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A NEW BASIS FOR DISCRETE ANALYTIC POLYNOMIALS

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Abstract

A new basis $\{\pi_k(z)\}_{k=0}^{\infty}$ for discrete analytic polynomials is presented for which the series $\Sigma_{k=0}^{\infty} a_k \pi_k(z)$ converges absolutely to a discrete analytic function in the upper right quarter lattice whenever $\overline{\lim} | a_k |^{1/k} = 0$.

Introduction

Let Z be the group of integers and consider functions

$$f: Z \times Z \rightarrow \mathbf{C}$$

such that

$$f(x, y) + if(x+1, y) - f(x+1, y+1) - if(x, y+1) = 0 \tag{1.1}$$

for every $(x, y) \in Z \times Z$. Such functions are termed *discrete entire*. If (1.1) holds only for $(x, y) \in G$, $G \subset Z \times Z$, then we say that f is *discrete analytic* in G.

Discrete analytic functions were introduced by Ferrand (1944) and the theory was developed by Duffin (1956) and others.

Duffin (1956) introduced the following basis for discrete analytic polynomials

$$\rho_k(z) = \frac{d^k}{dt^k} \left\{ \left(\frac{2+t}{2-t} \right)^x \left(\frac{2+it}{2-it} \right)^y \right\} \Bigg|_{t=0} \tag{1.2}$$

($z = x + iy$), which he called *pseudo-powers*.

Each $\rho_k(z)$ is a discrete entire function and a polynomial of degree k in (x, y). Duffin (1956) showed that every discrete analytic polynomial can be expressed as a linear combination of these pseudo-powers.

Duffin and Peterson (1968) introduced an analogue of the McClaurin series in terms of these pseudo-powers. However, their analogue has the disadvantage that the convergence of $\Sigma_0^\infty a_n\xi^n$ on **C** does not ensure the convergence of $\Sigma_0^\infty a_n\rho_n(z)$ on $Z\times Z$. In fact they showed that $\Sigma_0^\infty a_n\rho_n(z)$ converges on the whole lattice $Z\times Z$ only if

$$\overline{\lim}\,(|a_n|\,n!)^{1/n}<2.$$

In Section 2 other "reasonable" bases for discrete analytic polynomials will be considered. These will be called *systems of pseudo-powers*, and it will be shown that the above drawback of Duffin's basis $\{\rho_n(z)\}$ as regards the convergence of $\Sigma\, a_n\rho_n(z)$ cannot be removed by using other systems of pseudo-powers.

On the other hand, we shall construct a system of pseudo-powers $\{\pi_k(z)\}_0^\infty$ such that $\Sigma_0^\infty a_k\pi_k(z)$ converges absolutely on the quarter lattice $Z^+\times Z^+=\{(x+iy);\, x \text{ and } y \text{ integers}, x\geqq 0, y\geqq 0\}$ whenever $\Sigma_0^\infty a_k\xi^k$ converges on the entire plane. (The divergence of $\Sigma_0^\infty(2^n/n!)\rho_n(1,0)$ shows that this property is not enjoyed by the Duffin–Peterson series.)

In Section 3 we shall consider the existence and uniqueness of the expansion $\Sigma_0^\infty a_k\pi_k(z)$. The discrete analogue of 'multiplication by z' corresponding to the above basis will also be dealt with.

In Section 4, we discuss the lattice $Z_h^+\times Z_h^+$ where $Z_h^+=hZ^+$, $h>0$ and show that if $\{\pi_k^h(z)\}_0^\infty$ is the corresponding basis then

$$\sum_0^\infty a_k\pi_k^h(z)\to\sum_0^\infty a_kz^k$$

when $h\downarrow 0$ along a sequence for which $z\in Z_h\times Z_h$, provided $\Sigma_0^\infty a_k\xi^k$ is an entire function of exponential type.

