

Forced convection in two-phase core-annular flows

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Predicting the temperature distribution in laminar two-phase flows is essential in a wide range of engineering applications, like heat dissipation of electronic equipment and thermal design of biological reactors. Motivated by this, we extend the classical Graetz problem, studying the heat transfer between two flowing phases in a core-annular flow configuration. Using a rigorous two-scale asymptotic analysis, we derived two coupled one-dimensional advection–diffusion heat-transfer equations (one for each phase) embedding the effects of advection, diffusion (both axial and transverse) and viscous dissipation. Specifically, the heat-transfer mechanisms are described through effective velocity and effective diffusion coefficients, while the interaction between the phases is accounted for via *ad hoc* coupling and source terms, respectively. The dynamics of the problem is controlled by seven dimensionless groups: the Péclet and Brinkman numbers, the heat flux, the viscosity, thermal diffusivity and thermal conductivity ratios, and the volume fraction. Our analysis reveals the existence of two main regimes, depending on the disparity in thermal conductivity between the phases. When the conductivity ratio is of order one, the problem is strongly coupled; otherwise, the phases are thermally decoupled. Interestingly, we investigate the evolution of the heat-transfer coefficient in the thin-film limit, shedding light on the most common assumptions underlying extensively used models in the context of film flows. Finally, we derived closed-form scaling laws for the Nusselt number clarifying the impact of the phases topology on heat-transfer dynamics. Since our model has been derived by first principles, we hope that it will improve the understanding of two-phase forced convection.

Key words: core-annular flow, multiphase flow

1. Introduction

Predicting heat transfer in laminar two-phase flows is of practical relevance, with applications ranging from heat sinks for electronic cooling (Mudawar 2011; Kottke *et al.* 2015), biomedical engineering (Wang & Fan 2010), building refrigeration (Armatis & Fronk 2017), thermal design of chemical reactors and heat exchangers (Neveu *et al.* 2013) to geophysical sciences (Hasan & Kabir 2010).

Among those challenges, the growing demand for high-performance computing and miniaturisation makes the effective heat flux removal the major ambition in the design and thermal management of electronic devices (Abdollahi, Sharma & Vatani 2017). Unfortunately, the penetration of well-established technologies, such as wire coil inserts, twisted tapes or helical ribs, is limited by fouling and high pressure losses (Ghajar & Tang 2010). A promising solution consists in promoting two-phase flow conditions to improve the heat transfer. This can be achieved either via the injection of gas bubbles into a pipe filled with refrigerant (e.g. the heat transfer coefficient can increase up to ten times as proposed by Celata *et al.* 1999), using mixtures of immiscible liquids (e.g. an aqueous and an organic phases (Brauner 2002)), or promoting evaporation/condensation (Kim *et al.* 2015; Adera *et al.* 2021). In these contexts, the design of heat exchangers is more complex than in the case of a single phase. The complexity arises from the fact that the heat transfer depends not only on the flow parameters (i.e. flow rate, pressure drop, slip ratio, etc.) and the pipe geometry, but also on the phase topology (i.e. the flow regime: core-annular, stratified, intermittent and dispersed). In addition, while gas–liquid systems are characterised by a low density and viscosity ratio, in liquid–liquid systems, the viscosity ratio can span several orders of magnitude. Thus, understanding how heat transfer is enhanced or retarded due to the presence of more than one phase becomes necessary for the design of modern heat exchangers.

So far, a wide range of investigations has focused primarily on forced convection of single-phase flow, starting from Graetz's and Nusselt's pioneering works (Graetz 1882; Nusselt 1910), where the thermal entrance length inside a circular pipe has been solved neglecting the effect of axial heat conduction and viscous dissipation. Thereafter, the Graetz problem has been studied extensively, see Shah & London (1978). For example, when the flow Péclet number is sufficiently small, streamwise conduction becomes important (see Pahor & Strnad (1956); Reynolds (1963)) as is the case of compact heat exchangers employing liquid metals as working fluids or in micro-channels (Nonino *et al.* 2009). The effect of viscous dissipation on forced convection leads to the so-called Graetz–Brinkman problem (Brinkman 1951; Ou & Cheng 1973), indicating that the internal friction becomes relevant only for highly viscous fluids in capillaries (Morini & Spiga 2006) even at moderate flow rates. Other generalisations include the case of hydrodynamically developing flows (Boussinesq 1890), non-circular arbitrary geometries (Barrera *et al.* 2016), non-Newtonian fluids (Cotta & Özişik 1986; Ali & Khan 2018; Asghar *et al.* 2023), specified-flux (Sellars, Tribus & Klein 1956) and mixed-type (Hsu 1968) boundary conditions or a combination of those (Colle 1988; Hirbodi, Yaghoubi & Warsinger 2022).

Extensions of the Graetz's problem to the case of two-phase flow are rather limited and, unfortunately, a generalised understanding of the effect of the presence of more than one phase on heat transfer dynamics is still missing. Most existing studies focus on core-annular flow and are restricted to steady-state conditions. Specifically, Leib, Fink & Hasson (1977) and Stockman & Epstein (2001) studied the steady-state and fully developed thermal problem of a vertical core-annular flow when a uniform heat flux is imposed at the pipe wall, while Bentwich & Sideman (1964*a,b*) proposed a simplified description

of stratified and core-annular liquid–liquid flows when the wall temperature is imposed. Nogueira & Cotta (1990) and Su (2006) solved the heat-transfer Sturm–Liouville-type problem for core-annular flows subjected to fixed temperature and mixed boundary conditions neglecting both axial conduction and viscous dissipation. Later on, Lindemer, Advani & Prasad (2015) studied the impact of convection-type boundary condition combining analytical and numerical solutions (using the method of quasi-orthogonal functions), while Chalhub, Corrêa & Teixeira (2022) proposed a solution of the steady-state Graetz’s problem (neglecting axial diffusion) based on integral transform. In those solutions, the fluid temperature is obtained in the form of an infinite series and, therefore, macro-scale thermal parameters, such as the heat transfer coefficient (or the Nusselt number), cannot be obtained in a closed form, keeping the underlying physical scaling difficult to unravel. In fact, such macro-scale parameters are usually estimated using experiments and semi-empirical correlations (Dungan & Shapiro 1990). To sum up, a general theoretical framework that embeds both transient and steady-state effects, and all the relevant physical mechanisms (e.g. both axial and transverse diffusion), is still missing in the context of two-phase flows.

To fill this gap, the goal of this paper is to study the impact of the flow topology on heat transfer clarifying the competition between the main heat transfer mechanisms (diffusion, advection, an imposed heat flux at the channel walls and viscous dissipation) and flow parameters, like the volume fraction and the viscosity ratio. To do so, we extend the Graetz–Brinkman problem to core-annular flows (see figure 1, § 2) that represent the basic flow pattern in microchannel and it is often used as an idealisation of more complex flow regimes such as elongated bubbles (see Collier & Thome (1994); Thome, Dupont & Jacobi (2004); Picchi & Battiatto (2018); Picchi, Ullmann & Brauner (2018)). Our goal is to derive an effective description of the heat-transfer problem addressing both transient and steady-state conditions.

With this aim, we generalise the Aris–Taylor dispersion theory to heat-transfer in the context of two-phase flows. In fact, this theory has the great advantage to be a satisfactory compromise between accuracy, compactness and physical interpretation of the final solution. Specifically, the long-standing theory of Taylor dispersion (Taylor 1953; Aris 1956) is a rigorous perturbation scheme that has been first applied to elementary geometries and later generalised to a disparate class of physical systems (Brenner 1980), including periodic obstructions (Farah *et al.* 2020), a non-premixed flame established in a narrow duct (Liñán *et al.* 2020), radiant pipes (Batycky, Edwards & Brenner 1994), soils (Auriault & Lewandowska 1996), capillary-tissue exchange kinetics (Levitt 1972; Fallon & Chauhan 2005), suspensions and porous media (Brenner & Stewartson 1980; Rubinstein & Mauri 1986; Battiatto & Tartakovsky 2011; Parmigiani *et al.* 2011; Griffiths, Howell & Shipley 2013; Dejam, Hassanzadeh & Chen 2014; Bourbatache, Millet & Moyne 2020; Scholz & Bringedal 2022), and thin films (Picchi & Poesio 2022). Conveniently, it provides a reduced-order mathematical description where the competitive influence of advection, diffusion and boundary phenomena is taken into account via a set of effective coefficients (Frankel & Brenner 1989). So far, to the best of our knowledge, an attempt to adapt this approach to the case of core-annular flows has not been proposed yet.

Starting from transport equations (§ 2.1), we derive a set of coupled evolution equations for the average temperatures as functions of space and time by means of two-scale asymptotic expansions (§ 3). Our analysis (the full derivation is presented in Appendix C) is complemented by a rigorous specification of the theoretical bounds of validity of the model (§ 3.1). In the case of large thermal conductivity contrast between the phases, the system reduces to a single equation (§ 3.2), which admits an analytical solution (§ 4.1), including transient effects. Instead, when the thermal conductivities are of the same

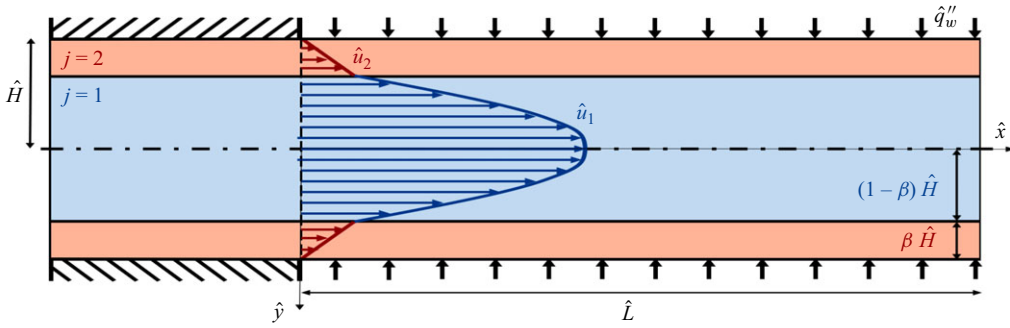


Figure 1. Schematic diagram of the extended Graetz–Brinkman problem for a laminar two-phase core-annular flow in a plane slender channel ($\hat{H} \ll \hat{L}$) with a semi-infinite heating section.

order of magnitude, the problem is described by two coupled advection–heat-transfer equations and an analytical solution only exists at the steady-state (§ 4.5). Finally, we derive analytical scaling laws for the asymptotic Nusselt number (§§ 4.2, 4.7) and discuss the thermal coupling between the phases (Brenner 1982; Chu, Sposito & Jury 1983; Auriault & Lewandowska 1994; Moyne & Murad 2006; Shelukhin, Yeltsov & Paranichev 2011). These results shed light on heat transfer phenomena in the context of multi-phase flows and may serve as a starting step to generalise the heat transfer model to other flow regimes.

2. Problem formulation

We consider an infinitely long and horizontal planar channel with a uniform cross-section of height $2\hat{H}$, as sketched in figure 1; the hat operator will be used to denote dimensional quantities. Two immiscible Newtonian fluids flow from left to right as a core-annular flow regime driven by a constant pressure gradient: the inner fluid, which is not in contact with the channel wall, will be denoted by subscript ‘1’, while the outer fluid that is in contact with the channel wall will be denoted by subscript ‘2’. The fluid–fluid interface is supposed to be flat, and both velocity profiles are assumed to be fully developed and laminar. The channel is heated by a constant and uniform wall heat flux $\hat{q}_w''$ imposed at both plates ($\hat{y} = \pm \hat{H}$) over the entire semi-infinite domain ($\hat{x} \geq 0$). We primarily consider liquid–liquid systems or gas–liquid systems with sufficiently small temperature differences so that the thermal expansion of the gaseous phase can be neglected. The dimensional velocity profile in each fluid domain is derived from the incompressible Navier–Stokes equation for co-current core-annular flows in a planar geometry:

$$\hat{u}_1(\hat{y}) = \frac{1}{2\mu_1} \frac{d\hat{p}}{d\hat{x}} \left\{ \hat{y}^2 - \frac{\hat{H}^2 [\mu_2 - \beta(2 - \beta)(\mu_2 - \mu_1)]}{\mu_2} \right\}, \quad \frac{\hat{y}}{\hat{H}} \in [0; 1 - \beta], \quad (2.1a)$$

$$\hat{u}_2(\hat{y}) = \frac{1}{2\mu_2} \frac{d\hat{p}}{d\hat{x}} \left\{ \hat{y}^2 - \hat{H}^2 \right\}, \quad \frac{\hat{y}}{\hat{H}} \in [1 - \beta; 1], \quad (2.1b)$$

where $d\hat{p}/d\hat{x} < 0$ is the axial pressure gradient, μ_j is the dynamical viscosity of the two fluids, $j = \{1, 2\}$, and β is the volume fraction of the outer phase ($0 < \beta < 1$); the detailed

derivation of the velocity profiles is given in [Appendix A](#). Specifically, half the thickness of the inner and the outer phase is equal to $(1 - \beta)\hat{H}$ and $\beta\hat{H}$, respectively.

We obtain the averaged speed of the inner and outer phase by integrating the velocity profiles (2.1),

$$\hat{U}_1 = \frac{\int_0^{(1-\beta)\hat{H}} \hat{u}_1(\hat{y}^*) d\hat{y}^*}{(1-\beta)\hat{H}} = -\frac{\hat{H}^2}{6\mu_1\mu_2} \frac{d\hat{p}}{d\hat{x}} [2\mu_2 - \beta(2-\beta)(2\mu_2 - 3\mu_1)], \quad (2.2a)$$

$$\hat{U}_2 = \frac{\int_{(1-\beta)\hat{H}}^{\hat{H}} \hat{u}_2(\hat{y}^*) d\hat{y}^*}{\beta\hat{H}} = -\frac{\hat{H}^2}{6\mu_2} \frac{d\hat{p}}{d\hat{x}} [\beta(3-\beta)], \quad (2.2b)$$

yielding an overall expression for the averaged speed over the entire channel,

$$\hat{U} = (1-\beta)\hat{U}_1 + \beta\hat{U}_2 = -\frac{\hat{H}^2}{3\mu_1\mu_2} \frac{d\hat{p}}{d\hat{x}} [\mu_2 - \beta(\beta^2 - 3\beta + 3)(\mu_2 - \mu_1)]. \quad (2.3)$$

2.1. Governing equations for heat transfer

The internal energy balance for each phase written in terms of temperature $\hat{T}_j(\hat{x}, \hat{y}, \hat{t})$ is given by

$$\rho_j c_{p,j} \left(\frac{\partial \hat{T}_j}{\partial \hat{t}} + \hat{u}_j \frac{\partial \hat{T}_j}{\partial \hat{x}} \right) = \kappa_j \left(\frac{\partial^2 \hat{T}_j}{\partial \hat{x}^2} + \frac{\partial^2 \hat{T}_j}{\partial \hat{y}^2} \right) + \hat{\mathcal{W}}_j, \quad (2.4)$$

with ρ_j , $c_{p,j}$, κ_j being the density, the isobaric mass heat capacity and the thermal conductivity of each phase, respectively. The term $\hat{\mathcal{W}}_j$ represents the rate of viscous dissipation of mechanical energy per unit mass and, for laminar unidirectional and fully developed flows, it reduces to

$$\hat{\mathcal{W}}_j = \mu_j \left(\frac{d\hat{u}_j}{d\hat{y}} \right)^2. \quad (2.5)$$

At the channel wall, $\hat{y} = \hat{H}$, we assume that a uniform heat flux (UWF) is imposed (Shah & London 1978), leading to

$$\hat{q}_w'' = -\kappa_2 \left(\hat{\mathbf{n}}_2 \cdot \nabla \hat{T}_2 \right) \Big|_{\hat{y}=\hat{H}} = -\kappa_2 \frac{\partial \hat{T}_2}{\partial \hat{y}} \Big|_{\hat{y}=\hat{H}}, \quad (2.6)$$

where the unit vector is given by $\mathbf{n}_2 = (0; 1)$. From the practical point of view, an imposed uniform wall heat flux (for $\hat{x} \geq 0$) can be seen as an approximation of many experimental facilities where the channel is wrapped with a heat tape or wire resistance heater – see, for instance, Murphy, Alimohammadi & O'Shaughnessy (2024). In (2.6), we adopt as convention that the imposed heat flux is positive, $\hat{q}_w'' > 0$, when it exits the fluid. At the fluid–fluid interface, $\hat{y} = (1-\beta)\hat{H}$, the normal vectors are given by $\mathbf{n}_1 = (0; 1) = -\mathbf{n}_2$, and the continuity of temperature and heat flux across the flat interface gives

$$\kappa_1 \frac{\partial \hat{T}_1}{\partial \hat{y}} \Big|_{\hat{y}=(1-\beta)\hat{H}} = \kappa_2 \frac{\partial \hat{T}_2}{\partial \hat{y}} \Big|_{\hat{y}=(1-\beta)\hat{H}}, \quad (2.7a)$$

$$\hat{T}_1 \Big|_{\hat{y}=(1-\beta)\hat{H}} = \hat{T}_2 \Big|_{\hat{y}=(1-\beta)\hat{H}}. \quad (2.7b)$$

In the duct cross-section at $\hat{y} = 0$, the symmetry of the thermal problem results in

$$\left. \frac{\partial \hat{T}_1}{\partial \hat{y}} \right|_{\hat{y}=0} = 0. \quad (2.8)$$

The problem is closed imposing the initial condition at the channel entry as a uniform temperature distribution at $\hat{t} = 0$ and $\hat{x} = 0$ for both phases:

$$\hat{T}_j(0, \hat{y}, 0) = \hat{T}_{in, j}. \quad (2.9)$$

As it will be discussed in § 2.2, we are not focused on describing the early-time evolution of the system and, therefore, the initial condition (2.9) is unimportant for the purposes of our analysis.

Then, we recast the energy balance (2.4), written with respect to an absolute reference frame $(\hat{x}, \hat{y})$, introducing a new axial coordinate $\hat{z} = \hat{x} - \hat{V} \hat{t}$ that advects in the direction of the flow with an arbitrary speed $\hat{V}$ (with $\hat{\tau} = \hat{t}$) yielding

$$\rho_j c_{p, j} \left[\frac{\partial \hat{T}_j}{\partial \hat{\tau}} + (\hat{u}_j - \hat{V}) \frac{\partial \hat{T}_j}{\partial \hat{z}} \right] = \kappa_j \left(\frac{\partial^2 \hat{T}_j}{\partial \hat{z}^2} + \frac{\partial^2 \hat{T}_j}{\partial \hat{y}^2} \right) + \hat{\mathcal{W}}_j. \quad (2.10)$$

In general, the choice of the reference frame speed $\hat{V}$ is arbitrary (Brady 1975), but it affects the definition of the integral variables (e.g. the effective diffusion coefficients). Specifically, $\hat{V} = 0$ corresponds to a description given in the absolute coordinate system; this reference frame will be adopted to compute the bulk temperature and the Nusselt number. Choosing $\hat{V} = \hat{U}$ identifies the volume-fixed reference frame, i.e. a relative coordinate system moving a speed equal to the mean flow velocity, see (2.3). This reference frame will be used when the focus is on revealing the impact of advection on multiphase heat transfer (i.e. the equivalent of hydrodynamic dispersion in mass transfer problems (Taylor 1953)). In cases where the two phases are thermally decoupled, it is convenient to choose a reference frame attached to the annulus with $\hat{V} = \hat{U}_2$. A rigorous justification of this argument is provided both in the framework of statistical moments (Aris 1956) or perturbation analysis enforcing the Fredholm-type solvability condition (Mauri 1991, 1995, 2015; Mikelić, Devigne & van Duijn 2006). Note that other choices of the reference frame can be made (e.g. molar or mass mean velocity), but their use is limited to multicomponent systems (Hooyman 1956; Brady 1975; Piña 1979; Taylor & Krishna 1993; Kozlova *et al.* 2019).

2.2. Dimensionless formulation

The governing equations introduced in § 2.1 are made dimensionless to facilitate the derivation of an effective description of two-phase flow heat transfer. The relevant scale along the $\hat{y}$ -direction is chosen as the half-distance between the parallel plates $\hat{H}$, while we denote by $\hat{L}$ a characteristic macroscopic/observation length scale along the $\hat{x}$ -axis. We look at systems that can be treated as a shallow geometry, $\hat{L} \gg \hat{H}$, and, therefore, a small scale parameter indicating the separation between transverse and axial variables arises naturally,

$$\varepsilon = \frac{\hat{H}}{\hat{L}} \ll 1. \quad (2.11)$$

The scale separation assumption, $\varepsilon \ll 1$, is a key requirement to simplify the momentum equations and ensures that the lubrication approximation for the flow holds (see [Appendix A](#)).

The determination of the relevant time scale is not unique (Mauri [1991](#); Auriault & Adler [1995](#); Mikelić *et al.* [2006](#); Battiatto & Tartakovsky [2011](#); Bourbatache *et al.* [2020](#); Picchi & Poesio [2022](#)). In fact, advection introduces three advective time scales depending on whether the focus is on a specific phase or the overall system. Assuming that the characteristic scale for the velocity is the overall flow speed $\hat{U}$ given in (2.3), we can define the axial advective time as $\hat{\tau}_a = \hat{L}/\hat{U}$, while using the phase speed, U_j , we can introduce two additional time scales as $\hat{\tau}_{a,j} = \hat{L}/\hat{U}_j$. Thermal diffusion introduces two additional time scales for each phase: the axial, $\hat{\tau}_{\hat{L},j} = \hat{L}^2/\alpha_j$, and the transverse, $\hat{\tau}_{\hat{H},1} = (1-\beta)^2 \hat{H}^2/\alpha_1$, $\hat{\tau}_{\hat{H},2} = \beta^2 \hat{H}^2/\alpha_2$, diffusion times, with $\alpha_j = \kappa_j/\rho_j c_{p,j}$ being the thermal diffusivity of the j th phase. Since we are primarily interested in transport dynamics in a time frame much larger than the transverse diffusion time and close to the advection time, we choose $\hat{\tau}_a$ as the reference scale for the time variable (Mauri [1991](#); Griffiths *et al.* [2013](#); Mauri [2015](#); Ling *et al.* [2016](#); Liñán *et al.* [2020](#)). This choice allows the investigation of heat-transfer regimes characterised by the competition between advection and axial diffusion. In fact, as elucidated by Taylor's analysis of solute dispersion (Taylor [1953](#)), the advective time scale corresponds to a time window that is intermediate between the times characterising transverse and axial diffusion, i.e. $\tau_{H,j} \ll \tau_a \ll \tau_{L,j}$, defining a hierarchy of time scales that can be uniquely determined in terms of the scale parameter ε and the Péclet number of the problem (see § 4.3). This choice of the relevant time window also implies that the model is not capable of describing the early-time relaxation of the system's initial configuration due to transverse diffusion (known as pre-asymptotic regime) and, therefore, the initial condition given in (2.9) will not be considered. Note that the description of the pre-asymptotic regime (see e.g. Taghizadeh, Valdés-Parada & Wood ([2020](#))) is out of the scope of this work.

