dot{\mathbf{r}}_i\cdot\dot{\mathbf{r}}_i)(\sigma\delta^2\mathbf{r}_i + \mathbf{Q})$$
$$= \sum_{i=1}^{N}\sum_{j=1, j\neq i}^{N} \frac{m_i m_j \frac{r_{ij}^2}{2}(\sigma\delta^2\mathbf{r}_i + \mathbf{Q})}{r_{ij}^3\Big(1 - \frac{\sigma\delta^2 r_{ij}^2}{4}\Big)^{3/2}} - \sum_{i=1}^{N} m_i(\dot{\mathbf{r}}_i\cdot\dot{\mathbf{r}}_i)(\sigma\delta^2\mathbf{r}_i + \mathbf{Q}),$$

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which is 0 for any solution only if $\delta = 0$. By integrating in the case $\delta = 0$, we obtain the three integrals of the linear momentum,

$$\sum_{i=1}^{N} m_i \dot{\mathbf{r}}_i = \mathbf{a}, \tag{6.1}$$

where $\mathbf{a} = (a_1, a_2, a_3)$ is an integration vector. By integrating (6.1), we are led to the integrals of the centre of mass,

$$\sum_{i=1}^{N} m_i \mathbf{r}_i - \mathbf{a}t = \mathbf{b}, \tag{6.2}$$

where $\mathbf{b} = (b_1, b_2, b_3)$ is another integration vector. Obviously, these integrals do not show up for $\delta \neq 0$, a fact that puts into the evidence the bifurcations these integrals encounter at $\delta = 0$.

From the dynamical point of view, the integrals (6.1) and (6.2) express the fact that the centre of mass of the particle system moves uniformly along a straight line when $\mathbf{a} \neq \mathbf{0}$. By taking the origin of the coordinate system at the centre of mass, which implies that $\mathbf{a} = \mathbf{b} = \mathbf{0}$, the above integrals become, respectively,

$$\sum_{i=1}^{N} m_i \dot{\mathbf{r}}_i = \mathbf{0}, \tag{6.3}$$

$$\sum_{i=1}^{N} m_i \mathbf{r}_i = \mathbf{0}. \tag{6.4}$$

Their physical interpretation is that the centre of mass is fixed relative to the coordinate system. This means that the forces acting on the centre of mass cancel each other. In general, no such physical properties occur when $\delta \neq 0$. In particular, there is no point at which the forces acting on it cancel each other. Nevertheless, some particular solutions of the equations of motion have this property, as shown in previous work [6, 8].

6.2 The Integral of Energy

We further obtain the integral of energy for (5.3) and then use the change of variables (5.4) and the change of parameter to derive this integral for (5.6). The standard approach is to take $m_i \ddot{\mathbf{q}}_i \cdot \dot{\mathbf{q}}_i$ and sum up from $i = 1$ to $i = N$, *i.e.*,

$$\sum_{i=1}^{N} m_i \ddot{\mathbf{q}}_i \cdot \dot{\mathbf{q}}_i = \sum_{i=1}^{N} \dot{\mathbf{q}}_i \cdot \nabla_{\mathbf{q}_i} V_\kappa(\mathbf{q}) - \sum_{i=1}^{N} m_i (\dot{\mathbf{q}}_i \cdot \dot{\mathbf{q}}_i)(\kappa \mathbf{q}_i \cdot \dot{\mathbf{q}}_i) = \frac{d}{dt} V_\kappa(\mathbf{q}).$$

By integration we obtain the energy integral, $H_\kappa(\mathbf{q}, \dot{\mathbf{q}}) := T_\kappa(\mathbf{q}, \dot{\mathbf{q}}) - V_\kappa(\mathbf{q}) = h$, where H_κ is the Hamiltonian function, $T_\kappa(\mathbf{q}, \dot{\mathbf{q}}) := \frac{1}{2} \sum_{i=1}^{N} \kappa m_i q_i^2 (\dot{\mathbf{q}}_i \cdot \dot{\mathbf{q}}_i)$ is the kinetic energy, and h is an integration constant. Using the transformations (5.4) and the change of parameter, the kinetic energy T_κ becomes

$$\mathcal{T}_\kappa(\mathbf{r}, \dot{\mathbf{r}}) = \frac{1}{2} \sum_{i=1}^{N} m_i (\sigma \delta^2 r_i^2 + 2|\delta|\omega_i + 1)(\dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_i),$$

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so the integral of energy for (5.5) takes the form

$$\frac{1}{2}\sum_{i=1}^{N} m_i(\sigma\delta^2 r_i^2 + 2|\delta|\omega_i + 1)(\dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_i) - \sum_{1\le i<j\le N} \frac{m_i m_j\left(1 - \frac{\sigma\delta^2 r_{ij}^2}{2}\right)}{r_{ij}\left(1 - \frac{\sigma\delta^2 r_{ij}^2}{4}\right)^{1/2}} = h.$$

For $\delta = 0$, we recover the well-known integral of the Newtonian equations,

$$\frac{1}{2}\sum_{i=1}^{N} m_i(\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2) - \sum_{1\le i<j\le N} \frac{m_i m_j}{r_{ij}} = h,$$

so no bifurcations occur in this case.