The analogous problem of representing monodiffric functions (that is functions satisfying

$$(i-1)f(x,y)-if(x+1,y)+f(x,y+1)\equiv 0)$$

by a series of polynomials was considered by Atadžanov (1974).

2. The new basis

DEFINITION. *A basis $\{p_n(z)\}_0^\infty$ for the discrete analytic polynomials is called a system of pseudo-powers if the following properties are satisfied:*

(A1) $p_n(0)=0$ *for every* $n>0$

(A2) $\{p_n(z)\}$ *satisfies the binomial identity*

$$p_n(z_1+z_2)=\sum_{k=0}^n\binom{n}{k}p_k(z_1)p_{n-k}(z_2)$$

(A3) *$p_0 \equiv 1$ and for $n \geqq 0$ $p_n(z) = z^n + \tilde{p}_{n-1}(x, y)$ where $\tilde{p}_{n-1}$ is a polynomial of degree $\leqq n-1$.*

It is readily checked that Duffin's basis $\{\rho_n(z)\}$ constitutes a system of pseudo-powers. On the other hand, Duffin's basis fails to satisfy the following:

(*) $\sum_0^\infty a_n p_n(z)$ converges absolutely for every $z \in Z \times Z$

if $\sum_0^\infty a_n \xi^n$ converges in the whole ξ-plane.

One may ask: Does there exist a system of pseudo-powers satisfying (*)? That no such system exists follows from the next lemma.

LEMMA 2.1. *Let $\{p_k\}$ be any system of pseudo-powers. Then there exists a point z_0 in the half lattice $\{x + iy, y \geqq 0\}$ and a complex number ζ_0 such that*

$$\sum_0^\infty \frac{\zeta_0^k}{k!} p_k(z_0) \quad \textit{fails to converge absolutely.}$$

PROOF. Suppose that the statement is false, i.e., there exists a system of pseudo-powers $\{p_k\}$ such that

$$e(\zeta; z) = \sum_{k=0}^\infty \frac{\zeta^k}{k!} p_k(z)$$

converges absolutely for every point in the half lattice and for every complex number ζ. Then, for every such z, $e(\zeta, z)$ is an entire function in ζ and by (A2)

$$\begin{aligned} e(\zeta; z_1)e(\zeta; z_2) &= \left(\sum_0^\infty \frac{\zeta^k}{k!} p_k(z_1)\right)\left(\sum_0^\infty \frac{\zeta^r}{r!} p_r(z_2)\right) \\ &= \sum_0^\infty \frac{\zeta^n}{n!}\left[\sum_0^n \binom{n}{k} p_k(z_1) p_{n-k}(z_2)\right] \\ &= \sum_0^\infty \frac{\zeta^n}{n!} p_n(z_1 + z_2) = e(\zeta; z_1 + z_2). \end{aligned}$$

Thus $e(\zeta; x + iy) = f(\zeta)^x g(\zeta)^y$ where $f(\zeta) = e(\zeta; 1)$, $g(\zeta) = e(\zeta; i)$.

Since $e(\zeta; z)$ is discrete analytic in the upper half lattice (1.1) must be satisfied there:

$$f(\zeta)^x g(\zeta)^y \{1 + if(\zeta) - f(\zeta)g(\zeta) - ig(\zeta)\} = 0.$$

Thus

$$g(\zeta) = \frac{1 + if(\zeta)}{f(\zeta) + i}$$

and

$$e(\zeta; x+iy) = f(\zeta)^x \left(\frac{1+if(\zeta)}{f(\zeta)+i}\right)^y.$$

Since $e(\zeta; z)$ is entire in ζ for each fixed z in the half lattice and in particular for $z = 1, -1, i$ we see that $f(\zeta)$, $1/f(\zeta)$ and $(1+if(\zeta))/(f(\zeta)+i)$ are entire. But this implies that $f(\zeta)$ is entire and excludes the values 0 and $-i$. By the "little" Picard theorem [Rudin (1966), p. 324] this is too much to ask from a non-constant entire function. Evidently $f(\zeta)$ cannot be constant and so we arrive at a contradiction and the lemma is proved.