We can recast the energy balance (2.10) introducing the following dimensionless variables:

$$z = \frac{\hat{z}}{\hat{L}}, \quad y = \frac{\hat{y}}{\hat{H}}, \quad \tau = \frac{\hat{\tau}}{\hat{\tau}_a}, \quad \{u_j; V\} = \frac{\{\hat{u}_j; \hat{V}\}}{\hat{U}}, \quad \vartheta_j = \frac{\hat{T}_j - \hat{T}_{ref}}{\Delta \hat{T}_{ref}}, \quad (2.12a-e)$$

where $\hat{T}_{ref}$ and $\Delta \hat{T}_{ref}$ are arbitrary reference values for the temperature and its variation, respectively. For example, the reference temperature $\hat{T}_{ref}$ can be chosen as the initial temperature at a given location ($\hat{x} = 0$), $\hat{T}_{in}$, while the reference temperature difference $\Delta \hat{T}_{ref}$ can be defined based on the condition of uniform heat flux at the wall, as $\hat{q}_w'' \hat{L}/\kappa_2$ (Shah & London [1978](#)) or $\hat{q}_w'' \hat{L}/(\rho_2 c_{p,2} \hat{U} \hat{H})$ (Chen *et al.* [2022](#)). This procedure yields

$$\varepsilon \mathcal{A} Pe \left[\frac{\partial \vartheta_1}{\partial \tau} + (u_1 - V) \frac{\partial \vartheta_1}{\partial z} \right] = \left(\varepsilon^2 \frac{\partial^2 \vartheta_1}{\partial z^2} + \frac{\partial^2 \vartheta_1}{\partial y^2} \right) + \frac{m}{\mathcal{K}} Br \left(\frac{du_1}{dy} \right)^2, \quad y \in [0; 1 - \beta], \quad (2.13a)$$

$$\varepsilon Pe \left[\frac{\partial \vartheta_2}{\partial \tau} + (u_2 - V) \frac{\partial \vartheta_2}{\partial z} \right] = \left(\varepsilon^2 \frac{\partial^2 \vartheta_2}{\partial z^2} + \frac{\partial^2 \vartheta_2}{\partial y^2} \right) + Br \left(\frac{du_2}{dy} \right)^2, \quad y \in [1 - \beta; 1], \quad (2.13b)$$

where dimensionless absolute velocity profiles $u_j(y)$ are given in [Appendix A.1](#). In the following, $V = 0$ corresponds to a description given in the absolute coordinate system, $V = 1$ identifies the relative coordinate system moving at the mean flow speed, while $V = U_2$ identifies the reference frame attached to the annulus. The normalisation introduces the viscosity ratio m , the thermal conductivity ratio $\mathcal{K}$ and the thermal diffusivity ratio $\mathcal{A}$:

$$m = \frac{\mu_1}{\mu_2}, \quad \mathcal{K} = \frac{\kappa_1}{\kappa_2}, \quad \mathcal{A} = \frac{\alpha_2}{\alpha_1} = \frac{1}{\mathcal{K}} \frac{\rho_1}{\rho_2} \frac{c_{p,1}}{c_{p,2}}. \quad (2.14a-c)$$

The viscosity ratio accounts for the different viscosity of the flowing phases: when $m \rightarrow 0$, the outer fluid is much more viscous compared with the inner one as typical of the majority of gas–liquid systems (see [table 1](#)); we will refer to this limit as the free-surface limit. When $m \rightarrow \infty$, the inner fluid is much more viscous than the outer one and it behaves like a rigid body, see [figure 3\(a\)](#); we will refer to this limit as the rigid-core limit. The product $\mathcal{K} \mathcal{A}$ can be seen as the thermal capacity ratio: gas–liquid systems are characterised by lower values of $\mathcal{K} \mathcal{A}$ with respect to liquid–liquid systems (see [table 1](#)). Note that, although in our model, the physical properties of the two phases appear in the three property ratios given in (2.14), other sets of dimensionless parameters can provide an equivalent description. For example, in [Appendix B](#), we show how the property ratios (2.14) can be recast in terms of the heat-capacity flow rate ratio, commonly used to describe heat-exchanger problems.

In (2.13), the Péclet number Pe and the Brinkman number Br for the outer phase are given by

$$Pe_2 = Pe = \frac{\hat{H} \hat{U}}{\alpha_2}, \quad Br = \frac{\mu_2 \hat{U}^2}{\kappa_2 \Delta \hat{T}_{ref}}. \quad (2.15a-b)$$

The Péclet number expresses the ratio between advection and axial diffusion. When $Pe \ll 1$, diffusion dominates over advection, while for $Pe \gg 1$, advection plays a major role with respect to diffusion. The Péclet number Pe_1 of the inner phase is given by the product $Pe_1 = \mathcal{A} Pe$. The Brinkman number can be seen as the competition between the thermal power per unit mass produced by viscous dissipation and the one transferred due to conduction by the outer phase through the channel walls; $Br = 0$ only in non-dissipative flows.

The dimensionless boundary conditions at the wall (2.6), at the interface (2.7) and at the channel axis (2.8) read

$$\left. \frac{\partial \vartheta_2}{\partial y} \right|_{y=1} = - \frac{\hat{q}_w'' \hat{H}}{\kappa_2 \Delta \hat{T}_{ref}} = q_w, \quad (2.16a)$$

$$\left. \frac{\partial \vartheta_2}{\partial y} \right|_{y=1-\beta} = \mathcal{K} \left. \frac{\partial \vartheta_1}{\partial y} \right|_{y=1-\beta}, \quad (2.16b)$$

$$\vartheta_2|_{y=1-\beta} = \vartheta_1|_{y=1-\beta}, \quad (2.16c)$$

$$\left. \frac{\partial \vartheta_1}{\partial y} \right|_{y=0} = 0. \quad (2.16d)$$

Note that (2.16a) introduces the dimensionless wall heat flux q_w as a further governing dimensionless group.

3. Two-scale asymptotic analysis

The present work aims at obtaining a one-dimensional approximation of the energy equations (2.13) coupled with boundary conditions (2.16). Specifically, we look for one-dimensional advection–diffusion heat-transfer (ADHT) equations accounting for advection, diffusion and heat exchange between the two phases. To do so, we adopt a two-scale asymptotic expansion procedure (Hornung 1997; Boutin, Auriault & Geindreau 2010; Bensoussan, Lions & Papanicolaou 2011).

- (i) The starting point is the local description of the dimensionless heat-transfer problem, see (2.13), (2.16), together with the definition of the scale parameter ε given in (2.11).
- (ii) We focus on an asymptotic regime where the temperature profile can be considered almost uniform in the transverse direction (transverse diffusion has smeared out the initial profile) and axial diffusion is part of the game in the spirit of classical Taylor-dispersion theory (Taylor 1953). To this end, we look at axial variations in temperature in between the duct half-height, $\hat{H}$, and the duct length, $\hat{H}/\varepsilon$, of the order $1/\sqrt{\varepsilon}$. Therefore, the longitudinal coordinate z is re-scaled introducing a stretched spacial variable ξ (similarly to Griffiths *et al.* (2013); Ling *et al.* (2016); Liñán *et al.* (2020)):

$$\xi = \frac{z}{\sqrt{\varepsilon}}. \quad (3.1)$$

- (iii) The dimensionless groups (2.15), (2.16a) and the property ratios are evaluated in asymptotic terms expressing their magnitude as integer or half-integer power of the scale parameter ε :

$$\mathcal{K} = K \varepsilon^k \quad \text{with} \quad K = \mathcal{O}(1) \quad \text{and} \quad k \in \frac{1}{2}\mathbb{Z}, \quad (3.2a)$$

$$\mathcal{A} = \varepsilon^a \quad \text{and} \quad Pe = \varepsilon^{-p} \quad \text{with} \quad a, p \in \frac{1}{2}\mathbb{Z}, \quad (3.2b)$$

$$Br = B \varepsilon^b \quad \text{with} \quad B = \mathcal{O}(1) \quad \text{and} \quad b \in \frac{1}{2}\mathbb{Z}, \quad (3.2c)$$

$$q_w = Q \varepsilon^f \quad \text{with} \quad Q = \mathcal{O}(1) \quad \text{and} \quad f \in \frac{1}{2}\mathbb{Z}. \quad (3.2d)$$

- (iv) The temperature field ϑ_j in the energy equations (2.13), (2.16) is written as a half-integer power-series expansion in the scale parameter:

$$\vartheta_j(\xi, y, \tau; \varepsilon) = \vartheta_j^{(0)}(\xi, y, \tau) + \sqrt{\varepsilon} \vartheta_j^{(1)}(\xi, y, \tau) + \varepsilon \vartheta_j^{(2)}(\xi, y, \tau) + \mathcal{O}(\varepsilon\sqrt{\varepsilon}), \quad (3.3)$$

where $\vartheta_j^{(n)}$ is the n th-order term in the asymptotic expansion of the temperature field ϑ_j , with $j = \{1, 2\}$. We follow the classical multiscale approach where the main variables are expanded with respect to space and the axial coordinate is stretched according to (3.1) to properly describe the asymptotic regime (as explained in item (ii)) (Mauri 1991; Griffiths *et al.* 2013; Mauri 2015; Ling *et al.* 2016; Liñán *et al.* 2020). An equivalent approach would be to fully expand the independent variables in both time and space (Mauri 1995; Mazzino 1997; Mauri 2003; Mei & Vernescu 2010).

- (v) Then, we collect the terms of the same order obtaining a cascade of boundary-value problems. These are sequentially solved to obtain the effective description of the original problem in terms of the depth-averaged temperatures, $\langle \vartheta_j \rangle(\xi, \tau)$,

plus higher-order corrections. To do so, we introduce the cross-sectional averaging operator over each fluid domain

$$\langle \star_j \rangle = \begin{cases} \frac{1}{1-\beta} \left(\int_0^{1-\beta} \star_j \, dy \right) & \text{if } j = 1, \\ \frac{1}{\beta} \left(\int_{1-\beta}^1 \star_j \, dy \right) & \text{if } j = 2, \end{cases} \quad (3.4)$$

consistent with the classical two-scale asymptotic expansion procedure (Hornung 1997; Boutin *et al.* 2010; Bensoussan *et al.* 2011). Other types of mean relevant for heat-transfer problems, such as the mixing-cup (or bulk) temperature, require an *a priori* knowledge of the temperature field, and, therefore, they will be defined once the model's solution is determined, see § 3.3.

It will be shown, see Appendix C.2 and (C3a), that the leading-order term in (3.3) is independent on y and, therefore, $\vartheta_j^{(0)}(\xi, \tau) \equiv \langle \vartheta_j^{(0)} \rangle(\xi, \tau)$. In this type of approach, the goal is to describe the macroscopic behaviour of the system in terms of the zeroth-order terms $\vartheta_j^{(0)}$ while interpreting higher-order terms in (3.3) as small fluctuations around the zeroth-order values. Thus, without any lack of generality (Mauri 1995), we impose a gauge condition of the type

$$\langle \vartheta_j^{(n)}(\xi, y, \tau) \rangle \equiv 0 \quad \text{for } n \geq 1, \quad (3.5)$$

implying that the higher orders do not impact on the averaged temperature

$$\langle \vartheta_j^{(n)} \rangle \equiv \vartheta_j^{(0)} \delta_0^n, \quad (3.6)$$

where δ_0^n is the Kronecker delta indicator introduced to make the problem derivation more concise

$$\delta_{i_2}^{i_1} = \begin{cases} 0 & \text{if } i_1 \neq i_2, \\ 1 & \text{if } i_1 = i_2. \end{cases} \quad (3.7)$$

- (vi) The effective model is obtained imposing the compatibility condition, (also known as solvability condition or Fredholm alternative) which is the sufficient and necessary condition for the existence of solutions to the successive boundary-value problems. Satisfying the compatibility condition results in setting to zero the average of the expanded governing equation over its domain (Rubinstein & Mauri 1986; Auriault 2002; Mikelić *et al.* 2006).

The full derivation following the aforementioned procedure is provided in Appendix C, while the final results are given in upcoming sections, distinguishing between two different scenarios based of the order of magnitude of the thermal conductivity ratio $\mathcal{K}$, see (3.2a).

3.1. One-dimensional ADHT equations – coupled regime

The system of ADHT equations, describing the spatial and temporal evolution of the averaged temperatures, is given in (C36), with the coefficients defined in (C37). The

corresponding equation in the core is

$$\begin{aligned} & \mathcal{A} Pe \left(1 + \frac{1-\beta}{K\beta} \right) \frac{\partial \langle \vartheta_1 \rangle}{\partial \tau} + \mathcal{A} Pe \overbrace{(U_1 - V + \omega_{11}^*)}^{a_{11}^*} \frac{\partial \langle \vartheta_1 \rangle}{\partial z} + Pe a_{12}^* \frac{\partial \langle \vartheta_2 \rangle}{\partial z} \\ &= \varepsilon \underbrace{\left[1 + \frac{1-\beta}{K\beta} + \mathcal{A}^2 Pe^2 (D_1 + \delta_{11}^*) \right]}_{d_{11}^*} \frac{\partial^2 \langle \vartheta_1 \rangle}{\partial z^2} + \varepsilon Pe^2 d_{12}^* \frac{\partial^2 \langle \vartheta_2 \rangle}{\partial z^2} \\ &+ \frac{1}{\varepsilon} (g_1^* q_w + w_1^* Br) - \frac{1}{\varepsilon} e_1^* (\langle \vartheta_1 \rangle - \langle \vartheta_2 \rangle), \end{aligned} \quad (3.8)$$

while for the annulus, we get

$$\begin{aligned} & Pe \left(1 + \frac{K\beta}{1-\beta} \right) \frac{\partial \langle \vartheta_2 \rangle}{\partial \tau} + Pe \overbrace{(U_2 - V + \omega_{22}^*)}^{a_{22}^*} \frac{\partial \langle \vartheta_2 \rangle}{\partial z} + \mathcal{A} Pe a_{21}^* \frac{\partial \langle \vartheta_1 \rangle}{\partial z} \\ &= \varepsilon \underbrace{\left[1 + \frac{K\beta}{1-\beta} + Pe^2 (D_2 + \delta_{22}^*) \right]}_{d_{22}^*} \frac{\partial^2 \langle \vartheta_2 \rangle}{\partial z^2} + \varepsilon (\mathcal{A} Pe)^2 d_{21}^* \frac{\partial^2 \langle \vartheta_1 \rangle}{\partial z^2} \\ &+ \frac{1}{\varepsilon} (g_2^* q_w + w_2^* Br) + \frac{1}{\varepsilon} e_2^* (\langle \vartheta_1 \rangle - \langle \vartheta_2 \rangle), \end{aligned} \quad (3.9)$$

with $a_{jj}^*(U_j, V, \beta, m, K)$ and $d_{jj}^*(V, \beta, m, K, Pe)$ being the effective coefficients for advection and diffusion of the inner ($j=1$) and the outer phase ($j=2$). Those coefficients are a function of the speed of the reference frame, V , the phase speeds U_j , the volume fraction β , the viscosity ratio m , and the conductivity ratio K – their full expressions are given in [Appendix C.6](#). The ADHT equations (3.8), (3.9) are coupled through diverse physical mechanisms, namely advection and diffusion of the other phase via the coefficients $a_{12}^*(\beta, m, K)$ and $a_{21}^*(\beta, m, K)$, and $d_{12}^*(V, \beta, m, K)$ and $d_{21}^*(V, \beta, m, K)$, respectively; heat exchange between the phases through storage terms ($\propto \pm(\langle \vartheta_1 \rangle - \langle \vartheta_2 \rangle)$) and source terms proportional to q_w . All these mechanisms will be discussed in detail in §4.5.

The ADHT equations (3.8), (3.9) hold only in a specific range of the dimensionless groups (2.14), (2.15), (2.16a) to guarantee that the two scales can be decoupled. Such requirements are known as the applicability region and are sufficient (but not necessary) conditions for the effective model to be representative of the spatially averaged processes within the error bounds prescribed by the asymptotics. In our case, the upscaled equations hold only when:

- (i) $\varepsilon \ll 1$;
- (ii) $Pe \ll 1/\sqrt{\varepsilon}$;
- (iii) $\mathcal{A} Pe \ll 1/\sqrt{\varepsilon}$;
- (iv) $|q_w| \ll 1$;
- (v) $Br \ll 1$.

Condition (a) ensures that the spatial scale separation exists. Conditions (b) and (c) provide the upper bounds to the Péclet number of each phase, ensuring that advection does not prevail over diffusion. Condition (d) restricts the rate at which heat is transferred by the

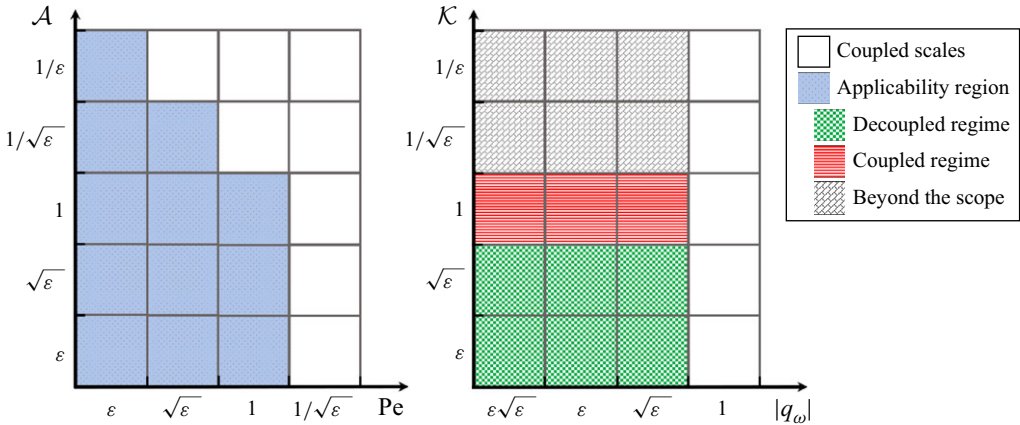


Figure 2. Parameter space of transport regimes in the (Pe, A) (left) and in the (q_w, K) (right) planes. Coloured areas of the maps correspond to regimes where the effective heat-transfer equation for the averaged temperatures can be formally written by means of two-scale expansion.

external energy source to/from the channel. Finally, condition (e) concerns the Brinkman number, guaranteeing that the impact of viscous shear heating is low enough not to conflict with the other transport phenomena.

The applicability region given by conditions (a)–(e) has been determined from the asymptotic solution developed in [Appendix C](#) selecting the order of the dimensionless parameters in such a way that they enter the problem at the highest order compatible with the separation of scales. In other words, the set of exponents p , a , k , b and f has been selected accordingly so that a given term enters into the corresponding order of magnitude depending on the transport regime. Those constraints are sufficient conditions which ensure a rigorous thermal decoupling between the transverse and the axial scales. If such constraints are not met, the idea of cross-sectional averaging (3.4) lacks significance and the accuracy of the upscaled model cannot be guaranteed (Boso & Battiato 2013). Note that there exists strategies to relax the applicability conditions, for example, using iterative hybrid numerical methods (see Battiato *et al.* (2011)).

Figure 2 shows the graphical representation of the conditions (b)–(d). Specifically, we can identify three main regimes based on the order of magnitude of the conductivity ratio K . When $K = \mathcal{O}(1)$, the thermal conductivities are of the same order of magnitude and the problem is described by two coupled ADHT equations (3.8), (3.9); this will be referred to as the coupled regime and it is typical of liquid–liquid systems (see [table 1](#)). Instead, when the thermal conductivity of the inner phase is small with respect to that of the outer one ($K \leq \mathcal{O}(\sqrt{\varepsilon})$), the thermal interaction between the phases is negligible and the problem is described by a single ADHT equation for the outer phase, see § 3.2; this will be referred to as the decoupled regime, typical of gas–liquid systems and a few liquid–liquid systems, see [table 1](#). There exists a third regime for $K \geq \mathcal{O}(1/\sqrt{\varepsilon})$ that leads to a single equation for the core averaged temperature. In this work, the latest regime will not be considered, since we are mainly interested in two-phase forced-convection configurations where the inner phase is less conductive with respect to the outer phase.

3.2. Decoupled regime

When the thermal conductivity of the outer phase is greater than that of the inner phase, $K \leq \mathcal{O}(\sqrt{\varepsilon})$, the thermal interaction is negligible and the fluid–fluid interface can be

Authors	Core (1)	Annulus (2)	m	ρ_1/ρ_2	$\mathcal{K}$	$\mathcal{A}$
Liquid–liquid						
Kruyer <i>et al.</i> (1967)	Transformer oil	Water	17	0.853	0.18	1.84
Hasson (1978)	Water	Kerosene	0.625	1.245	5.18	0.51
Arney <i>et al.</i> (1993)	No. 6 fuel oil	Water	2.70×10^3	0.989	0.20	2.31
Bannwart <i>et al.</i> (2004)	Heavy crude oil	Water	4.15×10^4	0.975	0.20	2.09
Gas–liquid						
Cioncolini (2023)	Air	Water	2.08×10^{-2}	1.19×10^{-3}	0.04	6.73×10^{-3}
Guevara & Gotham (1983)	Steam	Water	1.44×10^{-2}	6.00×10^{-4}	0.04	6.77×10^{-3}
Vuong <i>et al.</i> (2018)	Nitrogen	Isopar L	1.39×10^{-2}	2.19×10^{-2}	0.20	5.38×10^{-2}
Poesio <i>et al.</i> (2009)	Air	Turbine oil	1.54×10^{-5}	1.34×10^{-3}	0.19	3.44×10^{-3}

Table 1. Physical properties of liquid–liquid and gas–liquid core-annular systems taken from the literature. The property ratios m , $\mathcal{K}$, $\mathcal{A}$ denote the viscosity, thermal conductivity and thermal diffusivity ratios, respectively, as defined in (2.14).

considered adiabatic (see the derivation in Appendix C.5.1). Thus, in the moving reference frame (z, τ) , (3.8), (3.9) reduce to a single ADHT for the averaged temperature of the outer phase $\langle \vartheta_2 \rangle$:

$$\frac{\partial \langle \vartheta_2 \rangle}{\partial \tau} + (U_2 - V) \frac{\partial \langle \vartheta_2 \rangle}{\partial z} = D_2^* \frac{\partial^2 \langle \vartheta_2 \rangle}{\partial z^2} + S_2^*, \quad (3.10)$$

where

$$D_2^* = \frac{\varepsilon}{Pe} \left(1 + Pe^2 D_2 \right), \quad S_2^* = \frac{q_w + Br W_2^*}{\varepsilon Pe \beta}, \quad (3.11a-b)$$

are the effective diffusion coefficient and the effective source term, respectively, whose analytical expressions are given in Appendix C.6. The effective coefficients in (3.10) incorporate the impact of the non-uniformity of the velocity profile (2.1b) and thermal boundary conditions at the channel axis (2.8) and at the wall (2.6). Specifically, the shear flow spreads the temperature inhomogeneity along the axial direction, affecting axial heat diffusion at sufficiently large Péclet numbers via the coefficient $D_2^*(V, \beta, m, Pe)$. The effective sink/source term S_2^* consists of two distinct contributions: q_w is the heat flux imposed at the wall, while W_2^* embeds the effect of viscous dissipation. The trends of the effective coefficients and the analytical solution of the thermal problem in the decoupled regime will be discussed in § 4.1.