The energy integral also shows that (5.6) is Hamiltonian and that the Hamiltonian function is given by

$$\mathcal{H}_\kappa(\mathbf{r}, \dot{\mathbf{r}}) := \frac{1}{2}\sum_{i=1}^{N} m_i(\sigma\delta^2 r_i^2 + 2|\delta|\omega_i + 1)(\dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_i) - \sum_{1\le i<j\le N} \frac{m_i m_j\left(1 - \frac{\sigma\delta^2 r_{ij}^2}{2}\right)}{r_{ij}\left(1 - \frac{\sigma\delta^2 r_{ij}^2}{4}\right)^{1/2}}.$$

Notice, however, that if we want to derive the equations of motion from the above Hamiltonian, then the expression of the potential must be written in the ambient space, *i.e.*, the form (5.1) must be taken into account, before applying the partial derivatives to $\mathcal{H}_\kappa$. After that the constraints can be taken into consideration to see that, restricted to their respective manifolds, the equations of motion take the form derived in the previous section.

6.3 The Integrals of the Total Angular Momentum

As in the case of the energy integral, we can derive the integrals of the total angular momentum for (5.3) and use the transformation (5.4) and the change of parameter to obtain the integrals for (5.6). The total angular momentum is defined as

$$\sum_{i=1}^{N} m_i \mathbf{q}_i \wedge \dot{\mathbf{q}}_i,$$

where $\wedge$ represents the exterior product of the Grassman algebra over $\mathbb{R}^4$. From the physical point of view, this quantity measures the rotation of the system relative to the six planes given by every two of the four axes that form the coordinate system of $\mathbb{R}^4$. We further show that this quantity is conserved for the equations of motion (5.3), *i.e.*,

$$\sum_{i=1}^{N} m_i \mathbf{q}_i \wedge \dot{\mathbf{q}}_i = \mathbf{c}, \tag{6.5}$$

where $\mathbf{c} = c_{wx}\mathbf{e}_w \wedge \mathbf{e}_x + c_{wy}\mathbf{e}_w \wedge \mathbf{e}_y + c_{wz}\mathbf{e}_w \wedge \mathbf{e}_z + c_{xy}\mathbf{e}_x \wedge \mathbf{e}_y + c_{xz}\mathbf{e}_x \wedge \mathbf{e}_z + c_{yz}\mathbf{e}_y \wedge \mathbf{e}_z$, with the coefficients $c_{wx}, c_{wy}, c_{wz}, c_{xy}, c_{xz}, c_{yz} \in \mathbb{R}$, and

$$\mathbf{e}_x = (1,0,0,0), \quad \mathbf{e}_y = (0,1,0,0), \quad \mathbf{e}_z = (0,0,1,0), \quad \mathbf{e}_w = (0,0,0,1)$$

represent the vectors of the standard basis of $\mathbb{R}^4$. We obtain this conservation law by integrating the identity formed by the left and right expressions in the sequence of

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equations

$$\sum_{i=1}^N m_i\ddot{\mathbf{q}}_i\wedge\mathbf{q}_i=\sum_{i=1}^N\sum_{j=1,j\neq i}^N\frac{m_im_j\mathbf{q}_j\wedge\mathbf{q}_i}{q_{ij}^3\big(1-\frac{\kappa q_{ij}^2}{4}\big)^{3/2}}$$
$$-\sum_{i=1}^N\Bigg[\frac{m_im_j\big(1-\frac{\kappa q_{ij}^2}{2}\big)}{q_{ij}^3\big(1-\frac{\kappa q_{ij}^2}{4}\big)^{3/2}}-\kappa m_i(\dot{\mathbf{q}}_i\cdot\dot{\mathbf{q}}_i)\Bigg]\mathbf{q}_i\wedge\mathbf{q}_i=\mathbf{0},$$

which follows after $\wedge$-multiplying the equations of motion (5.3) by $m_i\mathbf{q}_i$ and summing up from $i=1$ to $i=N$. The last of the above identities follows from the skew-symmetry of the $\wedge$ operation and, consequently, from the fact that $\mathbf{q}_i\wedge\mathbf{q}_i=\mathbf{0}$, $i=1,\ldots,N$. On components, the six integrals in (6.5) can be written as