We saw that there is no system of pseudo-powers satisfying ($*$). The next theorem will demonstrate a system of pseudo-powers satisfying the following weaker property.

(A4) $\Sigma_0^\infty a_n p_n(z)$ converges absolutely for every

$$z \in Z^+ \times Z^+ = \{x+iy; x \geqq 0, y \geqq 0\} \quad \text{if} \quad \sum_0^\infty a_n \xi^n$$

converges in the whole ξ-plane.

The divergence of $\Sigma(2^n/n!)\rho_n(1)$ shows that Duffin's basis does not satisfy (A4).

THEOREM 2.2. *The sequence of functions*

$$\pi_k(x,y) = \frac{d^k}{d\zeta^k}\{[(1+i)e^{\zeta/(i+1)} - i]^x[(1-i)e^{-\zeta/(1+i)} + i]^y\}\Big|_{\zeta=0} \tag{2.1}$$

$$k = 0, 1, 2, \cdots$$

constitutes a system of pseudo-powers satisfying (A4).

PROOF. The discrete analyticity of $\pi_k(x,y)$ is readily checked. (A1) is trivial, while (A2) follows from Leibnitz' formula. Also, by a straightforward computation

$$\pi_{k+1}(x,y) = \frac{1}{1+i}\{(x-y)\pi_k(x,y) + ix\pi_k(x-1,y) + iy\pi_k(x,y-1)\}. \tag{2.2}$$

Since $\pi_0(x,y) \equiv 1$ it follows by induction that each $\pi_k(x,y)$ is a polynomial of degree k and that (A3) holds. Since Duffin (1956) showed that the dimension of the space of discrete analytic polynomials of degree $\leqq k$ is $k+1$, it follows that $\{\pi_r\}_0^k$ is a basis for the discrete analytic polynomials of degree $\leqq k$ and consequently that $\{\pi_k\}_0^\infty$ is a basis for the discrete analytic polynomials. Thus $\{\pi_k\}$ is a system of pseudo-powers.

Now, let us note that for a fixed $z = x + iy \in Z^+ \times Z^+$

$$e(\zeta ; x + iy) = \sum_0^\infty \pi_k(x, y)\frac{\zeta^k}{k!} = [(1+i)e^{\zeta/(i+1)} - i]^x[(1-i)e^{-\zeta/(1+i)} + i]^y.$$

Since x and y are non-negative integers, the right hand side is an entire function of exponential type and the Taylor coefficients being $\pi_k(x, y)/k!$ you have (Boas (1954), p. 11) that there exist constants C and T (depending on (x, y)) such that

$$|\pi_k(x, y)| \leqq CT^k.$$

Thus $\Sigma_0^\infty a_k\pi_k(x, y)$ converges absolutely whenever $\overline{\lim}|a_k|^{1/k} = 0$, since $\Sigma_0^\infty a_k T^k$ does. This holds for every $(x, y) \in Z^+ \times Z^+$ and it follows that $\{\pi_k\}$ is a system of pseudo-powers satisfying (A4).

By Theorem (2.2) it follows that whenever $\Sigma_0^\infty a_k\xi^k$ is an entire function, i.e., whenever $\overline{\lim}|a_k|^{1/k} = 0$, then $\Sigma_0^\infty a_k\pi_k(z)$ converges to a discrete analytic function in $Z^+ \times Z^+$ (substitute in (1.1) and rearrange terms, using the fact that each $\pi_k(z)$ is discrete analytic).