3.3. Determination of the convective heat-transfer coefficient

The local convective heat-transfer coefficient h is defined via Newton's law of cooling as the ratio between the thermal power per unit area entering the system and the difference in temperature between the solid surface and a representative temperature for the fluid $\hat{T}_b$ (Incropera 2007):

$$h(\hat{x}, \hat{t}) = \frac{-\hat{q}_w''}{\hat{T}_2|_{\hat{y}=\hat{H}} - \hat{T}_b}, \quad (3.12)$$

where $\hat{T}_b$ is the bulk temperature; the estimation of h is provided in the absolute reference frame ($V = 0$). The bulk (or mixing-cup or flow average) temperature is the temperature reached when a certain amount of fluid reaches the equilibrium without heat

loss to the surroundings (Tamir & Taitel 1972). It is defined as an enthalpy-weighted average of the phase temperature and for the core-annular flow configuration in figure 1, reads (Su 2006; Chalhub *et al.* 2022)

$$\hat{T}_b(\hat{x}, \hat{t}) = \frac{\int_0^{(1-\beta)\hat{H}} \rho_1 c_{p,1} \hat{u}_1 \hat{T}_1 d\hat{y} + \int_{(1-\beta)\hat{H}}^{\hat{H}} \rho_2 c_{p,2} \hat{u}_2 \hat{T}_2 d\hat{y}}{\int_0^{(1-\beta)\hat{H}} \rho_1 c_{p,1} \hat{u}_1 d\hat{y} + \int_{(1-\beta)\hat{H}}^{\hat{H}} \rho_2 c_{p,2} \hat{u}_2 d\hat{y}}. \quad (3.13)$$

Equation (3.12) is made dimensionless using the same scales defined in § 2.2, which gives

$$h(x, t) = \frac{\kappa_2 \left. \frac{\partial \vartheta_2}{\partial y} \right|_{y=1}}{\hat{H} (\vartheta_2|_{y=1} - \vartheta_b)}, \quad (3.14)$$

where the dimensionless bulk temperature ϑ_b is given combining (3.13) with (2.12), (3.4),

$$\vartheta_b(x, t) = \frac{\mathcal{K} \mathcal{A} (1 - \beta) \langle u_1(y) \vartheta_1(y) \rangle + \beta \langle u_2(y) \vartheta_2(y) \rangle}{\mathcal{K} \mathcal{A} (1 - \beta) U_1 + \beta U_2}. \quad (3.15)$$

Interestingly, the dimensionless group $\mathcal{K} \mathcal{A}$ in (3.15) is the volumetric heat capacity ratio: when $\mathcal{K} \mathcal{A} \ll 1$, the bulk temperature simplifies significantly and does not depend anymore on the core properties (this is true for the decoupled regime in § 4.2). Since the averaged temperature $\langle \vartheta_2 \rangle$ is independent of y , the temperature (and its derivative) at the wall is calculated considering the power-series truncated to the first order as

$$\vartheta_2|_{y=1} = \langle \vartheta_2 \rangle + \sqrt{\varepsilon} \left. \vartheta_2^{(1)} \right|_{y=1} + \mathcal{O}(\varepsilon), \quad (3.16a)$$

$$\left. \frac{\partial \vartheta_2}{\partial y} \right|_{y=1} = \sqrt{\varepsilon} \left. \frac{\partial \vartheta_2^{(1)}}{\partial y} \right|_{y=1} + \mathcal{O}(\varepsilon) = q_w + \mathcal{O}(\varepsilon). \quad (3.16b)$$

In problems of internal forced convection, the heat transfer coefficient is made dimensionless defining the local Nusselt number computed at the wall (Incropera 2007)

$$Nu(x, t) = \frac{h \mathcal{L}}{\kappa_2}, \quad (3.17)$$

where $\mathcal{L}$ is a characteristic length, typically the hydraulic diameter d_{eq} . For the planar configuration depicted in figure 1, the hydraulic diameter can be computed by considering a rectangular slit of width $\hat{w}$ and height $2\hat{H}$, and taking the limit for $\hat{w} \gg 2\hat{H}$ (see, for instance, Merritt 1991),

$$d_{eq} = \lim_{\hat{w} \rightarrow \infty} \frac{8 \hat{H} \hat{w}}{2(\hat{w} + 2\hat{H})} = 4 \hat{H}. \quad (3.18)$$

Ultimately, the Nusselt number (3.17) can be expressed as

$$Nu(x, t) = \frac{4 \left. \frac{\partial \vartheta_2}{\partial y} \right|_{y=1}}{\vartheta_2|_{y=1} - \vartheta_b}, \quad (3.19)$$

in view of (3.18) and (3.14). In the following, when the flow is thermally fully developed, we will refer to as the asymptotic (or limiting) Nusselt number Nu^∞ (Shah & London 1978).

Finally, for the sake of comparison with the literature, we recast the Brinkman number defined in (2.15b) (Brinkman 1951; Bird, Stewart & Lightfoot 1960; Boucher & Alves 1963; Shah & London 1978) into

$$Br' = \frac{\mu_2 \hat{U}^2}{(-\hat{q}_w'') \mathcal{L}}. \quad (3.20)$$

In this work, the Brinkman numbers Br and Br' are linked to each other by the following relation:

$$Br = 4 q_w Br'. \quad (3.21)$$

4. Results and discussion

In this section, we will discuss the heat-transfer model illustrating the main physical mechanisms and deriving a closed-form expression for the Nusselt number for laminar core-annular flows, distinguishing between the decoupled and the coupled regime.

4.1. Heat-transfer mechanisms and temperature field in the decoupled regime

As described in §3.2, the decoupled model (3.10) applies to cases where the thermal conductivity of the outer phase is much bigger compared with that of the inner phase, $\mathcal{K} \leq \mathcal{O}(\sqrt{\varepsilon})$. This implies that the inner and the outer phases are thermally decoupled and the model reduces to a ‘one-side’ approach. Such a scenario is typical of the majority of applications, including gas–liquid and liquid–liquid systems (see table 1).

By inspection of (3.10), we see that the temperature evolves in time and space due to three different mechanisms: advection equal to the average speed of the outer phase U_2 , effective thermal diffusion through the coefficient D_2^* , and a constant source term S_2^* due to the imposed heat flux at the channel wall and viscous dissipation. Specifically, D_2^* has the typical structure of the Aris–Taylor dispersion coefficient (it scales with the square of the Péclet number), being the sum of heat diffusivity, equal to the unity, and axial shear-induced diffusion, $D_2(V, \beta, m)$ whose expression is given in (C40b). The latter is an apparent diffusion mechanism for the averaged temperature and has its physical origin in the non-uniformity of the velocity in the outer phase: from the physical point of view, local advection enhances/reduces the axial diffusion in terms of the leading-order variables.

Figure 3(a) shows the evolution of D_2 in a coordinate system moving at the mean speed of the outer phase, $V = U_2$, according to Aris (Aris 1959). In fact, in the decoupled model, thermal dispersion is exclusively driven by the annulus. The presence of the other phase, which acts as a thermal insulator, reduces the effective heat-diffusion compared with the single-phase scenario where $D_2 = 2/105$; such a limit is reached only when the core vanishes, i.e. for $\beta \rightarrow 1$. The maximal shear-induced diffusion is obtained in the rigid core limit ($m \rightarrow \infty$), where the velocity profile of the annulus is almost linear (see figure 3a), maximising the spread of temperature in the axial direction. In the free surface limit ($m \rightarrow 0$), instead, the flow in the outer phase is so slow that the shear-induced diffusion becomes negligible.

Due to the many applications that involve confined thin-films (e.g. Craster & Matar (2009); Lavalley *et al.* (2021)), we highlight the thin-film region (TFR) in figure 3(a), defined arbitrarily as $\beta \leq \sqrt{\varepsilon} \ll 1$. In the TFR, the outer phase is ‘thin’ enough that D_2

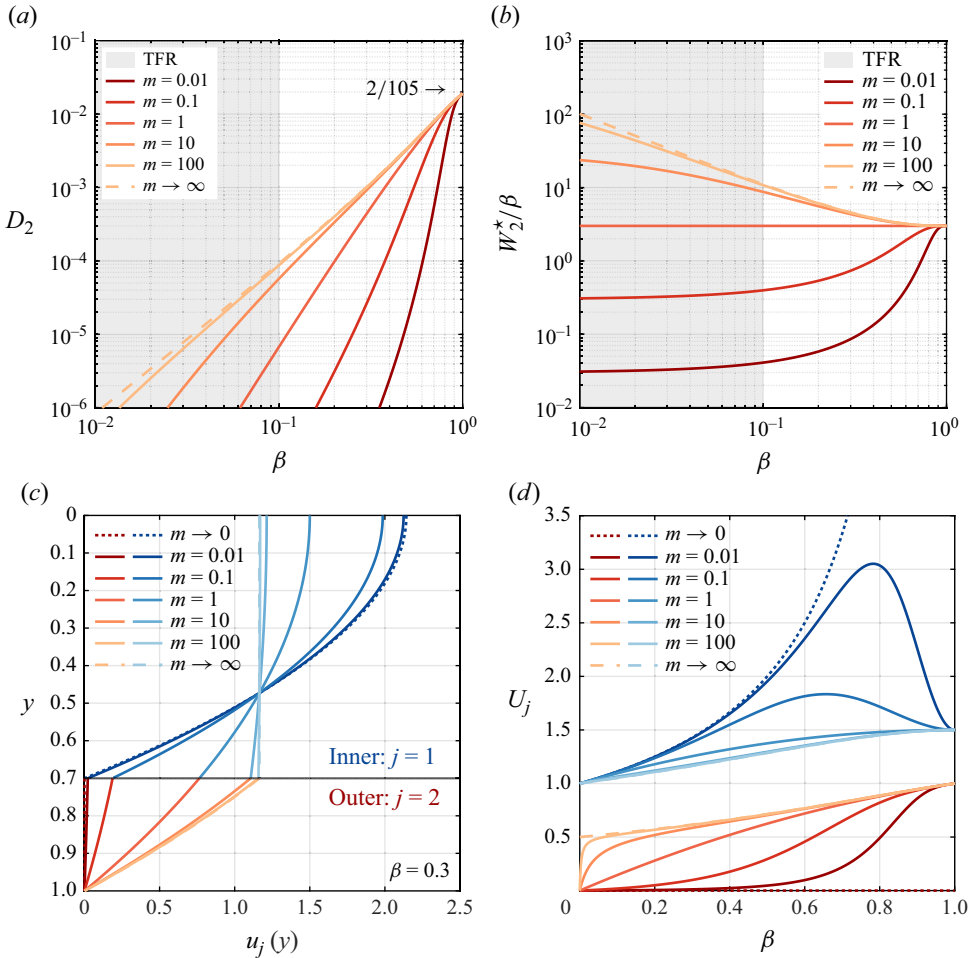


Figure 3. Effect of the viscosity ratio m on (a) D_2 (in the relative reference frame moving at the mean speed of the annulus, $V = U_2$) and (b) the source term related to viscous dissipation W_2^*/β , as functions of the volume fraction of the outer phase β . The thin film region (TFR) is highlighted by the grey area. For both the inner ($j = 1$) and the outer ($j = 2$) phase: (c) dimensionless velocity profiles $u_j(y)$ for different values of the viscosity ratio m and a fixed volume fraction $\beta = 0.3$; (d) average speed U_j as a function of β and for different values of m .

can be expanded as

$$D_2(V = U_2, \beta \rightarrow 0, m) = \begin{cases} \frac{3m^2}{40} \beta^4 + \mathcal{O}(\beta^5) & \text{if } m \text{ finite,} \\ \frac{1}{120} \beta^2 + \mathcal{O}(\beta^3) & \text{if } m \rightarrow \infty, \end{cases} \quad (4.1)$$

showing that the shear-induced diffusion becomes negligibly small in thin films. This can be explained by looking at figure 3(d): when $\beta \rightarrow 0$, the speed of the liquid becomes so small that the system is dominated by diffusion.

In general, advection and diffusion compete in the transient dynamics described by (3.10). To quantify this, similarly to Dejam *et al.* (2014) and Ling *et al.* (2016), we introduce the ratio between the effective diffusivity (3.11a) and its single phase limit

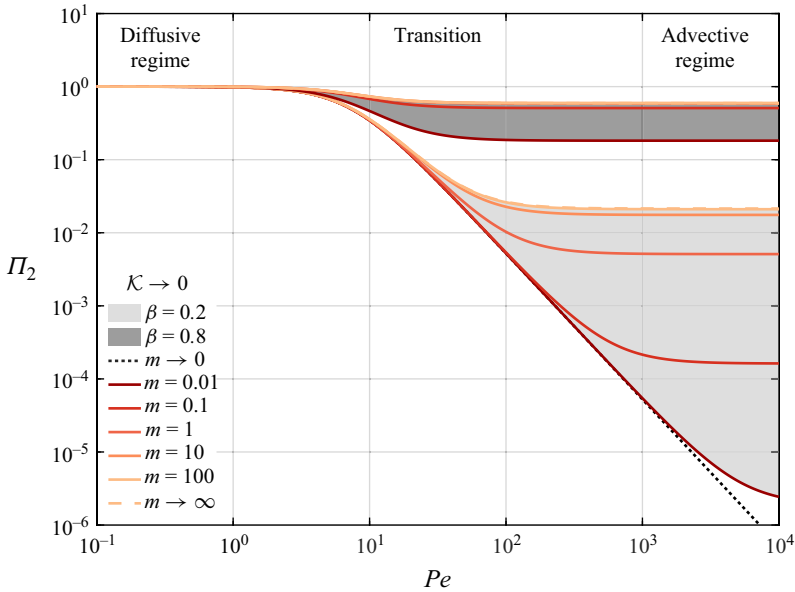


Figure 4. Normalised coefficient of shear-induced thermal diffusivity Π_2 for a decoupled system ($\mathcal{K} \rightarrow 0$) – see definition in (4.2) – against the flow Péclet number Pe , for different values of volume fraction β and viscosity ratio m .

(which corresponds to the analogous of the classical Aris–Taylor dispersion coefficient for the thermal problem), $\mathcal{D}_{AT} = 1 + \frac{2}{105} Pe^2$ (Horne & Rodriguez 1983; Berkowitz & Zhou 1996; Wang *et al.* 2012):

$$\Pi_2 = \frac{\varepsilon^{-1} Pe D_2^*}{\mathcal{D}_{AT}} = \frac{1 + Pe^2 D_2}{1 + \frac{2}{105} Pe^2}. \quad (4.2)$$

This normalisation facilitates the identification of different regimes, based on the competition between diffusion and advection, as shown in figure 4, where the evolution of Π_2 with respect to the Péclet number and the viscosity ratio at two fixed volume fractions, $\beta = \{0.2, 0.8\}$, is presented. Interestingly, for small Péclet numbers ($Pe < 1$), $\Pi_2 \rightarrow 1$ and it is independent of m and β , meaning that advective mixing of thermal energy in the outer phase is negligible compared with heat diffusion. When the Péclet number is sufficiently large, the advection dominates and Π_2 assumes always values lower than unity. This means that the core-annular flow pattern does not enhance heat diffusion compared with single-phase flow, in particular, when the volume fraction is small. Also, the mechanism of shear-induced diffusion becomes negligible in the free-surface limit ($m \rightarrow 0$) since the velocity in the outer layer is so small that, regardless of the film thickness β , Π_2 approaches zero without reaching a plateau as $Pe \rightarrow \infty$. The regime map in figure 4 can be used in transient multiphase heat-transfer applications to identify a specific set of operational conditions based on fluid and/or flow properties to enhance or reduce heat diffusion.

The source term S_2^* (3.11b) is affected by the dimensionless flux q_w and the Brinkman number Br . In particular, it scales with the inverse of the thickness of the outer phase $S_2^* \sim q_w/\beta$, meaning that heating a thin film is very effective since thermal energy is delivered to a small volume of fluid. The viscous dissipation $W_2^* = W_2^*(\beta, m)$ defined in (C28b, c), see (C46c), contributes to the source term, accounting not only for the viscous

heating produced in the annulus, but also for the production into the core that is exchanged across the interface. As shown in [figure 3\(b\)](#), in the free-surface limit ($m \rightarrow 0$), the velocity in the annulus is so slow that viscous dissipation is negligible, while for iso-viscous fluids ($m = 1$), $W_2^*/\beta = 3$ as for single-phase flow. However, in the rigid-core limit ($m \rightarrow \infty$), only the outer phase is dissipative and the heat generated due to friction becomes unbounded as the film gets thinner. Due to this singularity around $\beta = 0$, the scaling in the thin-film region can be obtained using Laurent series for $\beta \rightarrow 0$ as

$$\frac{W_2^*}{\beta} (\beta \rightarrow 0, m) = \begin{cases} 3m + \mathcal{O}(\beta) & \text{if } m \text{ finite,} \\ \beta^{-1} + 1 + \mathcal{O}(\beta) & \text{if } m \rightarrow \infty. \end{cases} \quad (4.3)$$

The ADHT equation (3.10) for the decoupled regime admits an analytical solution, setting the initial temperature of the outer phase to a uniform value over the entire channel half-length $x > 0$ and imposing the same temperature at the inlet $x = 0$ for $t > 0$, i.e. $\langle \vartheta_2 \rangle(x, 0) = \langle \vartheta_2 \rangle(0, t) \equiv 0 \forall x, t > 0$, we get – see Carslaw & Jaeger (1959) and Van Genuchten & Alves (1982) –

$$\begin{aligned} \langle \vartheta_2 \rangle(x, t) = S_2^* \left\{ t + \frac{(x - U_2 t)}{2 U_2} \operatorname{erfc} \left[\frac{x - U_2 t}{2 (D_2^* t)^{\frac{1}{2}}} \right] + \right. \\ \left. - \frac{(x + U_2 t)}{2 U_2} \exp \left(\frac{U_2 x}{D_2^*} \right) \operatorname{erfc} \left[\frac{x + U_2 t}{2 (D_2^* t)^{\frac{1}{2}}} \right] \right\}. \end{aligned} \quad (4.4)$$

At long times, the diffusion smears out the temperature profile and the steady-state temperature $\langle \vartheta_2^\infty \rangle$ becomes a linear function of x (like in classical internal forced-convection problems, see Incropera (2007)) with slope equal to S_2^*/U_2 , i.e.

$$\langle \vartheta_2^\infty \rangle(x) = \lim_{t \rightarrow +\infty} \langle \vartheta_2 \rangle(x, t) = \frac{S_2^*}{U_2} x. \quad (4.5)$$

This solution will be used in § 4.2 to derived a closed-form model for the heat-transfer coefficient. Note that, given the solution for the mean temperature (i.e. its leading order), higher order corrections can be easily calculated plugging (4.4) into (C15), (C23).

4.2. Nusselt number in the decoupled regime

In the decoupled regime, since $\mathcal{K} \rightarrow 0$ and $\mathcal{K} \mathcal{A} \lesssim \varepsilon$, the bulk temperature (3.15) simplifies to

$$\vartheta_b(x, t) = \langle \vartheta_2 \rangle + \sqrt{\varepsilon} U_2^{-1} \left\langle u_2(y) \vartheta_2^{(1)}(y) \right\rangle + \mathcal{O}(\varepsilon). \quad (4.6)$$

Upon substitution of (4.6) in (3.14), the local Nusselt number (3.14) yields

$$Nu(x, t) = \frac{4 q_w / \sqrt{\varepsilon}}{\vartheta_2^{(1)} \Big|_{y=1} - U_2^{-1} \left\langle u_2(y) \vartheta_2^{(1)}(y) \right\rangle}. \quad (4.7)$$

Equation (4.7) shows how the Nusselt number evolves in space and time and is calculated by plugging the analytical solution (4.4) into the first-order correction (C15) and using the velocity profile for $u_2(y)$, given in (A5a). Here, the Nusselt number is studied with respect to the fixed reference frame, in accordance with the literature, where the bulk temperature is always defined in systems with $V = 0$.

The Nusselt number in laminar decoupled core-annular flows with uniform wall heat flux is a function of space, time, the volume fraction, the viscosity ratio, the dimensionless heat-flux, the Brinkman and the Péclet numbers only, i.e. $Nu = Nu(x, t, \beta, m, q_w, Br', Pe)$. The Nusselt number and, as a consequence, the heat-transfer coefficient h depend on the Péclet number only in the transient regime, while at the steady-state, $Nu^\infty = Nu^\infty(\beta, m, Br')$. Note that, as typical in the heat transfer community (Shah & London 1978), the Péclet number can be seen as the product of the Prandtl and Reynolds numbers, i.e. $Pe = Pr Re$, where $Pr = \mu_2/(\rho_2 \alpha_2)$ and $Re = \rho_2 \hat{U} \hat{H}/\mu_2$, respectively.

When the thermal response reaches the steady state, we get the following expression for the asymptotic Nusselt number Nu^∞ , obtained by combining (4.5), (C15) and (4.7),

$$Nu^\infty(\beta, m, Br') = \frac{280(3 - \beta)^2}{\beta (\chi Br' + 45 \beta^2 - 245 \beta + 336)}, \quad (4.8)$$

where χ accounts for the contribution of viscous dissipation as

$$\chi = \frac{18 m [3 m \beta (7 - 3 \beta)^2 - (5 \beta^2 - 35 \beta + 56) (1 - \beta)^3]}{[1 + \beta(\beta^2 - 3 \beta + 3)(m - 1)]^2}. \quad (4.9)$$

Equation (4.8) shows that, in the absence of viscous dissipation ($Br' = 0$) or – equivalently – in the inviscid limit ($m \rightarrow 0$), the asymptotic Nusselt number is uniquely a function of the volume fraction of the outer phase β . Such an expression embeds the effects of viscous flow in both phases and heat-transfer (advection and diffusion) in the outer phase and it has been obtained starting from first principles. In figure 5(a), the Nusselt number in the non-dissipative regime is represented by the solid black line, which converges to the single phase limit of $140/17$ as $\beta \rightarrow 1$. Instead, in the dissipative regime, the single-phase limit of Nu^∞ approaches $140/(108 Br' + 17)$ according to (278) of Shah & London (1978).

The viscosity ratio impacts Nu^∞ only in the dissipative regime, as shown in figure 5(a) for $Br' = 1$. Interestingly, when the viscosity ratio m is sufficiently large (but finite), there exists conditions where viscous heating ($Br' > 0$) prevails over the wall heat flux ($q_w > 0$), and the bulk temperature becomes greater than the wall temperature, $\vartheta_2|_{y=1} - \vartheta_b < 0$, reversing the direction of the heat exchange. When this happens, the heat-transfer coefficient h becomes negative, see (3.14), and the singularity of the Nusselt number represents the limiting situation where $\vartheta_2|_{y=1} = \vartheta_b$. The critical value of volume fraction $\bar{\beta} = \bar{\beta}(m, Br')$ which determines the position of the vertical asymptote can be calculated by finding the zeros of the denominator in (4.8) numerically. To ensure that the Nusselt number remains positive in the entire film thickness range ($0 < \beta < 1$), the viscosity ratio and the Brinkman number have to satisfy the inequality $\chi Br' > -91/36$.