$$\sum_{i=1}^N m_i(x_i\dot{y}_i-\dot{x}_iy_i)=c_{xy},\qquad \sum_{i=1}^N m_i(x_i\dot{z}_i-\dot{x}_iz_i)=c_{xz},$$
$$\sum_{i=1}^N m_i(y_i\dot{z}_i-\dot{y}_iz_i)=c_{yz},\qquad \sum_{i=1}^N m_i(w_i\dot{x}_i-\dot{w}_ix_i)=c_{wx},$$
$$\sum_{i=1}^N m_i(w_i\dot{y}_i-\dot{w}_iy_i)=c_{wy},\qquad \sum_{i=1}^N m_i(w_i\dot{z}_i-\dot{w}_iz_i)=c_{wz}.$$

Using the transformations (5.4) and the change of parameter, we can see that for system (5.6) these integrals take the form

$$\sum_{i=1}^N m_i(x_i\dot{y}_i-\dot{x}_iy_i)=c_{xy},\quad \sum_{i=1}^N m_i(x_i\dot{z}_i-\dot{x}_iz_i)=c_{xz},\quad \sum_{i=1}^N m_i(y_i\dot{z}_i-\dot{y}_iz_i)=c_{yz},$$
$$\sum_{i=1}^N m_i\dot{x}_i+|\delta|\sum_{i=1}^N m_i(\omega_i\dot{x}_i-\dot{\omega}_ix_i)=|\delta|c_{\omega x},\ \sum_{i=1}^N m_i\dot{y}_i+|\delta|\sum_{i=1}^N m_i(\omega_i\dot{y}_i-\dot{\omega}_iy_i)=|\delta|c_{\omega y},$$
$$\sum_{i=1}^N m_i\dot{z}_i+|\delta|\sum_{i=1}^N m_i(\omega_i\dot{z}_i-\dot{\omega}_iz_i)=|\delta|c_{\omega z},$$

where $c_{\omega x}=c_{wx}$, $c_{\omega y}=c_{wy}$, $c_{\omega z}=c_{wz}$.

So for $\delta\neq0$, system (5.6) has six integrals of the total angular momentum. But at $\delta=0$, only the first three integrals of the total angular momentum survive; the others become the three integrals of the linear momentum (6.3) obtained when the origin of the coordinate system is taken at the centre of mass of the particle system. Thinking of this phenomenon geometrically, it is not so surprising to see that some of the angular momentum integrals of the sphere and hyperbolic sphere become linear momentum integrals when passing from curved spaces to the Euclidean space. So an interesting kind of bifurcation occurs in this case as we pass through the value $\delta=0$ of the parameter.

7 Conclusions

The study of the new equations of motion (5.6) is neither simpler nor more complicated than that of the systems provided in intrinsic or extrinsic coordinates in Sections

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2, 3, and 4, although certain problems might be more approachable with some of the equations presented in this paper than with others. Nevertheless, system (5.6) has the advantage of unifying the cases $\delta \neq 0$ and $\delta = 0$, thus offering a larger perspective for the Newtonian equations of the N-body problem. As we showed above, some interesting bifurcations occur for the integrals of motion as we pass through the value $\delta = 0$ of the curvature parameter. Since the geometric shape of the physical space still eludes us, a further study of these equations might bring more insight on this question.

All the solutions obtained for $\delta = 0$ can now be viewed from the point of view of $\delta \neq 0$ and vice versa, in the sense of studying whether such solutions bifurcate or occur for all values of the curvature parameter δ, under what circumstances they show up, and whether the stability of these solutions changes with δ. Consequently system (5.6) opens new perspectives of research that were not possible with the previously derived equations of motion. In particular, the study of the Lagrangian and the Eulerian orbits of the 3-body problem can be considered in the future in the context of this larger framework. But these are only a couple of the questions among the many new exciting problems that can be considered from this novel point of view.

It is also interesting to remark the change in the number of first integrals that occurs when passing through the value $\delta = 0$ of the parameter. Although this phenomenon occurs in other systems of differential equations, it seems to be quite rare. Nevertheless, the message is well known: fewer integrals imply fewer symmetries and therefore suggest more complicated behaviour. However, a fact never encountered before, as far as our knowledge goes, is the change of the dimension of the space relative to the extrinsic coordinates we introduced here when the parameter is varied. Though one may argue that the dimension of the involved manifolds is the same, the background space in which they are embedded is not. Whether this fact will lead to new general insights, is something we still have to understand.

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