Let $\mathscr{A}$ be the algebra of entire functions and let $\mathscr{D}$ be the set of discrete analytic functions on $Z^+ \times Z^+$. Define a mapping

$$T : \mathscr{A} \to \mathscr{D}$$

by

$$T\left(\sum_0^\infty a_n\xi^n\right) = \sum_0^\infty a_n\pi_n(z).$$

Let $\mathscr{F} \subset \mathscr{D}$ be the range of T. $\mathscr{F}$ can be made into an algebra by requiring T to be a homomorphism:

$$\left(\sum_0^\infty a_k\pi_k(z)\right)\left(\sum_0^\infty b_k\pi_k(z)\right) = \sum_{n=0}^\infty\left(\sum_{k=0}^n a_kb_{n-k}\right)\pi_n(z).$$

Thus in our class $\mathscr{F}$, multiplication is defined for every pair $f, g \in \mathscr{F}$. This is an improvement on the multiplication in the Duffin–Peterson class,

$$\mathscr{F}_{DP} = \left\{\sum_0^\infty a_n\rho_n(z);\ \overline{\lim}(|a_n|\,n!)^{1/n} < 2\right\}$$

which is only defined on a subset of $\mathscr{F}_{DP} \times \mathscr{F}_{DP}$. In particular $\exp f$ is well defined in our class:

$$\exp\left(\sum_0^\infty a_k\pi_k(z)\right) = T\left(\exp\left(\sum_0^\infty a_k\xi^k\right)\right).$$

3. Existence and uniqueness of Taylor expansion

Formula (2.2) motivates the following analogue for the continuous "multiplication by z"

$$(3.1)\qquad \mathfrak{z}f(x,y)=\frac{1}{1+i}\{(x-y)f(x,y)+ixf(x-1,y)+iyf(x,y-1)\}.$$

It is readily checked that if f is discrete analytic, then so is $\mathfrak{z}f$ and, by (2.2)

$$\mathfrak{z}\pi_k=\pi_{k+1}\qquad \mathfrak{z}e(\xi\,;x+iy)=\frac{d}{d\xi}e(\xi\,;x+iy).$$

Let us restrict attention to $\mathscr{D}$, the class of discrete analytic functions on $Z^+\times Z^+$. It was shown in Zeilberger (to appear) that each $f\in\mathscr{D}$ is uniquely determined by the pair of formal power series (ϕ_f,ψ_f) where

$$\phi_f(X)=\sum_{x=0}^{\infty}f(x,0)X^x,\qquad \psi_f(Y)=\sum_{y=0}^{\infty}f(0,y)Y^y,$$

and we write $f=(\phi_f,\psi_f)$.

Since $\mathfrak{z}f(x,0)=1/(1+i)\{xf(x,0)+ixf(x-1,0)\}$

$$\sum_{x=0}^{\infty}\mathfrak{z}f(x,0)X^x=\frac{1}{1+i}\sum_{x=0}^{\infty}x(f(x,0)+if(x-1,0))X^x$$

$$=\frac{X}{1+i}\frac{d}{dX}[(1+iX)\phi_f(X)].$$

Similarly

$$\sum_{y=0}^{\infty}\mathfrak{z}f(0,y)Y^y=\frac{Y}{1+i}\frac{d}{dY}[(iY-1)\psi_f(Y)].$$

So the operation of $\mathfrak{z}$ in terms of formal power series is

$$(3.2)\quad(\phi_f,\psi_f)\to\frac{1}{1+i}\left(X\frac{d}{dX}[(1+iX)\phi_f(X)],\qquad Y\frac{d}{dY}[(iY-1)\psi_f(Y)]\right).$$

Thus $\mathfrak{z}f\equiv 0$ iff

$$\phi_f(X)=\frac{C}{1+iX};\qquad \psi_f(Y)=\frac{C}{1-iY}.$$

(The constants agree since $\phi_f(0)=f(0,0)=\psi_f(0)$.) So, unfortunately, $\mathfrak{z}$ has a non-trivial kernel.