Finally, it is interesting to observe that the asymptotic Nusselt number in the TFR scales as follows:

$$Nu^\infty(\beta \rightarrow 0, m, Br' \neq 0) = \begin{cases} \frac{15}{2(1 - 3mBr')} \beta^{-1} + \mathcal{O}(1) + \dots & \text{if } m \text{ finite,} \\ \frac{60}{7Br'} + \mathcal{O}(\beta) & \text{if } m \rightarrow \infty, \end{cases} \quad (4.10)$$

while for non-dissipative flows ($Br' = 0$), we get that $Nu^\infty \sim 15/(2\beta)$. In the finite-viscosity regimes, the asymptotic Nusselt number Nu^∞ shows a singularity. Specifically,

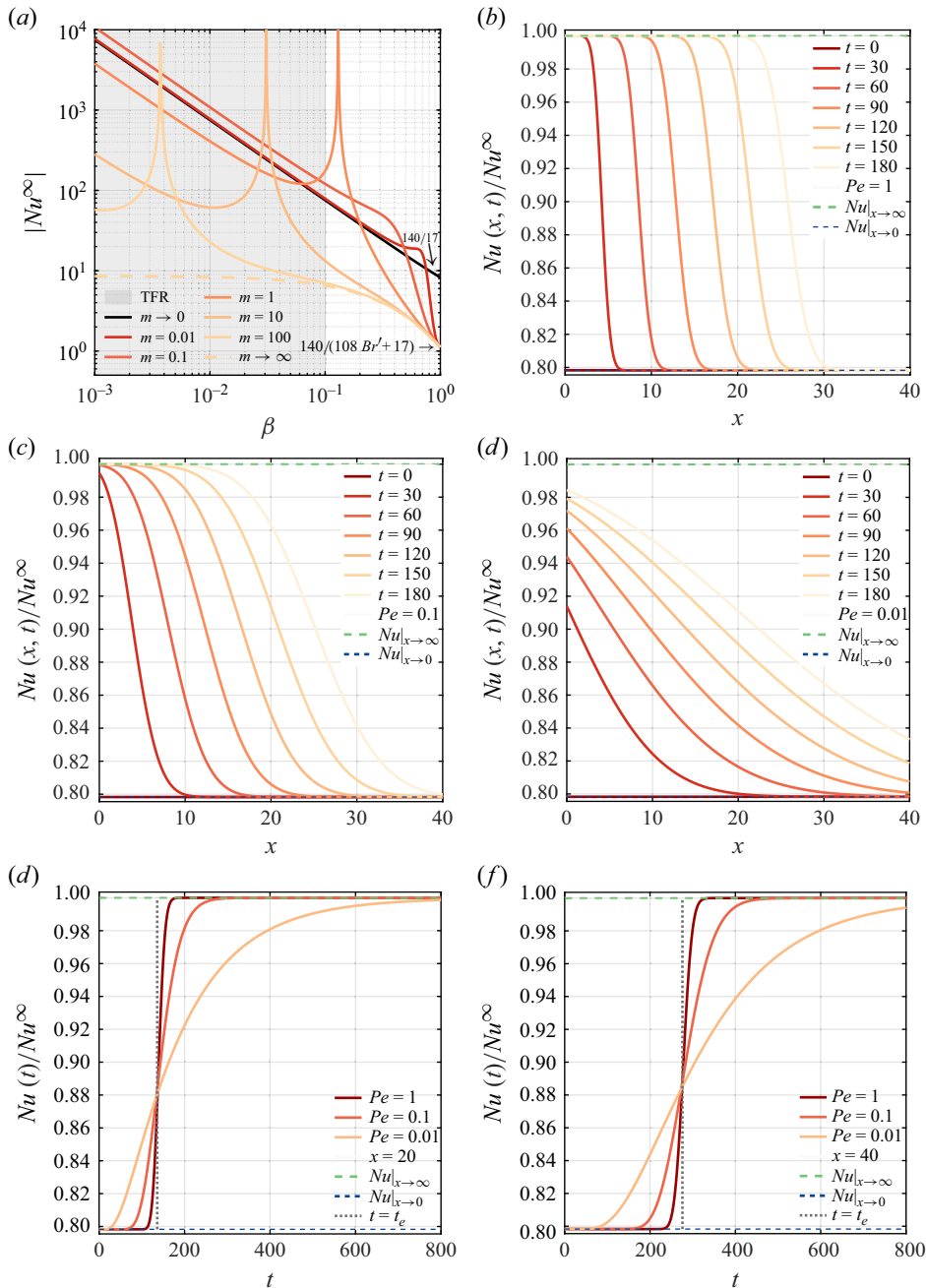


Figure 5. (a) Dependence of the asymptotic Nusselt number Nu^∞ (absolute value) on the volume fraction β , for fixed viscosity ratios m and $Br' = 1$. Evolution of the normalised Nusselt number Nu/Nu^∞ : over space at fixed times and Péclet numbers – (b) $Pe = 1$, (c) $Pe = 0.1$, (d) $Pe = 0.01$ – over time and across the transport regimes at fixed axial locations – (e) $x = 20$, (f) $x = 40$, with $\varepsilon = 0.01$, $Br' = 0$, $q_w = \beta = 0.1$, $m = 1$.

increasing m shifts the position of the vertical asymptote towards lower values of β , and a larger number of terms in the expansion (4.10) is required to ensure the convergence of the series to (4.8).

The time and spatial evolution of the local Nusselt number, normalised with respect to its asymptotic value for $t \rightarrow \infty$, i.e. Nu/Nu^∞ , is shown in [figure 5\(b–d\)](#) for different Péclet numbers. The resulting plots have the form of S-shaped breakthrough curves and they are bounded between two horizontal asymptotes: the upper one denotes the asymptotic Nusselt number, while the lower one corresponds to the Nusselt number predicted by (4.7) at the initial time instant, $t = 0$. At sufficiently high Péclet numbers, the S-shaped curves are almost rigidly transported by the flow over time, indicating an advective-dominated behaviour; a reduction of the Péclet number leads to a diffusion-dominated regime (see [figure 4](#)).

To determine how the advective and diffusive mechanisms compete in the transient dynamics, we plot the normalised Nusselt number for the different transport regimes considered in [figure 5\(b–d\)](#) over an extended time interval at two fixed positions along the channel: $x = 20$ (half-length of the channel) and $x = 40$ (end of the channel). At high Péclet numbers ($Pe = 1$), the heat is primarily advected downstream and the system quickly reaches the fully developed condition. This means that the heat transfer is quite efficient due to the strong influence of advection. Conversely, decreasing the Péclet number ($Pe = \{0.1, 0.01\}$) results in a slower increase of the Nusselt number due to the effect of diffusion. An interesting feature in both [figures 5\(e\)](#) and [5\(f\)](#) is the existence of a common intersection point between all the curves at $t = t_e$ that is independent of the value of the Péclet number. For $t > t_e$, the Nusselt number increases with the Péclet number, whereas for $t < t_e$, the behaviour of the system is reversed. At a fixed axial position, the curves appear to pivot around this point.

4.3. Hierarchy of time scales in thermal dispersion

The trends illustrated in [figure 5](#) also reflect the development of the forced-convective thermal boundary layer, similarly to the single-phase scenario (Siegel & Sparrow 1959; Siegel & Perlmutter 1963; Fakoor-Pakdaman *et al.* 2014). As time elapses, the thermal boundary layer progressively penetrates and fills up the thickness β occupied by the annulus. As a result, increasingly larger distances after $x = 0$ become thermally fully developed. In fact, since a finite amount of time, $t^* = x^*/U_2$, is required for the entrance fluid to be advected downstream and reach the axial position x^* , beyond this coordinate, $x \geq x^*$, there has not been any penetration of the fluid which was originally outside the channel before the start of the transient (at $t = 0$). For $x < x^*$, the thermal problem reaches its long-time thermal behaviour (linear temperature profile and constant Nusselt number).

Similarly to mass transfer (Allaire, Mikelić & Piatnitski 2010; Feder, Flekkøy & Hansen 2022), the thermal mixing is enhanced in regimes where diffusion has had time to even out the transverse variations of temperature (across the layer) while there are still longitudinal variations at the large scale (along it). Referring to the time scales introduced in § 2.2 and choosing the advective characteristic time scale $\hat{\tau}_a$ as the reference time, we obtain the following time scale hierarchy valid for the decoupled model (with $j = 2$ and $\mathcal{K} \rightarrow 0$):

$$\tau_{\hat{H},2} \ll \tau_{a,2} \ll \tau_{L,2} \quad \Rightarrow \quad \varepsilon Pe \beta^2 \ll U_2 \ll \frac{Pe}{\varepsilon}. \quad (4.11)$$

By inspection of (4.11), we can conclude that if the channel is sufficiently shallow ($\varepsilon \rightarrow 0$), these three time scales are sharply separated over a wide range of Péclet numbers, as shown in [figure 6](#). In particular, since the mean velocity of the annulus is bounded between 0 and 1, i.e. $0 \leq U_2 \leq 1$ (see [figure 3d](#)), thermal dispersion is observed for times of the order of $\hat{\tau}_a$ whenever $Pe \gtrsim \varepsilon$. For smaller Péclet numbers, the problem approaches its purely diffusive limit, where mixing is not influenced by shear-induced mechanisms. Interestingly, in the inviscid limit ($m \rightarrow 0$), the outer phase is almost at rest

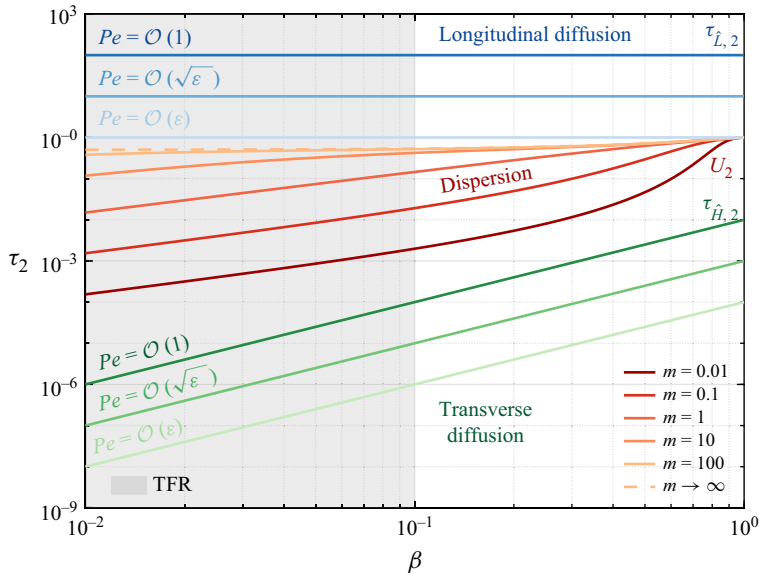


Figure 6. Time scale hierarchy and corresponding dominant heat-transfer mechanisms for the decoupled model ($\mathcal{K} \rightarrow 0$) as a function of the volume fraction β at different viscosity ratio m and Péclet number Pe . $\tau_2 = \{\tau_{\hat{H},2}, U_2, \tau_{\hat{L},2}\}$ are the characteristic times of transverse diffusion, advection (average speed of the annulus) and longitudinal diffusion, see (4.11), respectively. The dispersion regime is characterised by the dynamical competition between longitudinal advection and diffusion. $\varepsilon = 0.01$.

($U_2 \rightarrow 0$) and does not experience any shear-induced mixing, while in the TFR limit, the outer phase is so thin that transverse diffusion is almost instantaneous ($\tau_{\hat{H},2} \rightarrow 0$ as $\beta \rightarrow 0$) at any Pe .

Finally, it is worth recalling that our analysis cannot describe the early-time dynamics of the system – which may be referred to as pre-asymptotic dispersion regime (Young & Jones 1991; Taghizadeh *et al.* 2020) – and the upscaled model (3.10) holds only at times that are much greater than $\tau_{\hat{H},2}$, i.e. $t \gg \varepsilon Pe \beta^2$, holding no memory from the initial conditions (Ananthakrishnan, Gill & Barduhn 1965; Sankarasubramanian, Gill & Benjamin 1973; Fallon & Chauhan 2005).

4.4. Ramifications for the modelling of thin films and Taylor bubbles

In some circumstances, the modelling of flow patterns, such as intermittent slug flow, relies on an idealisation of the liquid film region, borrowing closure relations from core-annular flows (see for example, Balestra, Zhu & Gallaire (2018); Picchi *et al.* (2018)). Taylor bubble flow, in fact, consists of a sequence of liquid slugs and elongated bubbles, and – if the bubble is sufficiently long – a region of uniform film thickness forms, which can be considered, as a first approximation, a (local) region of fully developed core-annular flow. This approach neglects the evolution of the thin-film typical of Bretherton's problem (Bretherton 1961), while preserving model simplicity. In the context of heat transfer, the widely used three-zone model developed by Thome *et al.* (2004) and Dupont, Thome & Jacobi (2004) assumes that the film region of an evaporating/condensing Taylor bubble can be treated in this way.

Specifically, the heat transfer coefficient between the thin film surrounding a Taylor bubble and the channel wall is modelled as $h = \kappa_2/\delta$ (where δ is the film thickness) considering only the heat conduction in the transverse direction (across the film) at the

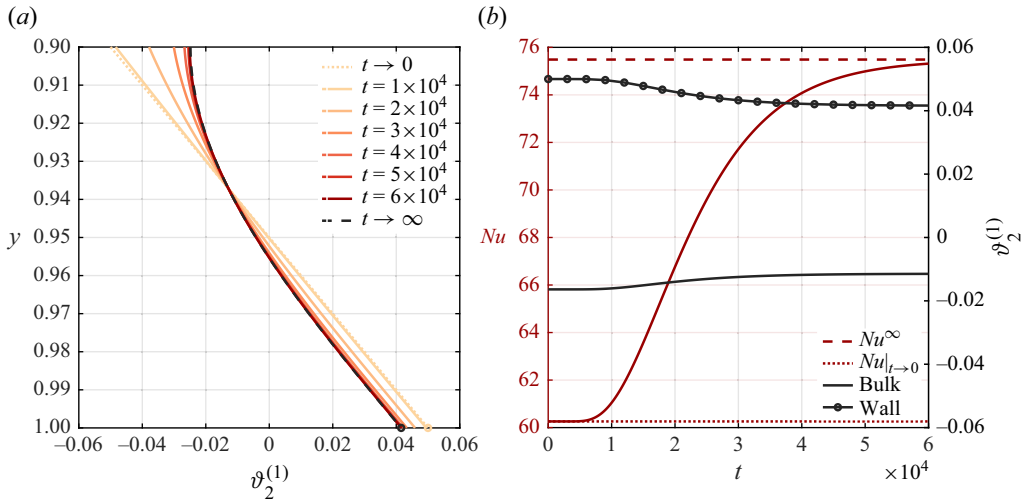


Figure 7. (a) Transient evolution of the first-order correction of the temperature profile of the annulus $\vartheta_2^{(1)}$ in the decoupled regime at $x = 40$, for $\beta = 0.1$, $m = 0.01$, $\varepsilon = 0.01$, $Pe = 1$, $q_w = 0.1$ and $Br = Br' = 0$. (b) Right ordinate: time evolution of the wall temperature ($\vartheta_2^{(1)}|_{y=1}$, black solid lines with circles) and the bulk temperature ($U_2^{-1} \langle u_2 \vartheta_2^{(1)} \rangle$, black solid line). Left ordinate: Nusselt number (4.7). The lower asymptote (dotted line) identifies the starting Nusselt number, $Nu|_{t \rightarrow 0} = 16(3 - \beta)\beta^{-1}(8 - 3\beta)^{-1}$; the upper asymptote (dashed line) denotes the fully developed Nusselt number Nu^∞ (4.8).

steady-state and neglecting the impact of advection (see Dupont *et al.* (2004); Thome *et al.* (2004); Dai *et al.* (2015); Magnini & Thome (2017); Zhang & Nikolayev (2023)). In this way, the heat-transfer coefficient is obtained by assuming a linear temperature profile in the film. The results presented in § 4.2 can be used to check the validity of this hypothesis. Combining (4.10) with (3.14) and (3.19), we obtain the following scaling law for the steady-state heat-transfer coefficient of core-annular flows in a planar geometry:

$$h = \frac{15}{8} \frac{\kappa_2}{\beta \hat{H}}. \quad (4.12)$$

At the steady-state, h is a function of the fluid conductivity and the film thickness only, i.e. $h = h(\kappa_2, \beta \hat{H})$. Our analysis confirms that the heat-transfer is primarily driven by conduction in the film, and the temperature profile remains almost linear along the transverse direction: when the film is thin enough, the dynamic effects due to the flow are negligible, see figure 3(d), $U_2 \rightarrow 0$ as $\beta \rightarrow 0$. This can be easily shown combining (C15) with the steady-state axial derivative of the averaged temperature (4.4) for $Br = 0$,

$$\vartheta_2^{(1)}|_{t \rightarrow \infty} = \frac{q_w}{\sqrt{\varepsilon}} \left[\frac{5(3+y)(1-y)^3 - \beta^3(5-\beta)}{20\beta^2(3-\beta)} + y + \frac{\beta}{2} - 1 \right], \quad (4.13)$$

as plotted in figure 7(a). The temperature profile is almost linear in proximity of the wall, while it flattens out at the interface where the heat-flux is zero; this behaviour is embedded in the numerical factor 15/8 in (4.12).

However, in regimes where the film thickness is not small compared with the channel size (Aussillous & Qu  r   2000), the flow in the film is not negligible, and the Nusselt number should be estimated using (4.8) instead of (4.12). In addition, accounting for transient effects would introduce the dependence of the P  clet number into the problem – see figure 7(b).

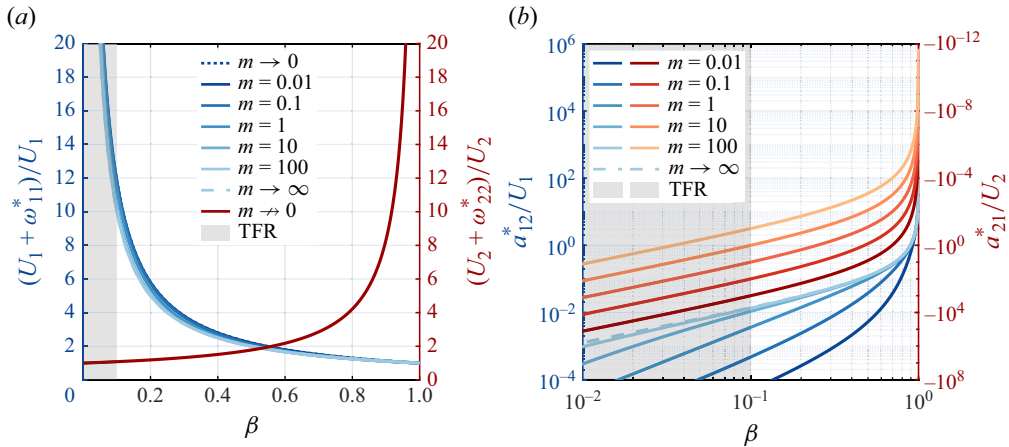


Figure 8. Effective coefficients of advection as functions of the volume fraction β , for different viscosity ratios m , when $\mathcal{K} = 1$ in the absolute reference frame ($V = 0$). (a) Advective coefficients normalised with the mean flow speed, i.e. a_{jj}^*/U_j : $j = 1$ (left) and $j = 2$ (right). (b) Coefficients of coupled advection a_{12}^*/U_1 (left) and a_{21}^*/U_2 (right). Those coefficients are independent of the speed of the reference frame.

4.5. Heat-transfer mechanisms and temperature field in the coupled regime

When the thermal conductivities of the two phases are of the same order, $\mathcal{K} = \mathcal{O}(1)$, the heat transfer problem is described by two coupled ADHT equations (3.8), (3.9). Differently from the decoupled regime, the phases can exchange energy through the fluid–fluid interface. By inspection of (3.8), (3.9), we see that the temperatures evolve in time and space due to three different mechanisms: advection, diffusion and storage/source terms. Specifically, each ADHT equation shows (i) a canonical and a coupled effective advection term; (ii) a canonical and a cross-coupling effective diffusion term; (iii) storage and source terms. In the following, we elucidate the physical interpretation of those effective coefficients to describe the main heat-transfer mechanisms of the coupled regime.

The evolution of the advection coefficient $a_{jj}^* = U_j + \omega_{jj}^*$ is shown in figure 8(a) in the fixed reference frame ($V = 0$). For each phase, the advection can be written as the sum of the mean phase velocity and an extra contribution due to phase coupling. When β is small, the lubrication effect of the thin annulus enhances the effective advection of the core that scales as β^{-1} and it is not affected by the viscosity ratio. The coupling advective coefficients a_{ij}^* appear in each ADHT due to the continuity of temperature at the fluid–fluid interface (see § C.5.2) and link one phase with the advection of the other one. Those coupling coefficients are plotted in figure 8(b): when β is small (in the TFR), the coupled advection in the core becomes negligible, $a_{12}^* \rightarrow 0$, and, therefore, axial temperature gradients of the annulus do not directly affect the core. Instead, regardless of the value of β , $a_{21}^* \rightarrow 0$ in the rigid-core limit (for $m \rightarrow \infty$), where the velocity profile in the core is almost flat, see figure 3(c), and does not trigger any coupled advection mechanism.

To investigate the diffusive mechanisms, we can think of heat transfer in the flowing phases via an electrical circuit analogy, as shown in figure 9(a). Specifically, the core–annular flow is represented by a parallel connection of resistors with a specific thermal conductance equal to G_1 and G_2 ; the equivalent conductance is then $G_{eq} = G_1 + G_2$. In this framework, we can recast the diffusive coefficients in analogy with the Aris–Taylor

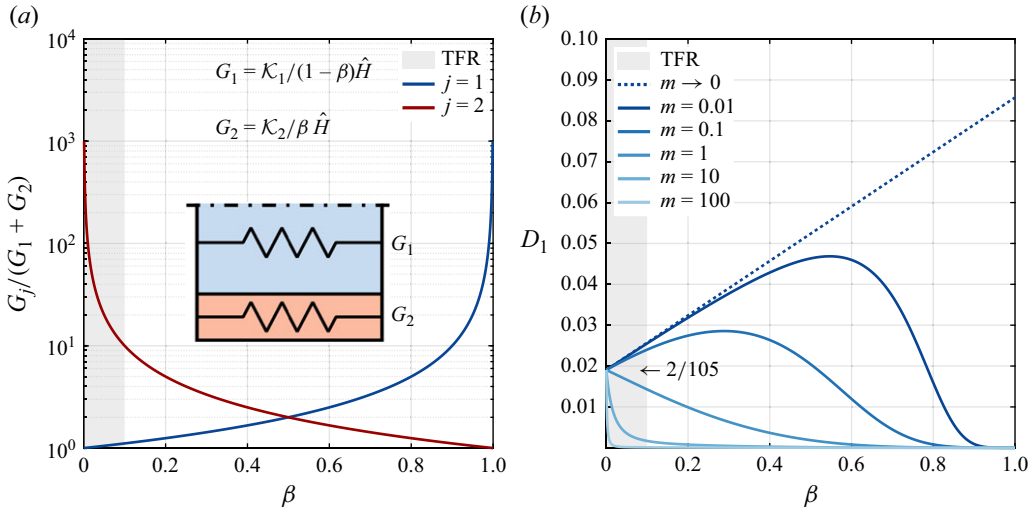


Figure 9. (a) Conductance ratios G_j/G_{eq} as functions of the volume fraction β , for $\mathcal{K} = 1$. (b) Effective coefficients of diffusion as functions of β , for different values of the viscosity ratio m , when $\mathcal{K} = 1$, in the reference frame moving at the mean flow speed ($V = 1$).

formalism to highlight the contribution of heat- and shear-induced diffusion as

$$d_{11}^* = \underbrace{\frac{G_1}{G_{eq}}}_{(i)} + Pe^2 \underbrace{(D_1 + \delta_{jj}^*)}_{(ii)}, \quad d_{22}^* = \underbrace{\frac{G_2}{G_{eq}}}_{(i)} + \mathcal{A}^2 Pe^2 \underbrace{(D_2 + \delta_{22}^*)}_{(ii)}, \quad (4.14a-b)$$

The first contribution, (i), is a volume-weighted average of the thermal conductivities of the individual phases and recalls the typical effective description proposed by Maxwell for the heat diffusion coefficient of composites (Maxwell 1873); its evolution with the volume fraction is shown figure 9(a), where we set $\mathcal{K} = 1$. Note that the ratio G_j/G_{eq} also appears in front of each time term in (3.8), (3.9), representing the thermal inertia of the phases, i.e. their capacity to store heat and delay its transmission. In the limit of $\mathcal{K} \rightarrow 0$, the ratio G_2/G_{eq} equals unity and the coefficient reduces to one discussed for the decoupled model. The second contribution in the effective diffusive coefficients, (ii), accounts for the shear-induced diffusion, D_1 and D_2 , and the coupling between the phases via the terms δ_{jj}^* . Both the effective diffusion coefficients scale with the square of the Péclet number, while the extra contribution due to the interaction is specific of this class of coupled-layers problems since it enters the ADHT equations by imposing the continuity of temperatures at the fluid–fluid interface (see the derivation in § C.5.2). Those terms are quadratic expressions in the speed of the moving reference frame, V , see (C41), consistent with previous works in the context of mass diffusion, e.g. see Aris (1959).

The evolution of D_2 with β and m has been discussed in § 4.1, figure 3(a), while the shear-induced diffusion for the core, D_1 , is plotted in figure 9(b) setting $V = 1$. Specifically, in the rigid-core limit, $m \rightarrow \infty$, the shear-induced diffusion becomes negligible since the inner phase has a flat velocity profile, see figure 3(c), preventing any extra spreading across the axial direction rather than heat diffusion. Conversely, in the free-surface limit, $m \rightarrow 0$, the outer phase is almost at rest and behaves like a static coating: D_1 increases linearly with the thickness β as

$$D_1(V = 1, m \rightarrow 0) = \frac{2}{105} + \frac{\beta}{15}, \quad (4.15)$$

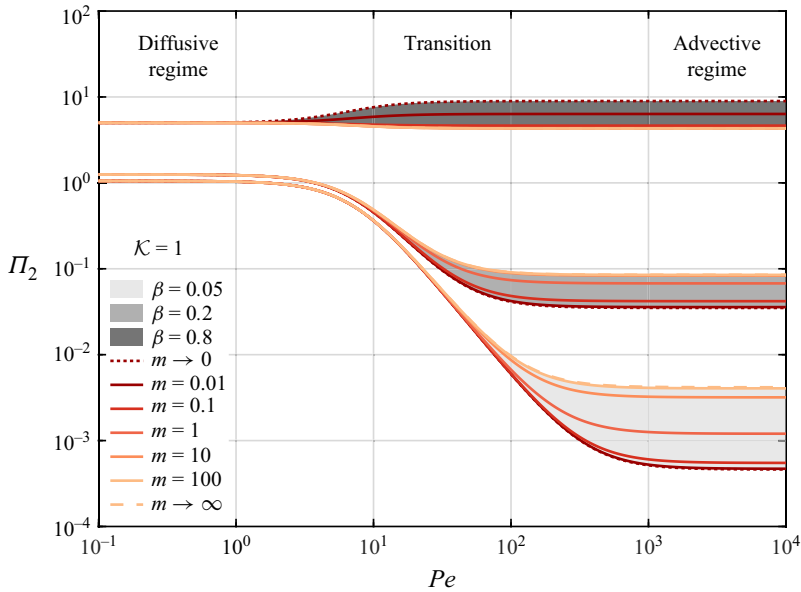


Figure 10. Normalised coefficient of shear-induced thermal conductivity Π_2 for the outer phase of core-annular flows ($\mathcal{K} = 1$) against the flow Péclet number Pe , for different values of volume fraction β and viscosity ratio m .

showing that a thinner core will result in higher transverse variations in its velocity profile, promoting the shear-induced axial diffusion mechanism. As expected, the single-phase limit of $2/105$ is recovered when $\beta \rightarrow 0$.

To explore the influence of the flow parameters (m , β , Pe , $\mathcal{K}$, $\mathcal{A}$) on the heat-transfer mechanisms in the coupled regime, we set $V = 1$ and study the ratio between the coefficient of shear-induced thermal diffusivity in (4.14) and its single-phase limit, $\Pi_j = d_{jj}^*/\mathcal{D}_{AT, j}$, similarly to what done in § 4.1:

$$\Pi_1 = \frac{1 + \frac{1-\beta}{K\beta} + \mathcal{A}^2 Pe^2 (D_1 + \delta_{11}^*)}{1 + \frac{2}{105} \mathcal{A}^2 Pe^2}, \quad \Pi_2 = \frac{1 + \frac{K\beta}{1-\beta} + Pe^2 (D_2 + \delta_{22}^*)}{1 + \frac{2}{105} Pe^2}. \quad (4.16a-b)$$

The evolution of Π_2 against the Péclet number is shown in figure 10, varying the viscosity ratio m for three different volume fractions $\beta = \{0.05, 0.2, 0.8\}$; the conductivity ratio $\mathcal{K}$ has been set to unity. The normalised diffusion coefficient of the outer phase has the form of an S-shaped curve and allows the identification of a diffusion-dominated and an advection-dominated regime. Differently from the decoupled model, the purely diffusive regime is characterised by a limiting value that exceeds unity, i.e. $\Pi_2 \rightarrow 1 + K\beta/(1-\beta)$ for $Pe \rightarrow 0$, and equal to the conductance ratio G_2/G_{eq} , see the expression of t_2^* given in (C38). The advective limit is strongly affected by the viscosity ratio and the volume fractions. Specifically, when the thickness β is small, the combined effect of advection and the coupling between phases reduces the diffusion coefficients compared with the case of single-phase flow. This effect is more pronounced in the free-surface than in the rigid-core limit. When the annulus is thick enough, instead, Π_2 can overcome the purely diffusive limit for sufficiently small values of the viscosity ratio m . The shear-induced diffusion ratio

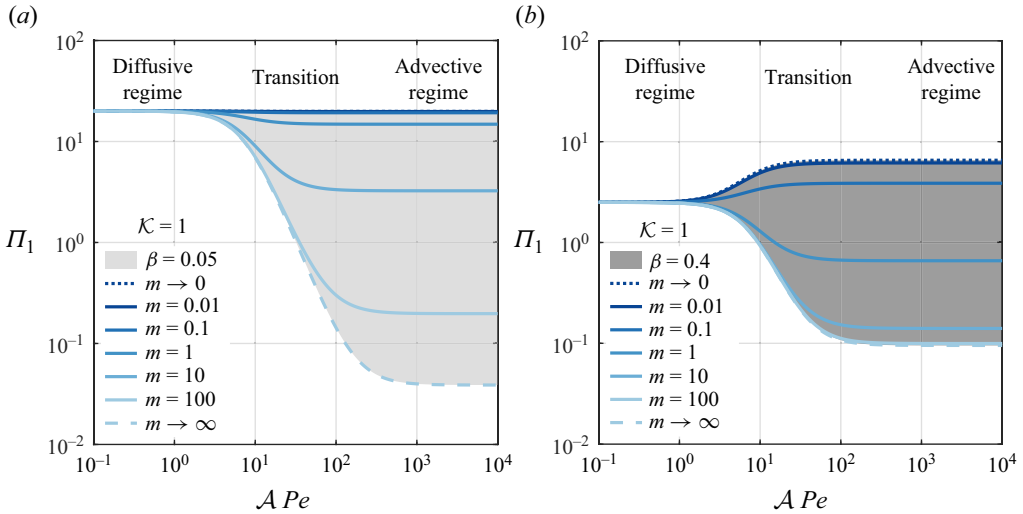


Figure 11. Normalised coefficient of shear-induced thermal diffusivity Π_1 for the inner phase of core-annular flows ($K = 1$) against the flow Péclet number $A Pe$, for different values of viscosity ratio m , setting the volume fraction to (a) $\beta = 0.05$ and (b) $\beta = 0.4$.

for the inner phase Π_1 shows a similar behaviour, as presented in figure 11: at low Péclet numbers, $\Pi_1 \rightarrow G_1/G_{eq} = 1 + (1 - \beta)/(K\beta)$, see the expression of t_1^* given in (C38). However, in the advective regime, the shear-induced diffusion in the core is minimal in the rigid-core limit, where transverse variations of velocity are negligible.

A similar analysis can be done for the cross-coupling diffusive terms $Pe^2 d_{12}^*$ and $A^2 Pe^2 d_{21}^*$ in (3.8), (3.9), introducing the following ratios:

$$\Pi_{12} = \frac{Pe^2 d_{12}^*}{1 + \frac{2}{105} Pe^2}, \quad \Pi_{21} = \frac{A^2 Pe^2 d_{21}^*}{1 + \frac{2}{105} A^2 Pe^2}. \quad (4.17a-b)$$

The physical origin of these coupling terms is due to advection only, as can be seen in figure 12. Both coefficients, in fact, tend to zero in the purely diffusive regime ($Pe \rightarrow 0$) and play a role only when advection is important, i.e. at high Péclet numbers. Note that the coupling terms enter the model when the temperature continuity is enforced, see Appendix C.5.2, suggesting that, at high Péclet numbers, heat diffusion in one phase is influenced by diffusion in the other one and *vice versa*. This mechanism can either play as negative diffusion (with respect to a reference frame that moves with the mean flow $V = 1$, and, therefore, the core is always faster than the annulus), as in the case of the annulus, see figure 12(a), or either enhance or reduce diffusion depending on m and β in the case of the core, see figure 12(b).

Finally, we study the source terms in (3.8), (3.9) to interpret how the external source of energy q_w affects the model. To do so, observing that the source terms are identical up to a factor equal to K^2 , i.e. $K^2 e_1^* = e_2^*$, see (C45), and are opposite-signed, we consider the linear combination, K^2 (3.8) + (3.9), so that the exchange terms cancel out to obtain

$$\frac{q_w}{\varepsilon} (K^2 g_1^* + g_2^*) = \frac{q_w \hat{H}}{\varepsilon K^2} (G_1 + G_2) = \frac{q_w \hat{H}}{\varepsilon K^2} G_{eq}. \quad (4.18)$$

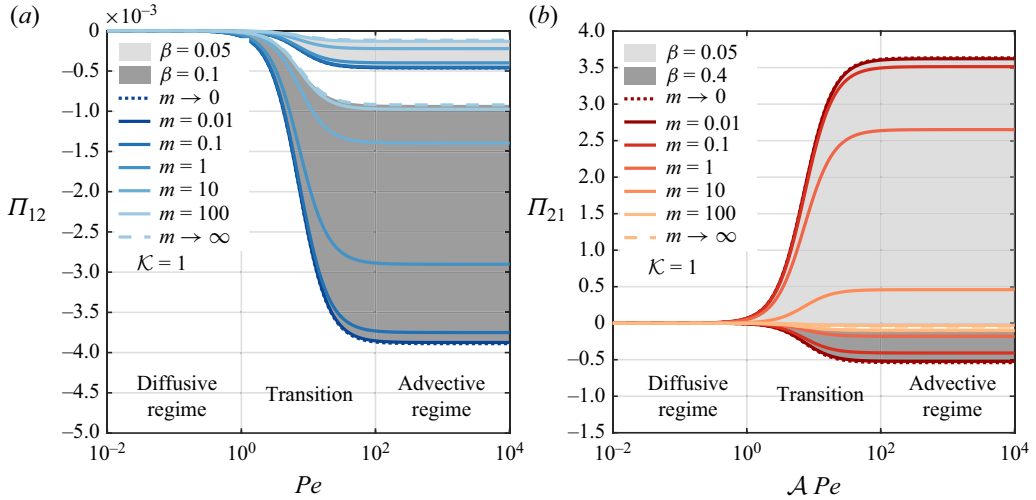


Figure 12. Normalised coefficients of cross-coupling diffusivity, (a) Π_{12} and (b) Π_{21} , for a core-annular system ($\mathcal{K} = 1$) against the corresponding flow Péclet number, for different values of viscosity ratio m and volume fraction β .

The resulting source term is the imposed heat flux at the channel wall multiplied by the sum of the two thermal conductances, see figure 9(a), confirming the consistency of our model. In other words, in the equation for the annulus, the source term is positive and greater with respect to the decoupled model by a factor $3K/2(1 - \beta) > 0$ – see (C44) – while in the core, the source term is negative, $-K/2(1 - \beta) < 0$, so as to preserve the continuity of the temperature and the thermal flux across the interface. Overall, the energy entering both layers is equal to (4.18) and consistent with the imposed boundary conditions (2.16).

4.6. Transient and steady-state temperature field in the coupled regime

In the coupled regime, the system of coupled ADHT equations (3.8), (3.9) can be solved numerically and admits an analytical solution only at the steady state.

At the steady state, in fact, diffusion smears out any sharp differences of temperature, and the solution $\langle \vartheta_j^\infty \rangle(x)$, $j = \{1, 2\}$, can be derived by setting to zero the time derivatives and diffusive terms in (3.8), (3.9), yielding an inhomogeneous system of linear first-order ordinary differential equations (ODEs), which can be recast via a linear combination as

$$\begin{cases} \frac{d\langle \vartheta_1^\infty \rangle}{dx} = \eta_1 (\langle \vartheta_1^\infty \rangle - \langle \vartheta_2^\infty \rangle) + \gamma_1, \\ \frac{d\langle \vartheta_2^\infty \rangle}{dx} = \eta_2 (\langle \vartheta_1^\infty \rangle - \langle \vartheta_2^\infty \rangle) + \gamma_2, \end{cases} \quad (4.19a,b)$$

where

$$\lambda = \varepsilon (a_{11}^* a_{22}^* - a_{12}^* a_{21}^*), \quad \eta_1 = -\frac{a_{22}^* e_1^* + a_{12}^* e_2^*}{\mathcal{A} Pe \lambda}, \quad \eta_2 = \frac{a_{21}^* e_1^* + a_{11}^* e_2^*}{Pe \lambda}, \quad (4.20a)$$

$$\gamma_1 = \frac{(a_{22}^* w_1^* - a_{12}^* w_2^*) Br + (a_{22}^* g_1^* - a_{12}^* g_2^*) q_w}{\mathcal{A} Pe \lambda}, \quad (4.20b)$$

$$\gamma_2 = \frac{(-a_{21}^* w_1^* + a_{11}^* w_2^*) Br + (-a_{21}^* g_1^* + a_{11}^* g_2^*) q_w}{Pe \lambda}. \quad (4.20c)$$

A particular solution to system (4.19) (see Kamke (1977); Polyanin & Zaitsev (2017)) is represented by two parallel lines of slope M and intercepts Q_j :

$$\langle \vartheta_j^\infty \rangle(x) = Mx + Q_j, \quad \text{with} \quad M = \frac{\eta_1 \gamma_2 - \eta_2 \gamma_1}{\eta_1 - \eta_2}, \quad Q_j = -\frac{\eta_j (\gamma_1 - \gamma_2)}{(\eta_1 - \eta_2)^2}, \quad (4.21)$$

meaning that, at the steady state, the temperature of both phases increases linearly in the axial direction with slope equal to $M(q_w/Pe, \beta, m, \mathcal{K}\mathcal{A}, Br')$ given by

$$M = \frac{2 q_w}{\varepsilon Pe} \left\{ \frac{(1 - \beta)^3 + m [\beta (\beta^2 - 3\beta + 3) + 12 Br']}{m \beta^2 (3 - \beta) + \mathcal{K}\mathcal{A} [2 (1 - \beta)^3 + 3 m \beta (1 - \beta) (2 - \beta)]} \right\}, \quad (4.22)$$

where the dimensionless group $\mathcal{K}\mathcal{A}$ represents the volumetric heat capacity ratio. Remarkably, in the free-surface limit, $m \rightarrow 0$, M is independent of β and equals $q_w/(\varepsilon Pe \mathcal{K}\mathcal{A})$, whereas the single-phase limits are

$$\frac{\varepsilon Pe M}{q_w} = \begin{cases} \frac{1 + 12 m Br'}{\mathcal{K}\mathcal{A}} & \text{if } \beta \rightarrow 0, \\ 1 + 12 Br' & \text{if } \beta \rightarrow 1, \end{cases} \quad (4.23)$$

recovering, in the absence of viscous dissipation ($Br' = 0$), the expected steady-state thermal behaviour, i.e. $M \propto \{1/(\mathcal{K}\mathcal{A}); 1\}$, see figure 13(a). The difference between the intercepts $Q_2 - Q_1$ corresponds to the steady-state thermal lag between the outer and the inner phase. Interestingly, in the free-surface limit, $m \rightarrow 0$, the normalised thermal lag increases linearly with β as $[4(1 - \beta) + 5 K \beta]/(10 K)$.

The transient formulation of (3.8), (3.9) can be solved numerically, as shown in figure 14(a, b), choosing a liquid–liquid system from table 1; the numerical solution has been obtained using the *pdepe* solver of MATLAB, setting an initial temperature equal to zero in both phases and imposing the steady-state heat flux M , given in (4.22), at the right boundary of the domain ($x = 20$). A second-order accurate spatial discretisation scheme is employed to convert the original problem into a set of ODEs, which are then integrated to obtain approximate solutions at the specified times (Skeel & Berzins 1990). Time discretisation is performed using a multistep variable-step variable-order (VSVO) solver based on numerical differentiation formulae (NDFs) of orders 1–5 (Shampine & Reichelt 1997; Shampine, Reichelt & Kierzenka 1999). We use equally spaced meshes in the intervals $0 \leq x \leq 20$ and $0 \leq t \leq 18$, with 1001 and 501 points, respectively. To ensure highly accurate results, the relative and absolute errors at each integration step have been set to 10^{-12} , with norm control enabled to manage the overall error in the solution vector. As time elapses, the solution evolves and an increasing region of the channel reaches the steady state.

Interestingly, we can use the numerical solution to check the model consistency and verify that the continuity of temperature at the fluid–fluid interface has been properly enforced. This issue represents a key aspect of the model, since such a boundary condition has been imposed in asymptotic terms, see § C.5.2. From figure 14(c), we can see that the

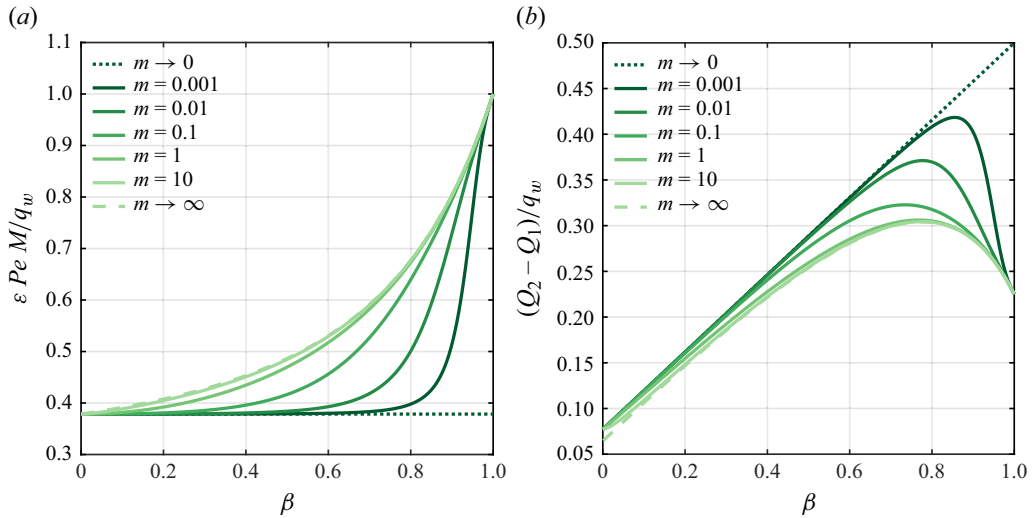


Figure 13. Normalised (a) slope and (b) difference in intercepts for the steady-state solutions (4.21) to the coupled model, as a function of the volume fraction of the outer phase β and for fixed values of the viscosity ratio m , corresponding to a liquid–liquid scenario with $\mathcal{A} = 0.51$ and $\mathcal{K} = 5.18$, see table 1.

temperature difference, $\vartheta_1 - \vartheta_2 = (\langle \vartheta_1 \rangle + \sqrt{\varepsilon} \vartheta_1^{(1)} + \varepsilon \vartheta_1^{(2)}) - (\langle \vartheta_2 \rangle + \sqrt{\varepsilon} \vartheta_2^{(1)} + \varepsilon \vartheta_2^{(2)})$ evaluated at the interface and keeping the first three terms of the expansions is within the $\mathcal{O}(\varepsilon\sqrt{\varepsilon})$ error, consistent with the two-scale asymptotic expansion in (3.3). Our model is in fact accurate up to the order $\mathcal{O}(\varepsilon)$.

4.7. Nusselt number in the coupled regime

In the coupled regime, the bulk temperature ϑ_b , see (3.15), simplifies to

$$\vartheta_b(x, t) = \frac{\mathcal{K} \mathcal{A} (1 - \beta) U_1 \left(\langle \vartheta_1 \rangle + \sqrt{\varepsilon} \vartheta_1^{(1)} \right) + \beta U_2 \left(\langle \vartheta_2 \rangle + \sqrt{\varepsilon} \vartheta_2^{(1)} \right)}{\mathcal{K} \mathcal{A} (1 - \beta) U_1 + \beta U_2} + \mathcal{O}(\varepsilon), \quad (4.24)$$

and the Nusselt number can be computed through the definition (3.19), using the numerical solution of (3.8), (3.9) for the averaged temperatures $\langle \vartheta_j \rangle$ to estimate the first-order terms in (4.24) via (C12) and (C15). The Nusselt number in laminar coupled core-annular flows with uniform wall heat-flux is a function only of $Nu = Nu(x, t, \beta, m, \mathcal{A}, \mathcal{K}, q_w, Br', Pe)$ and, based on the definition of the heat-transfer coefficient, see (3.12), it allows for obtaining the two-phase heat-transfer coefficient.

An example of the time and spatial evolution of $Nu(x, t)$ is given in figure 14(d): the Nusselt number evolves according to an S-shaped trend and is bounded between a lower and an upper horizontal asymptote, corresponding respectively to its limiting values for $t \rightarrow \infty$ and $t \rightarrow 0$. The impact of the Péclet number (i.e. the competition between advection and diffusion) is qualitatively similar to what is described in figure 5(b–f) for the decoupled regime and it is not shown here only for the sake of brevity.