Clearly, $\mathfrak{z}f(0)=0$ for every function f discrete analytic in $Z^+\times Z^+$. Let $g\in\mathscr{D}$, $g(0)=0$ then $f\in\mathscr{D}$ given by

$$\phi_f(X) = \frac{1+i}{1+iX}\int \frac{\phi_g(X)}{X}\,dX = \frac{1+i}{1+iX}\left[\sum_1^\infty \frac{g(x,0)X^x}{x} + C\right]$$

$$\psi_f(Y) = \frac{1+i}{1-iY}\int \frac{\phi_g(Y)}{Y}\,dY = \frac{1+i}{1-iY}\left[\sum_1^\infty \frac{g(0,y)Y^y}{y} + C\right]$$

solves $\mathfrak{z}f = g$.

We have thus obtained

THEOREM 3.1. *The operator*

$$\mathfrak{z} : \mathcal{D} \to \mathcal{D}$$

has range $\{f \in \mathcal{D}; f(0,0) = 0\}$ *and kernel* $\{Cf_0\}$ *where* $f_0 \in \mathcal{D}$ *is given by*

$$\phi_{f_0} = \frac{1}{1+iX}; \qquad \psi_{f_0} = \frac{1}{1-iY}.$$

Let us consider the class $\mathcal{F} \subset \mathcal{D}$ defined at the end of Section 2. It is not yet known whether the inclusion $\mathcal{F} \subset \mathcal{D}$ is proper or not; i.e., whether every discrete analytic function on $Z^+ \times Z^+$ possesses a discrete Taylor expansion

$$f(z) = \sum_{k=0}^{\infty} a_k \pi_k(z). \tag{3.3}$$

Theorem (3.1) implies that even if such a representation exists it need not be unique. However if attention is restricted to the class

$$\mathcal{F}_e = \left\{\sum_0^\infty a_k \pi_k(z); \qquad \overline{\lim}\,(k!\,|a_k|)^{1/k} < \infty\right\}$$

then the representation (3.3) is unique, as follows from the following

THEOREM 3.2. *If* $\Sigma_0^\infty a_k \pi_k(z) \equiv 0$ *in* $Z^+ \times Z^+$ *and* $\overline{\lim}\,(k!\,|a_k|)^{1/k} < \infty$ *then* $a_k = 0$ *for every* k.

PROOF. By definition (2.1)

$$\pi_k(x,y) = \frac{k!}{2\pi i}\int_\Gamma \frac{[(1+i)e^{\zeta/(1+i)} - i]^x[(1-i)e^{-\zeta/(1+i)} + i]^y}{\zeta^{k+1}}\,d\zeta$$

where Γ is any contour surrounding 0. So,

$$\begin{aligned} f(z) &= \sum_0^\infty a_k \pi_k(x,y) \\ &= \frac{1}{2\pi i}\int_\Gamma \left(\sum \frac{k!\,a_k}{\zeta^{k+1}}\right)[(1+i)e^{\zeta/(1+i)} - i]^x[(1-i)e^{-\zeta/(1+i)} + i]^y\,d\zeta \end{aligned}$$

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for any contour Γ for which

$$f_B(\zeta)=\sum_{k=0}^{\infty}\frac{k!a_k}{\zeta^{k+1}}$$

is defined. $f_B(\zeta)$ is the Borel transform of

$$f^c(\zeta)=\sum_{k=0}^{\infty}a_k\zeta^k$$

and $f_B(\zeta)$ converges for $|\zeta|\geqq$ type f^c (see Boas (1954), p. 73). Thus

$$f(z)=\frac{1}{2\pi i}\int_\Gamma f_B(\zeta)[(1+i)e^{\zeta/(1+i)}-i]^x[(1-i)e^{-\zeta/(1+i)}+i]^y d\zeta$$

and for some constant M

$$|f(x,0)|\leqq CM^x$$

and $\phi_f(t)=\sum_0^\infty f(x,0)t^x$ converges in the disc $|t|<1/M$. We have then

$$\phi_f(t)=\sum_{x=0}^{\infty}f(x,0)t^x=\frac{1}{2\pi i}\int_\Gamma f_B(\zeta)\sum_0^\infty[(1+i)e^{\zeta/(1+i)}-i]^x t^x$$

$$=\frac{1}{2\pi i}\int_\Gamma\frac{f_B(\zeta)d\zeta}{1-[(1+i)e^{\zeta/(1+i)}-i]t}.$$