At the steady state, the axial gradients of temperature $\partial \langle \vartheta_j \rangle / \partial x$ can be replaced by the slope M given in (4.21) and the Nusselt number $Nu^\infty = Nu^\infty(\beta, m, \mathcal{A}, \mathcal{K}, Br')$ can be written in a closed form (its full expression looks quite cumbersome and it is not reported here only for the sake of brevity). Figure 15 shows the evolution of Nu^∞ with respect to the volume fraction and the viscosity ratio for a liquid–liquid case, see table 1, keeping

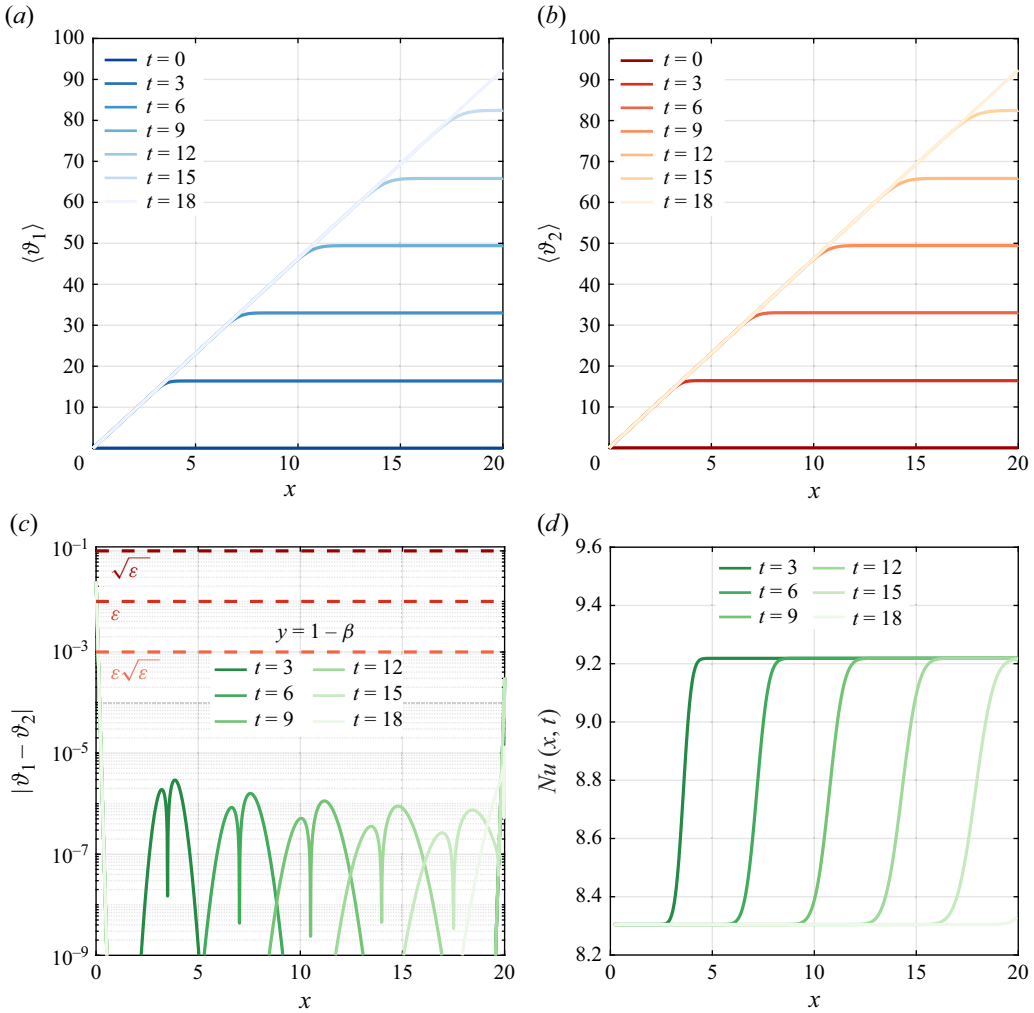


Figure 14. Transient evolution of the averaged temperature $\langle \vartheta_j \rangle(x, t)$ in the coupled regime, (a) $j=1$, (b) $j=2$. (c) Absolute value of the difference between the temperatures of the two phases at the interface $\vartheta_j|_{y=1-\beta}$, including corrections up to the second order. Dashed horizontal lines represent the $\mathcal{O}(\varepsilon^{1/2})$, ε , $\varepsilon^{3/2}$ tolerances choosing $\varepsilon = 0.01$. (d) Time evolution of the Nusselt number $Nu(x, t)$ against the axial coordinate x . For this liquid–liquid scenario (see table 1), the simulation parameters are: $Pe = 1$, $\mathcal{A} = 0.51$, $\mathcal{K} = 5.18$, $m = 0.625$, $q_w = 0.1$, $\beta = 0.5$, $Br' = 0$. Mesh resolution: $\Delta x = 0.02$.

the conductivity ratio as $\mathcal{K} = 1$; the panels (a)–(c) show the effect of the diffusivity ratio $\mathcal{A}$. First, the single-phase limits of the steady-state Nusselt number for a vanishing core (i.e. the channel section is entirely occupied by the outer phase) yields to

$$Nu^\infty(\beta \rightarrow 1, m, \mathcal{A}, \mathcal{K}, Br') = \begin{cases} 4 + \mathcal{O}(1 - \beta) & \text{if } m \rightarrow 0, \\ \frac{140}{108 Br' + 17} + \mathcal{O}(1 - \beta) & \text{otherwise,} \end{cases} \quad (4.25)$$

confirming that the result obtained as $m \not\rightarrow 0$ is consistent with the single-phase benchmark expression given by (278) of Shah & London (1978) ($Nu^\infty = 140/17$) for a

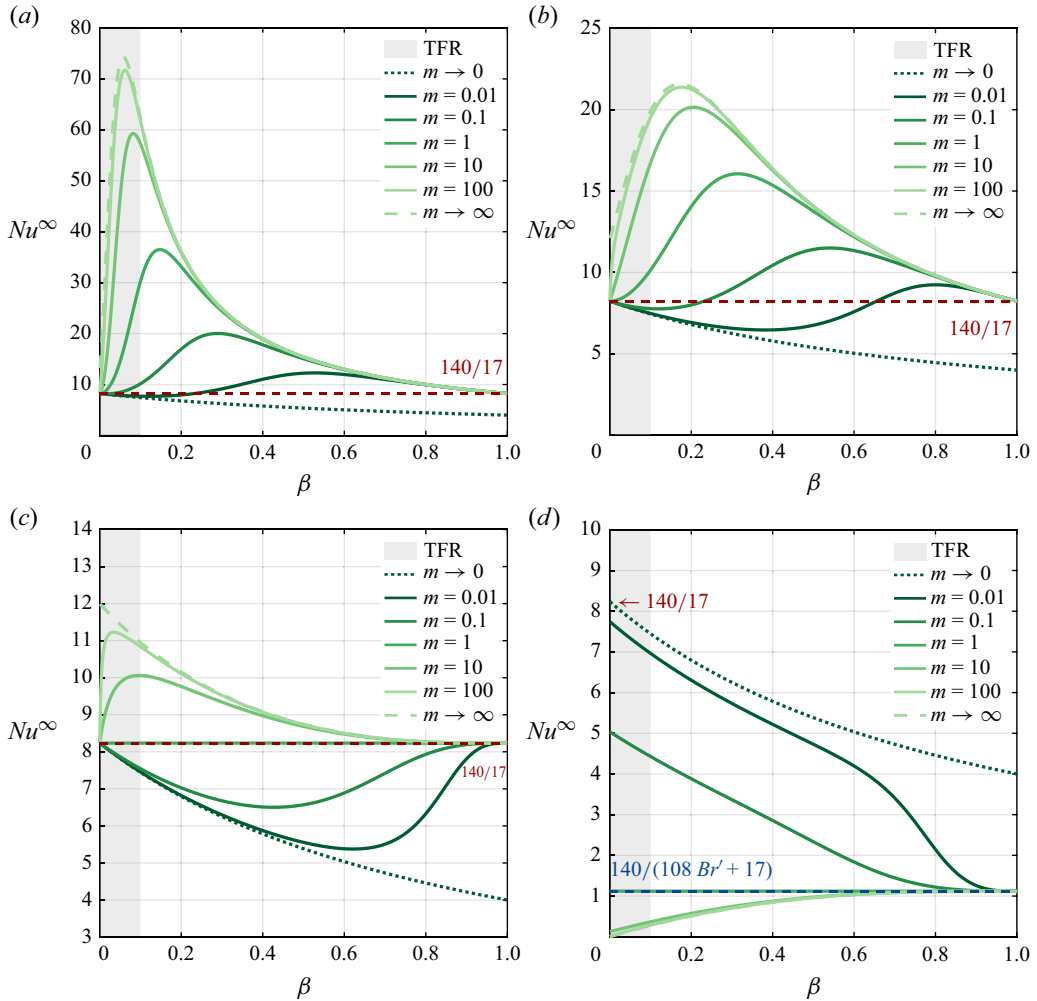


Figure 15. Limiting two-phase Nusselt number for a core-annular flow of unitary conductivity ratio, $\mathcal{K} = 1$, as a function of the volume fraction β , for fixed values of the viscosity ratio m . Panels (a) to (c) refer to a non-dissipative core-annular flow ($Br' = 0$) with increasing diffusivity ratios as increasing powers of the small-scale parameter, $\varepsilon = 0.01$: (a) $\mathcal{A} = \varepsilon$, (b) $\mathcal{A} = \sqrt{\varepsilon}$, (c) $\mathcal{A} = 1$. In (d), $\mathcal{A} = Br' = 1$.

uniform heat flux boundary condition imposed in a planar channel. Note that in the free-surface limit, $m \rightarrow 0$, the outer phase moves so slowly (see figure 3d) that the heat flux cannot be convected downstream and the Nusselt number converges to the value of 4, in agreement with the case of two different temperatures specified at each boundary, see (268) of Shah & London (1978).

Interestingly, when $m \rightarrow 0$, the convective heat-transfer is less efficient compared with the single-phase flow and $Nu^\infty < 140/17$ in the whole range of volume fractions, see figure 15(a–c). In the free-surface limit, in fact, the annulus moves so slowly (see figure 3c) that the asymptotic Nusselt number is unaffected by the thermal diffusivity ratio $\mathcal{A}$:

$$Nu^\infty(\beta, m \rightarrow 0, \mathcal{K}) = \frac{140 K}{35 K \beta + 17(1 - \beta)}. \quad (4.26)$$

Notice that, in the coupled regime ($k = 0$), the use of K or $\mathcal{K} = K \varepsilon^k$ is equivalent in (4.26). In the rigid core limit, instead, convective heat-transfer is favoured and is always enhanced compared with the single-phase flow, namely $Nu^\infty > 140/17$, in the whole range of β . This can be explained by the shape of the velocity profile (linear in the annulus and flat in the core, see figure 3c), that maximises heat transfer ensuring the most efficient replacement of fluid over the heated surface. This effect is amplified by lowering the diffusivity ratio $\mathcal{A}$. Specifically, the thermal diffusivity represents the promptness of a material in dissipating a temperature inhomogeneity relative to its tendency to store thermal energy, and, therefore, when $\mathcal{A}$ is less than one, the diffusivity of the core is greater than that of the annulus. In our case, a lower value of $\mathcal{A}$ results in a greater tendency of the outer phase to accumulate the heat received from the surroundings rather than diffusing it, enhancing the heat-transfer coefficient. This behaviour is favoured at low volume fractions, i.e. when the fluid in contact with the channel wall is sufficiently thin. In contrast, increasing β leads to a larger thermal resistance in the annulus and gives a greater difference between the wall and the bulk temperature defined in (4.24), reducing the heat-transfer coefficient h and the Nusselt number.

In other words, two-phase flows enhance convective heat transfer only under certain conditions. Specifically, only if the viscosity ratio is finite and the diffusivity ratio is small enough, we observe an enhanced heat-transfer coefficient, in particular at low volume fractions.

Finally, the effect of viscous dissipation on the Nusselt number is shown in figure 15(d), setting $Br' = 1$ while keeping $\mathcal{A} = 1$. When the two fluids have the same viscosity, i.e. $m = 1$, the Nusselt number does not depend on the volume fraction and viscous dissipation lowers the value of $140/(108 Br' + 17)$. The curves obtained for values of m larger than 1 are located below this threshold, while those where $0 < m < 1$ lie above it. In any case, viscous heating ($Br' > 0$) reduces the efficiency of convective heat-transfer and the dependence on the viscosity ratio is flipped compared with the non-dissipative scenario. This can be attributed to the combination of two factors (Shah & London 1978): (i) a reduction in the wall temperature gradient near the wall region due to viscous heating; and (ii) a slower rise of the bulk temperature along the channel axis, due to a reduced amount of heat transferred through the wall.

5. Conclusions

Forced convection in two-phase channel flows arises in a large variety of applications. In this paper, we derived an asymptotic one-dimensional model to describe the heat transfer in laminar core-annular flows in a planar geometry heated by a uniform heat-flux (extended Graetz-type problem).

The main heat-transfer mechanisms (advection and diffusion) occurring along the transversal and the longitudinal direction has been modelled via effective coefficients, which depend on the Péclet and Brinkman numbers, the dimensionless heat flux, the viscosity, thermal diffusivity and thermal conductivity ratios, and the volume fraction only. Specifically, the resulting diffusion coefficients provide a generalisation of the classical Aris–Taylor dispersion theory to two-phase flows. The model reveals the existence of two main regimes, depending on the thermal capacity of the two phases.

When the thermal conductivities of the two fluids are of the same order of magnitude, as in liquid–liquid systems, the system is described by two-coupled advection–diffusion heat-transfer equations. The coupling between the phases results in a canonical and a coupled effective advection term, a canonical and a cross-coupling effective diffusion term, and storage and source terms for each phase. We derived an analytical model for the

Nusselt number revealing that the heat transfer is enhanced with respect to the single-phase scenario only if the viscosity ratio is finite and the diffusivity ratio is small enough. In addition to that, we identified the dominant regime controlling thermal mixing depending on the magnitude of the Péclet number. In particular, at small Pe , the transport of thermal energy is dominated by diffusion. For intermediate values of the Péclet number, advection and diffusion compete, leading to a strong influence of both the flow condition (i.e. Péclet number) and the properties of the system (i.e. volume fraction, viscosity, thermal conductivity and thermal diffusivity ratios) on the dispersion coefficient. At high Pe , heat transport is fully dominated by advection.

When the annulus is more conductive than the core, as in most liquid–gas systems, the phases are thermally decoupled and the averaged temperature of the core evolves according to a single one-dimensional advection–diffusion heat-transfer equation. In this case, the limiting Nusselt number scales as the inverse of the annulus thickness and it is independent of the viscosity ratio. If the annulus is thin, our model confirms that, within the film, the heat-transfer is primarily driven by conduction and the temperature profile remains almost linear along the transverse direction. Interestingly, in the advection-dominated regime, the core-annular flow pattern does not enhance thermal mixing compared with single-phase flow, in particular when the volume fraction is small. Our analysis is complemented by the identification of a hierarchy between the characteristic time scales of advection and diffusion, helping in understanding the proper time window where thermal dispersion can be observed.

Our work clarifies the impact of the phases topology on forced convection and we hope that it may be extended with the aim of offering a more rigorous interpretation of the heat-transfer phenomena taking place in more complex two-phase flow patterns.

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Appendix A. Velocity profiles

Consider a core-annular flow as shown in [figure 1](#) at laminar and steady-state conditions: the system is composed by two immiscible and incompressible fluids with no interfacial instabilities or entrainment of one phase into the other (Joseph, Nguyen & Beavers 1984) and in the absence of any body forces (such as buoyancy). The channel is assumed to be shallow enough ($\hat{H}/\hat{L} \ll 1$) so that the momentum equations of each phase are simplified using the lubrication approximation and the flow is treated as one-dimensional (the velocity components in the transverse and normal directions are neglected). The momentum equation for each j th-Newtonian fluid, then, reduces to

$$0 = -\frac{d\hat{p}}{d\hat{x}} + \mu_j \frac{d^2\hat{u}_j}{d^2\hat{y}^2} \quad j = \{1, 2\}, \quad (\text{A1})$$

subjected to the following boundary conditions:

$$\left. \frac{d\hat{u}_1}{d\hat{y}} \right|_{\hat{y}=0} = 0, \quad (\text{A2a})$$

$$\mu_1 \left. \frac{d\hat{u}_1}{d\hat{y}} \right|_{\hat{y}=(1-\beta)\hat{H}} = \mu_2 \left. \frac{d\hat{u}_2}{d\hat{y}} \right|_{\hat{y}=(1-\beta)\hat{H}}, \quad (\text{A2b})$$

$$\hat{u}_1|_{\hat{y}=(1-\beta)\hat{H}} = \hat{u}_2|_{\hat{y}=(1-\beta)\hat{H}}, \quad (\text{A2c})$$

$$\hat{u}_2|_{\hat{y}=\hat{H}} = 0, \quad (\text{A2d})$$

namely, the symmetry of the inner velocity profile with respect to the channel axis (A2a), the continuity of tangential stresses (A2b) and velocities (A2c) at the interface, and, the no-slip condition at the channel wall. The set of equations (A1) admits the following analytical solution

$$\hat{u}_j(\hat{y}) = \frac{1}{2\mu_j} \frac{d\hat{p}}{d\hat{x}} \hat{y}^2 + \hat{A}_j \hat{y} + \hat{B}_j \quad j = \{1, 2\}, \quad (\text{A3})$$

where $\hat{A}_j$, $\hat{B}_j$ are constants to be determined imposing the boundary conditions given in (A2):

$$\hat{A}_1 = 0, \quad \hat{A}_2 = 0, \quad (\text{A4a})$$

$$\hat{B}_1 = -\frac{1}{2} \frac{d\hat{p}}{d\hat{x}} \frac{\hat{H}^2 [\mu_2 - \beta(2 - \beta)(\mu_2 - \mu_1)]}{\mu_1 \mu_2}, \quad \hat{B}_2 = -\frac{1}{2} \frac{d\hat{p}}{d\hat{x}} \frac{\hat{H}^2}{\mu_2}, \quad (\text{A4b})$$

leading to the dimensional velocity fields expressed in (2.1). Effects related to thermo-capillary Marangoni convection have not been considered in this work.

A.1 Dimensionless velocity profiles

The dimensionless velocity profiles $u_j(y)$ normalised using (2.12b,d) are given by

$$u_1(y) = 3 \Lambda [1 + \beta(2 - \beta)(m - 1) - y^2], \quad u_2(y) = 3 \Lambda m(1 - y^2), \quad (\text{A5a})$$

$$U_1 = \Lambda [2 + \beta(2 - \beta)(3m - 2)], \quad U_2 = \Lambda m\beta(3 - \beta), \quad (\text{A5b})$$

with

$$\Lambda^{-1} = 2[1 + \beta(\beta^2 - 3\beta + 3)(m - 1)]. \quad (\text{A6})$$

Figure 3(c) shows the velocity profiles (A5a) as a function of the viscosity ratio m . When $m \rightarrow \infty$, the inner fluid approaches the constant plug-like profile and we will refer to it as the rigid-core limit; conversely, the outer velocity profile becomes linear in the free-surface limit when $m \rightarrow 0$ (Balestra *et al.* 2018).

The evolution of the average speed U_j , see (A5b), against the volume fraction of the outer phase for different values of the viscosity ratio is presented in figure 3(d).

Appendix B. Heat capacity flow rate ratio

In Graetz-type heat exchange problems, the heat capacity flow rate ratio (or capacitance ratio) Cr is widely used. This dimensionless number is the ratio between the heat capacity rates (the product of mass flow rate $\dot{m}_j$ and specific heat capacity at constant pressure $c_{p,j}$)

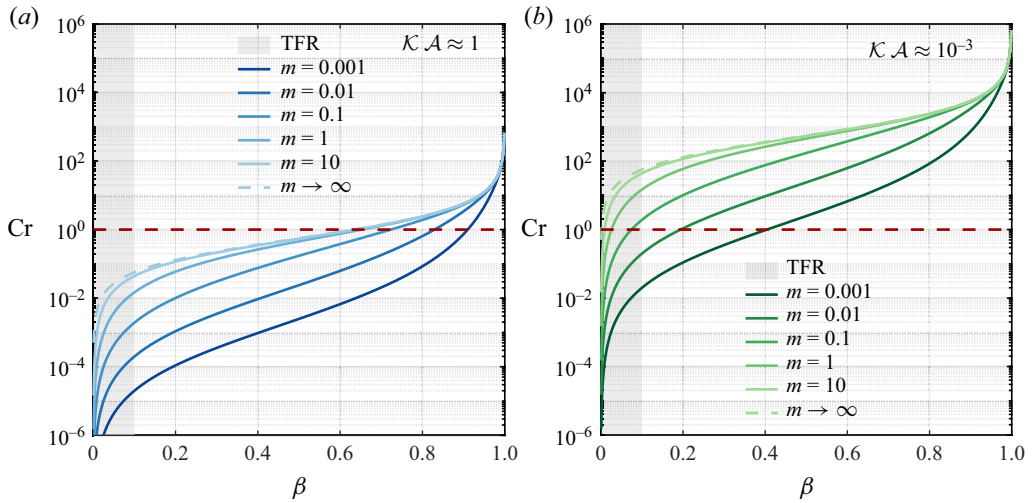


Figure 16. Evolution of the heat capacity flow rate ratio Cr as a function of the dimensionless thickness of the outer layer β for different values of the viscosity ratio m in (a) liquid–liquid and (b) gas–liquid systems. The two set of curves differ for the thermal capacity ratio $\mathcal{K}\mathcal{A}$. The TFR is highlighted by the grey area.

of the two phases flowing in a heat exchanger (Bai *et al.* 2018). For the core–annular configuration depicted in figure 1, Cr is computed combining (2.14c, A5b, A6) as

$$Cr = \frac{\dot{m}_2 c_{p,2}}{\dot{m}_1 c_{p,1}} = \frac{\rho_2}{\rho_1} \frac{\beta}{1-\beta} \frac{U_2}{U_1} \frac{c_{p,2}}{c_{p,1}} = \frac{1}{\mathcal{K}\mathcal{A}} \frac{\beta^2 m(3-\beta)}{(1-\beta)[2(1-\beta)^2 + 3m\beta(2-\beta)]}. \quad (\text{B1})$$

From the physical point of view, the heat capacity ratio governs the distribution of heat between the two phases and plays a crucial role in determining the overall effectiveness of heat transfer. Specifically, Cr describes how efficiently the energy is transferred from one phase to the other: the optimal performance is obtained when $Cr \approx 1$, while for $Cr \rightarrow 0$ or $Cr \rightarrow \infty$, its effectiveness is limited due to one phase dominating the heat transfer process. The Cr is typically related to the design and the optimisation of heat exchangers, e.g. the NTU (number of transfer units) method (Langerova & Matuska 2023).

In figure 16, we plot the capacitance ratio against the volume fraction of the outer phase for different values of the viscosity ratio for liquid–liquid and gas–liquid systems, identified by the order of magnitude of the thermal capacity ratio (i.e. the product $\mathcal{K}\mathcal{A}$). Referring to table 1, we can assume that $\mathcal{K}\mathcal{A} \approx 1$ in liquid–liquid and $\mathcal{K}\mathcal{A} \approx 10^{-3}$ for gas–liquid systems. Interestingly, Cr increases with m while keeping β fixed for both liquid–liquid and gas–liquid systems, and the optimal condition of $Cr = 1$ shifts to lower β , see figure 16.

Appendix C. Derivation of the heat-transfer model using two-scale asymptotic expansion

In the following sections, the derivation of the effective equations is carried out following the procedure discussed in § 3.

C.1 Expanded governing equations

After making the axial-coordinate stretching (3.1) and using (3.2), the governing equations (2.13) become

$$\varepsilon^{-p+\delta_1^j a} \left[\varepsilon \frac{\partial \vartheta_j}{\partial \tau} + \sqrt{\varepsilon} (u_j - V) \frac{\partial \vartheta_j}{\partial \xi} \right] = \left(\varepsilon \frac{\partial^2 \vartheta_j}{\partial \xi^2} + \frac{\partial^2 \vartheta_j}{\partial y^2} \right) + B \varepsilon^{b+\delta_1^j [\log_\varepsilon(m/K)-k]} \left(\frac{du_j}{dy} \right)^2, \quad (C1)$$

where $u_j(y)$ is the velocity profile of the j -phase given in (A5), V is the velocity of the moving reference frame, and $\delta_1^j = 1 - \delta_2^j$ is the Kronecker delta defined in (3.7) and introduced to make the model derivation more compact. Plugging the expansion (3.3) into (C1) leads to

$$\begin{aligned} -\varepsilon^{-p+\delta_1^j a} \left[\varepsilon \frac{\partial \vartheta_j^{(0)}}{\partial \tau} + \varepsilon \sqrt{\varepsilon} \frac{\partial \vartheta_j^{(1)}}{\partial \tau} + \sqrt{\varepsilon} (u_j - V) \frac{\partial \vartheta_j^{(0)}}{\partial \xi} + \varepsilon (u_j - V) \frac{\partial \vartheta_j^{(1)}}{\partial \xi} \right. \\ \left. + \varepsilon \sqrt{\varepsilon} (u_j - V) \frac{\partial \vartheta_j^{(2)}}{\partial \xi} \right] + \varepsilon \frac{\partial^2 \vartheta_j^{(0)}}{\partial \xi^2} + \varepsilon \sqrt{\varepsilon} \frac{\partial^2 \vartheta_j^{(1)}}{\partial \xi^2} \\ + \left(\frac{\partial^2 \vartheta_j^{(0)}}{\partial y^2} + \sqrt{\varepsilon} \frac{\partial^2 \vartheta_j^{(1)}}{\partial y^2} + \varepsilon \frac{\partial^2 \vartheta_j^{(2)}}{\partial y^2} \right) + B \varepsilon^{b+\delta_1^j [\log_\varepsilon(m/K)-k]} \left(\frac{du_j}{dy} \right)^2 = \mathcal{O}(\varepsilon^2). \end{aligned} \quad (C2)$$

Since the employed asymptotic expansion (3.3) was truncated to its first three terms, a cascade of equations for the unknown functions $\vartheta_j^{(n)}$, with $n = \{0, 1, 2\}$, originates from (C2). Specifically, we obtain

$$\mathcal{O}(1) : \frac{\partial^2 \vartheta_j^{(0)}}{\partial y^2} = 0, \quad (C3a)$$

$$\mathcal{O}(\sqrt{\varepsilon}) : \frac{\partial^2 \vartheta_j^{(1)}}{\partial y^2} = \varepsilon^{-p+\delta_1^j a} (u_j - V) \frac{\partial \vartheta_j^{(0)}}{\partial \xi} - B \varepsilon^{b-\frac{1}{2}+\delta_1^j [\log_\varepsilon(\frac{m}{K})-k]} \left(\frac{du_j}{dy} \right)^2, \quad (C3b)$$

$$\mathcal{O}(\varepsilon) : \frac{\partial^2 \vartheta_j^{(2)}}{\partial y^2} = \varepsilon^{-p+\delta_1^j a} \left[\frac{\partial \vartheta_j^{(0)}}{\partial \tau} + (u_j - V) \frac{\partial \vartheta_j^{(1)}}{\partial \xi} \right] - \frac{\partial^2 \vartheta_j^{(0)}}{\partial \xi^2}, \quad (C3c)$$

$$\mathcal{O}(\varepsilon\sqrt{\varepsilon}) : 0 = \varepsilon^{-p+\delta_1^j a} \left[\frac{\partial \vartheta_j^{(1)}}{\partial \tau} + (u_j - V) \frac{\partial \vartheta_j^{(2)}}{\partial \xi} \right] - \frac{\partial^2 \vartheta_j^{(1)}}{\partial \xi^2}, \quad (C3d)$$

where the order of magnitude of the governing parameters is within the applicability region discussed in § 3.1 and has been chosen in a way that ensures the least degeneracy of the closure problem (Van Dyke 1964). In other words, we keep the maximum possible number of non-vanishing small terms (Feuillebois & Lasek 1977) when equal powers of ε are gathered from the expanded governing equation (C2). Doing so, all the relevant physical effects enter the problem as soon as practicable and not beyond (Richard *et al.* 2016).

C.1.1 Boundary conditions

The boundary conditions (2.16) have to be expanded so that at each order in (C3), we obtain a boundary-value problem in the form of a second-order linear partial differential equation (PDE) that can be solved integrating twice in y .

The continuity of thermal fluxes across the interface (2.16b) reads

$$\left(\frac{\partial \vartheta_2^{(0)}}{\partial y} + \sqrt{\varepsilon} \frac{\partial \vartheta_2^{(1)}}{\partial y} + \varepsilon \frac{\partial \vartheta_2^{(2)}}{\partial y} \right) \Big|_{y=1-\beta} = \mathcal{K} \left(\frac{\partial \vartheta_1^{(0)}}{\partial y} + \sqrt{\varepsilon} \frac{\partial \vartheta_1^{(1)}}{\partial y} + \varepsilon \frac{\partial \vartheta_1^{(2)}}{\partial y} \right) \Big|_{y=1-\beta}, \quad (\text{C4})$$

suggesting the identification of several regimes based on the order of magnitude of the thermal conductivity ratio $\mathcal{K}$. In this work, we will focus on regimes where the inner phase exhibits a comparable ($k = 0$) or lower ($k > 0$) thermal conductivity with respect to that of the outer phase only. Therefore, depending on the order of magnitude of the conductivity ratio, the terms on the right-hand side of (C4) shift between orders.

The continuity of temperatures across the interface (2.16c) is not imposed order-by-order, *viz.* $\vartheta_1^{(n)}|_{y=1-\beta} = \vartheta_2^{(n)}|_{y=1-\beta}$, because it is not compatible with the chosen gauge (3.6) (namely, that the leading-order temperature coincides with the averaged temperature). Instead, similarly to Ling *et al.* (2016), we interpret the temperature continuity as an asymptotic equivalence of the type

$$\left(\vartheta_1^{(0)} - \vartheta_2^{(0)} \right) \Big|_{y=1-\beta} + \sqrt{\varepsilon} \left(\vartheta_1^{(1)} - \vartheta_2^{(1)} \right) \Big|_{y=1-\beta} + \varepsilon \left(\vartheta_1^{(2)} - \vartheta_2^{(2)} \right) \Big|_{y=1-\beta} = \mathcal{O}(\varepsilon\sqrt{\varepsilon}). \quad (\text{C5})$$

The adiabatic condition (2.16d) at $y = 0$ is straightforward and consistent with the symmetry of the problem, giving

$$\frac{\partial \vartheta_1^{(n)}}{\partial y} \Big|_{y=0} = 0, \quad n = \{0, 1, 2\}. \quad (\text{C6})$$

Lastly, to satisfy the gauge condition (3.6), the dimensionless external heat flux imposed at $y = 1$ needs necessarily to appear as an $\mathcal{O}(\sqrt{\varepsilon})$ quantity, which is equivalent to writing (2.16a) as

$$\frac{\partial \vartheta_2^{(n)}}{\partial y} \Big|_{y=1} = \begin{cases} 0 & \text{if } n = 0 \vee n \geq 2, \\ Q \varepsilon^{f-\frac{1}{2}} & \text{if } n = 1. \end{cases} \quad (\text{C7})$$

C.2 Order $\mathcal{O}(1)$: $\vartheta_j^{(0)}$ solutions

The leading-order equation (C3a) and the boundary conditions (C6), (C7) for $n = 0$ are homogeneous. This ensures that the boundary-value problem

$$\begin{cases} \frac{\partial^2 \vartheta_j^{(0)}}{\partial y^2} = 0, \\ \frac{\partial \vartheta_j^{(0)}}{\partial y} \Big|_{y=\delta_2^j} = 0, \end{cases} \quad (\text{C8a,b})$$

admits a trivial solution (i.e. that $\vartheta_j^{(0)}$ is independent of y), and, therefore, $\vartheta_j^{(0)}(\xi, \tau)$ corresponds to the average dimensionless temperature $\langle \vartheta_j \rangle(\xi, \tau)$ defined by (3.4). This is

true as long as $p < 1/2$, $p - a < 1/2$ and $f > 0$, namely $Pe \ll 1/\sqrt{\varepsilon}$, $\mathcal{A}Pe \ll 1/\sqrt{\varepsilon}$ and $q_w \ll 1$ – viscous dissipation also has to enter the problem as an $\mathcal{O}(\sqrt{\varepsilon})$ contribution or lower.

C.3 Order $\mathcal{O}(\sqrt{\varepsilon})$: $\vartheta_j^{(1)}$ solutions

At the order $\mathcal{O}(\sqrt{\varepsilon})$, the derivation will be carried out without choosing a specific value for the velocity of the reference frame V , which will remain arbitrary. This issue will be discussed in § 4.

Starting from the inner phase, (C3b) and (C6) with $n = 1$ and $j = 1$ give the following boundary-value problem:

$$\begin{cases} \frac{\partial^2 \vartheta_1^{(1)}}{\partial y^2} = \varepsilon^{-p+a} (u_1 - V) \frac{\partial \vartheta_1^{(0)}}{\partial \xi} - \frac{m B}{K} \varepsilon^{b-k-\frac{1}{2}} \left(\frac{du_1}{dy} \right)^2, \\ \left. \frac{\partial \vartheta_1^{(1)}}{\partial y} \right|_{y=0} = 0. \end{cases} \quad (\text{C9a}, b)$$

Integrating (C9a) with respect to y gives

$$\frac{\partial \vartheta_1^{(1)}}{\partial y}(y) = \frac{\varepsilon^a}{\varepsilon^p} \left(\int (u_1(y^*) - V) dy^* \right) \frac{\partial \vartheta_1^{(0)}}{\partial \xi} - \frac{m B}{K} \frac{\varepsilon^{b-k}}{\sqrt{\varepsilon}} \int \left(\frac{du_1(y^*)}{dy^*} \right)^2 dy^* + A_1^{(1)}, \quad (\text{C10})$$

where the integration constant $A_1^{(1)} = 0$ is determined by using the thermal symmetry condition (C9b) along with the corresponding velocity profiles (A5).

Then, we proceed by integrating (C10), obtaining

$$\vartheta_1^{(1)}(y) = \frac{\varepsilon^a}{\varepsilon^p} \left(\iint (u_1(y^*) - V) d^2 y^* \right) \frac{\partial \vartheta_1^{(0)}}{\partial \xi} - \frac{m B}{K} \frac{\varepsilon^{b-k}}{\sqrt{\varepsilon}} \iint \left(\frac{du_1(y^*)}{dy^*} \right)^2 d^2 y^* + B_1^{(1)}, \quad (\text{C11})$$

where the integration constant $B_1^{(1)}$ is determined enforcing the gauge-fixing condition of the type of (3.5), i.e. $\langle \vartheta_1^{(1)} \rangle = 0$ (see Mikelić *et al.* (2006)). Doing so, we get an expression for the first-order correction of the temperature of the inner phase

$$\vartheta_1^{(1)}(\xi, y, \tau; \varepsilon) = \varepsilon^{-p+a} \mathcal{M}_1(y) \frac{\partial \vartheta_1^{(0)}}{\partial \xi} - \frac{m B}{K} \varepsilon^{b-k-\frac{1}{2}} \mathcal{M}_1^{\mathcal{W}}(y), \quad (\text{C12})$$

where the functions

$$\begin{aligned} \mathcal{M}_1(y) = & \left[\frac{(1-\beta)^2 - 3y^2}{6} \right] V + \frac{\Lambda}{20} \left\{ -5y^4 + 30 \left[\beta m(2-\beta) + (1-\beta)^2 \right] y^2 + \right. \\ & \left. -(1-\beta)^2 \left[10 \beta m(2-\beta) + 9(1-\beta)^2 \right] \right\}, \end{aligned} \quad (\text{C13a})$$

$$\mathcal{M}_1^{\mathcal{W}}(y) = \frac{3\Lambda^2 [5y^4 - (1-\beta)^4]}{5}, \quad (\text{C13b})$$

are traditionally named B-fields after the seminal works of H. Brenner (Brenner & Stewartson 1980; Brenner & Edwards 1993) on Taylor dispersion. These functions represent the mean axial displacement at long times of a fluid particle located at y within

the channel cross-section, under the assumption that its initial positions are all equally probable (Mauri 2015).

For the outer phase, $j = 2$, (C3b) and (C7) for $n = 1$ give

$$\begin{cases} \frac{\partial^2 \vartheta_2^{(1)}}{\partial y^2} = \varepsilon^{-p} (u_2 - V) \frac{\partial \vartheta_2^{(0)}}{\partial \xi} - B \varepsilon^{b-\frac{1}{2}} \left(\frac{du_2}{dy} \right)^2, \\ \left. \frac{\partial \vartheta_2^{(1)}}{\partial y} \right|_{y=1} = Q \varepsilon^{f-\frac{1}{2}}. \end{cases} \quad (\text{C14})$$

Following the same mathematical procedure illustrated earlier yields a solution for $\vartheta_2^{(1)}$ in a similar form:

$$\vartheta_2^{(1)}(\xi, y, \tau; \varepsilon) = \varepsilon^{-p} \mathcal{M}_2(y) \frac{\partial \vartheta_2^{(0)}}{\partial \xi} - B \varepsilon^{b-\frac{1}{2}} \mathcal{M}_2^{\mathcal{W}}(y) + Q \varepsilon^{f-\frac{1}{2}} \mathcal{M}_2^q(y), \quad (\text{C15})$$

where

$$\mathcal{M}_2(y) = \left[\frac{\beta^2 - 3(1-y)^2}{6} \right] V + \frac{\Lambda m}{20} (-5y^4 + 30y^2 - 40y + \beta^4 - 5\beta^3 + 15), \quad (\text{C16a})$$

$$\mathcal{M}_2^{\mathcal{W}}(y) = \frac{3\Lambda^2 m^2}{5} (-5y^4 + 20y + \beta^4 - 5\beta^3 + 10\beta^2 - 15), \quad (\text{C16b})$$

$$\mathcal{M}_2^q(y) = y + \frac{\beta}{2} - 1. \quad (\text{C16c})$$

C.4 Order $\mathcal{O}(\varepsilon)$: $\vartheta_j^{(2)}$ solutions

At the order $\mathcal{O}(\varepsilon)$, the procedure to find $\vartheta_j^{(2)}$ can be simultaneously applied to both phases using the Kronecker delta defined in (3.7). First, since the previous-order solutions $\vartheta_j^{(1)}$, see (C12), (C15), depend on the axial coordinate ξ only via the leading-order temperatures $\vartheta_j^{(0)}$, we have

$$\frac{\partial \vartheta_j^{(1)}}{\partial \xi} = \varepsilon^{-p+\delta_1^j a} \mathcal{M}_j(y) \frac{\partial^2 \vartheta_j^{(0)}}{\partial \xi^2}. \quad (\text{C17})$$

Then, we can replace (C17) into (C3c) and use the boundary conditions (C6), (C7) to obtain

$$\begin{cases} \frac{\partial^2 \vartheta_j^{(2)}}{\partial y^2} = \varepsilon^{-p+\delta_1^j a} \frac{\partial \vartheta_j^{(0)}}{\partial \tau} + \left[\varepsilon^{2(-p+\delta_1^j a)} (u_j(y) - V) \mathcal{M}_j(y) - 1 \right] \frac{\partial^2 \vartheta_j^{(0)}}{\partial \xi^2}, \\ \left. \frac{\partial \vartheta_j^{(2)}}{\partial y} \right|_{y=\delta_2^j} = 0. \end{cases} \quad (\text{C18a,b})$$

Integrating (C18a) between δ_2^j and y and making use of (C18b) lead to

$$\frac{\partial \vartheta_j^{(2)}}{\partial y}(y) = \varepsilon^{-p+\delta_1^j a} \varphi_j(y) \frac{\partial \vartheta_j^{(0)}}{\partial \tau} + \left[\varepsilon^{2(-p+\delta_1^j a)} \psi_j(y) - \varphi_j(y) \right] \frac{\partial^2 \vartheta_j^{(0)}}{\partial \xi^2}, \quad (\text{C19})$$

with

$$\varphi_j(y) = \int_{\delta_2^j}^y dy^*, \quad \psi_j(y) = \int_{\delta_2^j}^y \mathcal{I}_j(y^*) dy^*, \quad \mathcal{I}_j(y) = (u_j(y) - V) \mathcal{M}_j(y). \quad (\text{C20})$$

Integrating (C19) indefinitely gives

$$\vartheta_j^{(2)}(y) = \varepsilon^{-p+\delta_1^j a} \tilde{\Phi}_j(y) \frac{\partial \vartheta_j^{(0)}}{\partial \tau} + \left[\varepsilon^{2(-p+\delta_1^j a)} \tilde{\Psi}_j(y) - \tilde{\Phi}_j(y) \right] \frac{\partial^2 \vartheta_j^{(0)}}{\partial \xi^2} + B_j^{(2)}, \quad (\text{C21})$$

where

$$\tilde{\Phi}_j(y) = \int \varphi_j(y^*) dy^*, \quad \tilde{\Psi}_j(y) = \int \psi_j(y^*) dy^*, \quad (\text{C22})$$

and $B_j^{(2)}$ is an integration constant, which can be found via the gauge-fixing condition $\langle \vartheta_j^{(2)} \rangle = 0$. Finally, we obtain the full expression for the second-order correction of the temperature for both phases:

$$\vartheta_j^{(2)}(\xi, y, \tau; \varepsilon) = \varepsilon^{-p+\delta_1^j a} \Phi_j(y) \frac{\partial \vartheta_j^{(0)}}{\partial \tau} + \left[\varepsilon^{2(-p+\delta_1^j a)} \Psi_j(y) - \Phi_j(y) \right] \frac{\partial^2 \vartheta_j^{(0)}}{\partial \xi^2}, \quad (\text{C23})$$

with

$$\Phi_j(y) = \tilde{\Phi}_j - \langle \tilde{\Phi}_j \rangle, \quad \Psi_j(y) = \tilde{\Psi}_j - \langle \tilde{\Psi}_j \rangle. \quad (\text{C24})$$

By inspection of (C23), we can notice that the $\mathcal{O}(\varepsilon)$ temperature profile describes the temporary thermal fluctuations occurring within the system through a time-derivative and a diffusive term. We do not report here the full analytic expression for the y -dependent functions that define the $\mathcal{O}(\varepsilon)$ solution, since we will follow the approach presented by Ling *et al.* (2016) for closing the problem (see § C.5), where we will make use only of the value that (C19), (C23) assume at the interface. To lighten notation, the inverted circumflex accent will be employed to denote the evaluation of a given function at the interface, namely $\check{\mathcal{M}}_j \equiv \mathcal{M}_j(y)|_{y=1-\beta}$.

C.5 Closure of the upscaled model

To derive the upscaled model, the cross-sectional average operator (3.4) is applied to (C2) while accounting for the boundary conditions. This is a necessary and sufficient condition for the cascade of linear equations (C3) to be solvable, thereby enforcing the solvability condition.

Thus, applying the gauge condition (3.6) to the expanded energy equation for the outer phase (C2), $j = 2$, we obtain

$$\begin{aligned} \varepsilon^{-p} \left[\varepsilon \frac{\partial \langle \vartheta_2 \rangle}{\partial \tau} + \sqrt{\varepsilon} \langle u_2 - V \rangle \frac{\partial \langle \vartheta_2 \rangle}{\partial \xi} + \varepsilon \overbrace{\left\langle (u_2 - V) \frac{\partial \vartheta_2^{(1)}}{\partial \xi} \right\rangle}^{(i)} + \varepsilon \sqrt{\varepsilon} \overbrace{\left\langle (u_2 - V) \frac{\partial \vartheta_2^{(2)}}{\partial \xi} \right\rangle}^{(ii)} \right] \\ = \varepsilon \frac{\partial^2 \langle \vartheta_2 \rangle}{\partial \xi^2} + \frac{\sqrt{\varepsilon}}{\beta} \left(\frac{Q \varepsilon^f}{\sqrt{\varepsilon}} - \underbrace{\frac{\partial \vartheta_2^{(1)}}{\partial y} \bigg|_{y=1-\beta}}_{(iii)} \right) + \frac{\varepsilon}{\beta} \left(0 - \underbrace{\frac{\partial \vartheta_2^{(2)}}{\partial y} \bigg|_{y=1-\beta}}_{(iv)} \right) \\ + \frac{B \varepsilon^b}{\beta} \int_{1-\beta}^1 \left(\frac{du_2(y^*)}{dy^*} \right)^2 dy^*, \end{aligned} \quad (C25)$$

where we used the wall boundary conditions (C7) at $\mathcal{O}(\sqrt{\varepsilon}, \varepsilon)$. Four terms in (C25) need to be closed: two non-local advective terms (i, ii), and two interfacial boundary terms (iii, iv). By means of (C17) and (C20), the term labelled by (i) can be expressed in terms of macro-scale quantities as $\varepsilon^{-p} \langle \mathcal{I}_2 \rangle \partial^2 \langle \vartheta_2 \rangle / \partial \xi^2$, while the term (ii) is found to be zero by averaging (C3d) with $j = 2$.

Based on the disparity in the thermal conductivities of the two phases, the continuity condition of thermal fluxes at the interface, see (C4), has different expressions depending on the order of magnitude of thermal conductivity ratio and, therefore, different closure relations are valid for terms (iii) and (iv). This leads to the identification of two different thermal models which may incorporate one or two equations (similarly to Quintard & Whitaker (1993); Lewandowska & Auriault (2004); Lewandowska, Szymkiewicz & Auriault (2005)): the cases where $k > 0$ (decoupled model) and the case where $k = 0$ (coupled model).

C.5.1 Decoupled model

When $k \geq 1$, term (iii) is obtained by replacing (C12) into (C4)

$$(iii): -\frac{m B \varepsilon^b}{\sqrt{\varepsilon}} \frac{d\check{\mathcal{M}}_1^{\mathcal{W}}}{dy}, \quad (C26)$$

while term (iv) is zero. Thus, the upscaled energy equation describing how temperature evolves in the outer phase does not directly contain any information related to the inner flow temperature, and the resulting model is termed decoupled:

$$\varepsilon^{-p} \frac{\partial \langle \vartheta_2 \rangle}{\partial \tau} + \frac{\varepsilon^{-p}}{\sqrt{\varepsilon}} \langle u_2 - V \rangle \frac{\partial \langle \vartheta_2 \rangle}{\partial \xi} = \left[1 + \varepsilon^{-2p} D_2 \right] \frac{\partial^2 \langle \vartheta_2 \rangle}{\partial \xi^2} + \frac{Q}{\beta} \varepsilon^{f-1} + \frac{B}{\beta} \varepsilon^{b-1} W_2^*, \quad (C27)$$

where the integral contributions to effective diffusivity and viscous dissipation have been denoted for brevity as follows:

$$D_2 = -\langle \mathcal{I}_2 \rangle, \quad W_2^* = W_2 + m \frac{d\check{\mathcal{M}}_1^{\mathcal{W}}}{dy}, \quad W_2 = \int_{1-\beta}^1 \left(\frac{du_2(y^*)}{dy^*} \right)^2 dy^*. \quad (C28a-c)$$

The full expressions of the above effective coefficients are given later on in [Appendix C.6](#), see [\(C40b\)](#) and [\(C46c\)](#). After multiplying both sides of [\(C27\)](#) by ε^p , reintroducing dimensionless groups according to [\(3.2\)](#) and scaling back the spatial variable $\xi \mapsto z$ via [\(3.1\)](#), we obtain a one-dimensional ADHT equation for the averaged temperature of the outer phase whose full expression is given in [\(3.10\)](#).

A special scenario is represented for the case where $k = 1/2$: the $\mathcal{O}(\sqrt{\varepsilon})$ solution for the inner phase [\(C12\)](#) yields the following closure:

$$(iv): \quad K \varepsilon^{-p+a} \frac{\check{\mathcal{M}}_1}{dy} \frac{\partial \langle \vartheta_1 \rangle}{\partial \xi}, \quad (C29)$$

while term (iii) is the same as for the case where $k = 0$. Using [\(C29\)](#) leads to a single-equation model of the form of [\(C25\)](#) but requires an *a priori* knowledge of the axial gradient of the temperature of the inner phase $\partial \langle \vartheta_1 \rangle / \partial \xi$, reducing to the ADHT equation valid for $k \geq 1$ only if the inner phase has a uniform temperature. Thus, this case will not be addressed in this work.