The right hand side defines an analytic function in any region in the t-plane for which the denominator of the integrand does not vanish in a neighborhood of Γ in the ζ-plane. In particular, this includes a neighborhood of the point i in the t-plane. Thus for any discrete analytic function of class $\mathscr{F}_e$

$$\phi_f(t)=\sum_{x=0}^{\infty}f(x,0)t^x$$

whose radius of convergence is in general smaller than 1, can be analytically continued through the boundary of the circle of convergence to a neighborhood of $t=i$.

Now $\sum_0^\infty a_k\pi_k(z)\equiv 0$ implies $a_0=0$ and

$$\mathfrak{z}\left(\sum_1^\infty a_k\pi_{k-1}(z)\right)=0.$$

Let $g_1(z)=\sum_1^\infty a_k\pi_{k-1}(z)$. Then $g_1\in\mathscr{F}_e$ and hence $\phi_{g_1}(t)$ can be analytically continued to a neighborhood of $t=i$. But $\mathfrak{z}g_1\equiv 0$ implies, by Theorem 3.1, that $\phi_{g_1}(t)=C/(1+it)$ for some constant C. This forces $C=0$ for, otherwise ϕ_{g_1} would have a pole at $t=i$. Thus,

$$g_1(z)=\sum_1^\infty a_k\pi_{k-1}(z)\equiv 0 \quad\text{and}\quad a_1=0.$$

Continuing inductively we get that $a_k = 0$ for every k and the theorem is proved.

4. Limiting behavior as $h \downarrow 0$

Let $h > 0$. For the lattice of mesh size h

$$Z_h \times Z_h = \{(hm, hn);\ m, n \in Z\}$$

discrete analyticity is defined by

$$F(x, y) + iF(x + h, y) - F(x + h, y + h) - iF(x, y + h) = 0. \tag{4.1}$$

The above discussion carries over to discrete analytic functions for such lattices (all it amounts to is a change of scale). Now we have the basis

$$\pi_k^h(x, y) = \frac{d^k}{d\zeta^k}\{[(1+i)e^{\zeta h/(1+i)} - i]^{x/h}[(1-i)e^{-\zeta h/(1+i)} + i]^{y/h}\}\Big|_{\zeta=0}. \tag{4.2}$$

And for discrete analytic functions on the lattice $Z_h \times Z_h$ the exponential function is

$$e_h(x, y) = \sum_{k=0}^{\infty} \pi_k^h(x, y)\frac{\zeta^k}{k!} = [(1+i)e^{\zeta h/(1+i)} - i]^{x/h}[(1-i)e^{-\zeta h/(1+i)} + i]^{y/h}.$$

Now as $h \downarrow 0$

$$[(1+i)e^{\zeta h/(1+i)} - i]^{1/h} \to e^{\zeta} \quad [(1-i)e^{-\zeta h/(1+i)} + i]^{1/h} \to e^{i\zeta}.$$

So $e_h(x, y) \to e^{\zeta(x+iy)}$ and consequently

$$\pi_k^h(z) \to z^k \quad \text{as} \quad h \downarrow 0.$$

Suppose $|a_n| \leqq C\zeta_0^n/n!$ for some constants C and ζ_0, by dominated convergence

$$f^h(z) = \sum_0^{\infty} a_k \pi_k^h(z) \to \sum_0^{\infty} a_k z^k$$

as $h \downarrow 0$. We obtained

LEMMA 4.1. *If* $\overline{\lim}\,(|a_k|\,k!)^{1/k} < \infty$ *then* $f^h(z) \to f^c(z) = \Sigma_0^{\infty} a_k z^k$ *along a sequence* $h \downarrow 0$ *for which* $z \in Z_h^+ \times Z_h^+$.

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