C.5.2 Coupled model

When $k = 0$, the thermal conductivities of the two phases are comparable and the upscaled model consists of two coupled equations in the unknowns $\langle \vartheta_j \rangle$. Specifically, the boundary terms (iii), (iv) in [\(C25\)](#) are closed using [\(C4\)](#), obtaining

$$(iii): \quad K \frac{\partial \vartheta_1^{(1)}}{\partial y} \Big|_{y=1-\beta} = K \varepsilon^{-p+a} \frac{\check{\mathcal{M}}_1}{dy} \frac{\partial \langle \vartheta_1 \rangle}{\partial \xi} - m B \varepsilon^{b-1} \frac{d\check{\mathcal{M}}_1^{\mathcal{W}}}{dy}, \quad (C30a)$$

$$(iv): \quad K \frac{\partial \vartheta_1^{(2)}}{\partial y} \Big|_{y=1-\beta} = K \left[\varepsilon^{-p+a} \check{\varphi}_1 \frac{\partial \langle \vartheta_1 \rangle}{\partial \tau} + \underbrace{\left(\varepsilon^{2(-p+a)} \check{\psi}_1 - \check{\varphi}_1 \right)}_{(c_{iii})} \frac{\partial^2 \langle \vartheta_1 \rangle}{\partial \xi^2} \right], \quad (C30b)$$

where we used [\(C12\)](#) and [\(C19\)](#) with $j = 1$. Using [\(C30b\)](#) in [\(C25\)](#) introduces a coupled-derivative term, i.e. $\propto \varepsilon^{-p+a} \partial \langle \vartheta_1 \rangle / \partial \tau$, giving an upscaled equation including the time derivative of $\langle \vartheta_1 \rangle$ and $\langle \vartheta_2 \rangle$ simultaneously. To avoid this issue, we can derive the $\mathcal{O}(\varepsilon)$ solution of [\(C23\)](#) for the inner phase, $\vartheta_1^{(2)}$, and, evaluating it at the interface, we get an expression for replacing the time derivative in [\(C30b\)](#):

$$\varepsilon^{-p+a} \frac{\partial \langle \vartheta_1 \rangle}{\partial \tau} = \frac{1}{\check{\varphi}_1} \left\{ \vartheta_1^{(2)} \Big|_{y=1-\beta} - \underbrace{\left[\varepsilon^{2(-p+a)} \check{\psi}_1 - \check{\varphi}_1 \right]}_{(c_{ii})} \frac{\partial^2 \langle \vartheta_1 \rangle}{\partial \xi^2} \right\}. \quad (C31)$$

The last step is to enforce the continuity of temperature at the interface [\(C5\)](#) asymptotically, replacing the interfacial temperature in [\(C31\)](#) as the following power series:

$$\begin{aligned} & \vartheta_1^{(2)} \Big|_{y=1-\beta} \\ &= \frac{1}{\varepsilon} \left\{ \underbrace{(\langle \vartheta_2 \rangle - \langle \vartheta_1 \rangle)}_{(a)} + \underbrace{\sqrt{\varepsilon} \left(\vartheta_2^{(1)} - \vartheta_1^{(1)} \right) \Big|_{y=1-\beta}}_{(b)} + \underbrace{\varepsilon \vartheta_2^{(2)} \Big|_{y=1-\beta}}_{(c_i)} + \mathcal{O}(\varepsilon \sqrt{\varepsilon}) \right\}. \end{aligned} \quad (C32)$$

This procedure shows that three different mechanisms are needed to adequately capture the thermal interplay between the phases, namely a storage-type term (a), coupled advective contributions due to axial gradients, term (b), and time-dependent/diffusive contributions, term (c). Specifically, term (b) is obtained as the difference between the $\mathcal{O}(\sqrt{\varepsilon})$ solutions, see (C12), (C15), evaluated at the interface, whereas term (c) can be detailed as follows: term (c_i) follows from (C23) (with $j = 2$), thus containing a time-derivative ($\propto \partial \langle \vartheta_2 \rangle / \partial \tau$) and a direct-diffusive term ($\propto \partial^2 \langle \vartheta_2 \rangle / \partial \xi^2$), while terms (c_{ii}) and (c_{iii}) results in cross-diffusion ($\propto \partial^2 \langle \vartheta_1 \rangle / \partial \xi^2$). Note that existing transport models between coupled layers have postulated the coupling only in the form of a storage term (Reichert & Wanner 1991; Kazezyılmaz-Alhan 2008) or, additionally, an advective contribution due to axial gradients (Ling *et al.* 2016, 2021, 2024), but our analysis highlights that also cross-diffusive terms emerge naturally from the derivation of the model via asymptotic expansions and should be accounted for to preserve the model consistency.

The upscaled energy equation for the inner phase can be obtained following the same procedure shown in § C.5 with $j = 1$ and using the symmetry condition (C6) at the channel centre line (the detailed derivation is not shown here for the sake of brevity); this gives

$$\begin{aligned} & \varepsilon^{-p+a} \left[\varepsilon \frac{\partial \langle \vartheta_1 \rangle}{\partial \tau} + \sqrt{\varepsilon} \langle u_1 - V \rangle \frac{\partial \langle \vartheta_1 \rangle}{\partial \xi} + \varepsilon \overbrace{\left\langle (u_1 - V) \frac{\partial \vartheta_1^{(1)}}{\partial \xi} \right\rangle}^{(i)} + \varepsilon \sqrt{\varepsilon} \overbrace{\left\langle (u_1 - V) \frac{\partial \vartheta_1^{(2)}}{\partial \xi} \right\rangle}^{(ii)} \right] \\ &= \varepsilon \frac{\partial^2 \langle \vartheta_1 \rangle}{\partial \xi^2} + \frac{\sqrt{\varepsilon}}{1-\beta} \left(\underbrace{\frac{\partial \vartheta_2^{(1)}}{\partial y} \Big|_{y=1-\beta}}_{(iii)} - 0 \right) + \frac{\varepsilon}{1-\beta} \left(\underbrace{\frac{\partial \vartheta_2^{(2)}}{\partial y} \Big|_{y=1-\beta}}_{(iv)} - 0 \right) \\ &+ \frac{m B \varepsilon^{b-k}}{K (1-\beta)} \int_0^{1-\beta} \left(\frac{du_1(y^*)}{dy^*} \right)^2 dy^*, \end{aligned} \quad (C33)$$

where, in analogy with (C28), we can define the following effective coefficients for diffusion and viscous dissipation – the full expressions are given in Appendix C.6, see (C40a) and (C46c):

$$D_1 = - \langle \mathcal{I}_1 \rangle, \quad W_1 = \int_0^{1-\beta} \left(\frac{du_1(y^*)}{dy^*} \right)^2 dy^*. \quad (C34a-b)$$

The closure of two advective terms (i), (ii) and two boundary terms (iii), i(v) in (C33) is obtained as before: regardless of the value of k , by means of (C17) and (C20), we find that term (i) is equal to $\varepsilon^{-p+a} \langle \mathcal{I}_1 \rangle \partial^2 \langle \vartheta_1 \rangle / \partial \xi^2$, while term (ii) is zero by averaging equation (C3d) with $j = 1$. Again, the closure of terms (iii) and (iv) is accomplished via the boundary conditions at the interface: for term (iii), it is sufficient to enforce at $\mathcal{O}(\sqrt{\varepsilon})$ the continuity of thermal fluxes, see (C4), while for term (iv), this condition at $\mathcal{O}(\varepsilon)$ has to be combined with the continuity of temperatures written in asymptotic terms, in analogy with (C32):

$$\begin{aligned} & \vartheta_2^{(2)} \Big|_{y=1-\beta} \\ &= \frac{1}{\varepsilon} \left\{ (\langle \vartheta_1 \rangle - \langle \vartheta_2 \rangle) + \sqrt{\varepsilon} \left(\vartheta_1^{(1)} - \vartheta_2^{(1)} \right) \Big|_{y=1-\beta} + \varepsilon \vartheta_1^{(2)} \Big|_{y=1-\beta} + \mathcal{O}(\varepsilon\sqrt{\varepsilon}) \right\}. \end{aligned} \quad (\text{C35})$$

Finally, we obtain that heat-transfer between two flowing phases is described by the following two coupled ADHT equations:

$$\left\{ \begin{aligned} & \varepsilon^{-p+a} t_1^* \frac{\partial \langle \vartheta_1 \rangle}{\partial \tau} + \frac{1}{\sqrt{\varepsilon}} \varepsilon^{-p+a} a_{11}^* \frac{\partial \langle \vartheta_1 \rangle}{\partial \xi} + \frac{1}{\sqrt{\varepsilon}} \varepsilon^{-p} a_{12}^* \frac{\partial \langle \vartheta_2 \rangle}{\partial \xi} = \\ & d_{11}^* \frac{\partial^2 \langle \vartheta_1 \rangle}{\partial \xi^2} + \frac{1}{\varepsilon^{2p}} d_{12}^* \frac{\partial^2 \langle \vartheta_2 \rangle}{\partial \xi^2} + \frac{1}{\varepsilon} g_1^* Q \varepsilon^f + \frac{1}{\varepsilon} w_1^* B \varepsilon^b - \frac{1}{\varepsilon} e_1^* (\langle \vartheta_1 \rangle - \langle \vartheta_2 \rangle), \\ & \varepsilon^{-p} t_2^* \frac{\partial \langle \vartheta_2 \rangle}{\partial \tau} + \frac{1}{\sqrt{\varepsilon}} \varepsilon^{-p} a_{22}^* \frac{\partial \langle \vartheta_2 \rangle}{\partial \xi} + \frac{1}{\sqrt{\varepsilon}} \varepsilon^{-p+a} a_{21}^* \frac{\partial \langle \vartheta_1 \rangle}{\partial \xi} = \\ & d_{22}^* \frac{\partial^2 \langle \vartheta_2 \rangle}{\partial \xi^2} + \frac{\varepsilon^{2a}}{\varepsilon^{2p}} d_{21}^* \frac{\partial^2 \langle \vartheta_1 \rangle}{\partial \xi^2} + \frac{1}{\varepsilon} g_2^* Q \varepsilon^f + \frac{1}{\varepsilon} w_2^* B \varepsilon^b + \frac{1}{\varepsilon} e_2^* (\langle \vartheta_1 \rangle - \langle \vartheta_2 \rangle), \end{aligned} \right. \quad (\text{C36a,b})$$

where the effective coefficients are defined as follows and their explicit expressions are given in Appendix C.6:

$$t_1^* = 1 - \frac{\check{\vartheta}_2 \check{\varphi}_1}{K (1 - \beta) \check{\varphi}_2}, \quad t_2^* = 1 + \frac{K \check{\varphi}_1 \check{\varphi}_2}{\beta \check{\varphi}_1}, \quad (\text{C37a})$$

$$a_{11}^* = \langle u_1 - V \rangle + \omega_{11}^*, \quad a_{22}^* = \langle u_2 - V \rangle + \omega_{22}^*, \quad (\text{C37b})$$

$$\omega_{11}^* = -\frac{\check{\vartheta}_2 \check{\mathcal{M}}_1}{K (1 - \beta) \check{\varphi}_2}, \quad \omega_{22}^* = \frac{K \check{\varphi}_1 \check{\mathcal{M}}_2}{\beta \check{\varphi}_1}, \quad (\text{C37c})$$

$$a_{12}^* = \frac{1}{K (1 - \beta)} \left(\frac{\check{\vartheta}_2 \check{\mathcal{M}}_2}{\check{\varphi}_2} - \frac{d\check{\mathcal{M}}_2}{dy} \right), \quad a_{21}^* = \frac{K}{\beta} \left(\frac{d\check{\mathcal{M}}_1}{dy} - \frac{\check{\varphi}_1 \check{\mathcal{M}}_1}{\check{\varphi}_1} \right), \quad (\text{C37d})$$

$$d_{11}^* = t_1^* + \frac{\varepsilon^{2a}}{\varepsilon^{2p}} (D_1 + \delta_{11}^*), \quad d_{22}^* = t_2^* + \frac{1}{\varepsilon^{2p}} (D_2 + \delta_{22}^*), \quad (\text{C37e})$$

$$\delta_{11}^* = \frac{\check{\varphi}_2 \check{\Psi}_1}{K (1 - \beta) \check{\Phi}_2}, \quad \delta_{22}^* = -\frac{K \check{\varphi}_1 \check{\Psi}_2}{\beta \check{\Phi}_1}, \quad (\text{C37f})$$

$$d_{12}^* = \frac{1}{K (1 - \beta)} \left(\check{\Psi}_2 - \frac{\check{\varphi}_2 \check{\Psi}_2}{\check{\Phi}_2} \right), \quad d_{21}^* = \frac{K}{\beta} \left(\frac{\check{\varphi}_1 \check{\Psi}_1}{\check{\Phi}_1} - \check{\Psi}_1 \right), \quad (\text{C37g})$$

$$g_1^* = \frac{1}{K (1 - \beta)} \left(\frac{d\check{\mathcal{M}}_2^q}{dy} - \frac{\check{\varphi}_2 \check{\mathcal{M}}_2^q}{\check{\Phi}_2} \right), \quad g_2^* = \frac{1}{\beta} \left(1 - \frac{K \check{\varphi}_1 \check{\mathcal{M}}_2^q}{\check{\Phi}_1} \right), \quad (\text{C37h})$$

$$w_1^* = \frac{1}{K (1 - \beta)} \left[m \left(W_1 - \frac{\check{\varphi}_2 \check{\mathcal{M}}_1^{\mathcal{W}}}{K \check{\Phi}_2} \right) - \frac{d\check{\mathcal{M}}_2^{\mathcal{W}}}{dy} + \frac{\check{\varphi}_2 \check{\mathcal{M}}_2^{\mathcal{W}}}{\check{\Phi}_2} \right],$$

$$w_2^* = \frac{1}{\beta} \left[W_2 + m \frac{d\check{\mathcal{M}}_1^{\mathcal{W}}}{dy} + \frac{\check{\varphi}_1}{\check{\Phi}_1} \left(K \check{\mathcal{M}}_2^{\mathcal{W}} - m \check{\mathcal{M}}_1^{\mathcal{W}} \right) \right], \quad (\text{C37i})$$

$$e_1^* = -\frac{\check{\varphi}_2}{K (1 - \beta) \check{\Phi}_2}, \quad e_2^* = \frac{K \check{\varphi}_1}{\beta \check{\Phi}_1}. \quad (\text{C37j})$$

Reintroducing dimensionless groups and scaling back the spatial variable, we obtain the coupled model in its final form, see (3.8), (3.9).

Note that the ADHT equations of the coupled model (C37) reduce to the decoupled model in the limit of $K \rightarrow 0$. This requires that in w_2^* , we artificially remove the contribution originated by the dissipative function within the inner stream $\check{\mathcal{M}}_1^{\mathcal{W}}$, which is not part of the decoupled formulation.

C.6 Model coefficients

The full expressions of the effective coefficients (C37) are reported below, while their physical interpretation will be given in § 4.

In particular, transient effects are taken into account via the following coefficients:

$$t_1^* = 1 + \frac{1 - \beta}{K \beta}, \quad t_2^* = 1 + \frac{K \beta}{1 - \beta}. \quad (\text{C38})$$

Advective coefficients appearing in (3.8), (3.9) read

$$\omega_{11}^* = \frac{3 \Lambda (1 - \beta) [5 m \beta (2 - \beta) + 4 (1 - \beta)^2]}{5 K \beta} - \frac{1 - \beta}{K \beta} V, \quad (\text{C39a})$$

$$\omega_{22}^* = \frac{3 K \Lambda m \beta^2 (15 - 4 \beta)}{20 (1 - \beta)} - \frac{K \beta}{1 - \beta} V, \quad (\text{C39b})$$

$$a_{12}^* = \frac{\Lambda m \beta^2 (15 - 8 \beta)}{20 K (1 - \beta)}, \quad a_{21}^* = -\frac{2 \Lambda K (1 - \beta)^3}{5 \beta}; \quad (\text{C39c})$$

while effective diffusion is described by the following set of coefficients:

$$D_1 = -\frac{2 \Lambda (1-\beta)^4}{15} V + \frac{2 \Lambda^2 (1-\beta)^4 [7 m \beta (2-\beta) + 6 (1-\beta)^2]}{35}, \quad (\text{C40a})$$

$$D_2 = \frac{\Lambda m \beta^3 (15-8\beta)}{60} V - \frac{\Lambda^2 m^2 \beta^4 (8 \beta^2 - 49 \beta + 63)}{140}; \quad (\text{C40b})$$

$$\begin{aligned} \delta_{11}^* = & \frac{(1-\beta)^3}{15 K \beta} V^2 + \frac{2 \Lambda (1-\beta)^3 [(1-m) \beta (2-\beta) - 1]}{5 K \beta} V + \\ & + \frac{\Lambda^2 (1-\beta)^3 [105 m^2 \beta^2 (2-\beta)^2 + 210 m \beta (2-\beta) (1-\beta)^2 + 104 (1-\beta)^4]}{175 K \beta}, \end{aligned} \quad (\text{C41a})$$

$$\delta_{22}^* = \frac{K \beta^3}{15 (1-\beta)} V^2 + \frac{K \Lambda m \beta^4}{8 (1-\beta)} V - \frac{K \Lambda^2 m^2 \beta^5 (32 \beta^2 - 105 \beta - 270)}{5600 (1-\beta)}, \quad (\text{C41b})$$

$$d_{12}^* = -\frac{\beta^3}{15 K (1-\beta)} V^2 + \frac{\Lambda m \beta^4 (45-16 \beta)}{120 K (1-\beta)} V - \frac{\Lambda^2 m^2 \beta^5 (288 \beta^2 - 1855 \beta + 2790)}{5600 K (1-\beta)}, \quad (\text{C42a})$$

$$\begin{aligned} d_{21}^* = & -\frac{K (1-\beta)^3}{15 \beta} V^2 + \frac{2 K \Lambda (1-\beta)^3 [3 m \beta (2-\beta) + 2 (1-\beta)^2]}{15 \beta} V + \\ & - \frac{K \Lambda^2 (1-\beta)^3 [105 m^2 \beta^2 (2-\beta)^2 + 140 m \beta (2-\beta) (1-\beta)^2 + 44 (1-\beta)^4]}{175 \beta}. \end{aligned} \quad (\text{C42b})$$

Among the effective coefficients, see (C37), the speed of the reference frame V only impacts effective diffusion and direct-coupling advection a_{jj}^* . Specifically, shifting from moving to the fixed reference frame via the inverse transformation $x = z + V \tau$ has no net impact on diffusion, i.e. $\partial^2 \langle \vartheta_j \rangle / \partial z^2 \equiv \partial^2 \langle \vartheta_j \rangle / \partial x^2$, but offers significant advantages in terms of the physical interpretation of the advective coefficients. In fact, each of these could be recast as the phase-averaged velocity augmented by a specific factor, namely $U_j^* = a_{jj}^* + V t_{jj}^* = U_j - V + \omega_{jj}^* + V t_{jj}^* = U_j + \Delta U_j^*$, where

$$\Delta U_1^* = \frac{3 \Lambda (1-\beta) [5 m \beta (2-\beta) + 4 (1-\beta)^2]}{5 K \beta}, \quad \Delta U_2^* = \frac{3 K \Lambda m \beta^2 (15-4 \beta)}{20 (1-\beta)}. \quad (\text{C43})$$

Interestingly, the final expressions for ΔU_j^* defined in (C43) are independent of V , implying that effective extra-advection is invariant with respect to the choice of the reference frame, i.e. $\omega_{jj}^*|_{V=0} \equiv \Delta U_j^*$.

The effective coefficients in front of the source terms in (C36) read

$$g_1^* = -\frac{1}{2 K (1-\beta)}, \quad g_2^* = \frac{1}{\beta} + \frac{3 K}{2 (1-\beta)}; \quad (\text{C44})$$

whereas those concerning storage are

$$e_1^* = \frac{3}{K \beta (1 - \beta)}, \quad e_2^* = \frac{3 K}{\beta (1 - \beta)}. \quad (\text{C45})$$

Finally, the analytical expressions for the effective coefficients related to viscous dissipation are

$$w_1^* = \frac{3 \Lambda^2 m \{4 (1 - \beta)^3 [5 K \beta + 3 (1 - \beta)] - K m \beta^3 (15 - 8 \beta)\}}{5 K^2 \beta (1 - \beta)}, \quad (\text{C46a})$$

$$w_2^* = \frac{3 \Lambda^2 m \{m [3 \beta^2 K (4 \beta^2 - 15 \beta + 20) + 20 \beta (1 - \beta) (\beta^2 - 3 \beta + 3)] + 8 (1 - \beta)^4\}}{5 \beta (1 - \beta)}, \quad (\text{C46b})$$

$$W_1 = 12 \Lambda^2 (1 - \beta)^3, \quad W_2 = 12 \Lambda^2 m^2 \beta (\beta^2 - 3 \beta + 3), \quad W_2^* = 6 \Lambda m. \quad (\text{C46c})$$

Appendix D. Model validation

To check the accuracy of the ADHT equations (3.8), (3.9), we present the validation against the steady-state solution of the problem. At the steady-state, in fact, the temperature gradients along the channel length are uniform and diffusion becomes negligible so that (in the absolute reference frame, $V = 0$)

$$\frac{d\vartheta_1}{dx} = \frac{d\vartheta_2}{dx} = J. \quad (\text{D1})$$

Consequently, the energy balances (2.13) reduce to

$$\varepsilon \mathcal{A} Pe J u_1 = \frac{d^2 \vartheta_1}{dy^2} + \frac{m}{\mathcal{K}} Br \left(\frac{du_1}{dy} \right)^2, \quad y \in [0; 1 - \beta], \quad (\text{D2a})$$

$$\varepsilon Pe J u_2 = \frac{d^2 \vartheta_2}{dy^2} + Br \left(\frac{du_2}{dy} \right)^2, \quad y \in [1 - \beta; 1], \quad (\text{D2b})$$

subjected to boundary conditions (2.16). Integrating (D2) gives

$$\frac{d\vartheta_1(y)}{dy} = \varepsilon \mathcal{A} Pe J \int u_1(y^*) dy^* - \frac{m}{\mathcal{K}} Br \int \left(\frac{du_1(y^*)}{dy^*} \right)^2 dy^* + C_1, \quad (\text{D3a})$$

$$\frac{d\vartheta_2(y)}{dy} = \varepsilon Pe J \int u_2(y^*) dy^* - Br \int \left(\frac{du_2(y^*)}{dy^*} \right)^2 dy^* + C_2, \quad (\text{D3b})$$

where C_1 and C_2 are integration constants. The value of C_1 is determined via the symmetry boundary condition (2.16d), whereas the boundary condition at the wall (2.16a) allows to express the other constant of integration as a function of the steady-state axial gradient J and the external heat flux q_w , i.e. implicitly $C_2 = C_2(q_w, J, \dots)$. The expression for J can be found using the thermal flux continuity condition at the interface (2.16b) and solving for $J = J(q_w, \dots)$. These calculations have been carried out using a

symbolic tool and are not reported here for the sake of brevity. It can be easily checked that J is equivalent to the slope of the steady-state solution, M , of the leading-order model given in (4.21), obtaining that $M|_{V=0} = \sqrt{\varepsilon} J$.

Finally, aiming at validating the steady-state Nusselt number with the analytical solution of (D2), we can first integrate (D3) to obtain the steady-state temperature profile. Then, the continuity of temperatures at the interface (2.16c) is used to eliminate one of the two additional integration constants. It is convenient to leave the constant for the outer phase temperature undetermined since it does not affect the definition of the Nusselt number (3.19): at the denominator of the local heat-transfer coefficient, only the difference between the wall and the bulk temperature appears. Doing so, the expressions of the steady-state Nusselt number obtained using the steady-state analytical solution and the upscaled model match exactly, confirming the consistency of our model.

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