FOURIER TRANSFORMS OF DISTRIBUTION FUNCTIONS

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A distribution function $\phi(x)$ is assumed to have the following properties:

(1) $$\phi(x) \text{ is non-decreasing}$$

(2) $$\lim_{x\to-\infty} \phi(x) = 0, \quad \lim_{x\to+\infty} \phi(x) = 1,$$

(3) $$\phi(x) = \lim_{y\to x+0} \phi(y) \text{ for every } x.$$

The Fourier transform of $\phi(x)$ is defined by the Stieltjes integral

(4) $$\Phi(t) = \int_{-\infty}^{\infty} e^{-itx}\, d\phi(x).$$

Let ϕ_1 and ϕ_2 be two distribution functions. Let a positive real number δ be given. We consider the question, does there exist a positive ϵ such that the condition

(5) $$|\Phi_1(t) - \Phi_2(t)| < \epsilon \text{ for all } t$$

implies

(6) $$|\phi_1(x) - \phi_2(x)| < \delta\ ?$$

There are three separate problems here. (i) We may allow ϵ to depend on δ, ϕ_1, and x. Then our question is, does the uniform convergence of Φ_2 to Φ_1 imply a point-wise convergence of ϕ_2 to ϕ_1?. The answer to this question is yes, as is well known; in fact Lévy [**1**, p. 49] proves a theorem which states considerably more than is needed for our problem. (ii) We may allow ϵ to depend on δ and ϕ_1, but not on x. Then our question is, does uniform convergence of Φ_2 to Φ_1 imply uniform convergence of ϕ_2 to ϕ_1? The answer to this question is also yes; we prove this in Theorem 1 below. (iii) We may allow ϵ to depend on δ only. In this case the answer is no, as we shall show by an example.

Counter-example for case (iii). Let a and b be real numbers with $b > a > 0$. We consider the distribution functions

(7) $$\phi_1(x) = \begin{cases} \frac{1}{2}\log\left(\dfrac{x^2+b^2}{x^2+a^2}\right) \Big/ \log\left(\dfrac{b}{a}\right), & x \leqslant 0 \\ 1, & x \geqslant 0. \end{cases}$$

(8) $$\phi_2(x) = 1 - \phi_1(-x).$$

Received June 10, 1952.

https://doi.org/10.4153/CJM-1953-062-x Published online by Cambridge University Press

Then

$$\phi_1(x) - \phi_2(x) = \tfrac{1}{2}\log\left(\frac{x^2+b^2}{x^2+a^2}\right)\Big/\log\left(\frac{b}{a}\right), \qquad \text{all } x, \tag{9}$$

and in particular

$$\phi_1(0) - \phi_2(0) = 1. \tag{10}$$

However, by (9) we have

$$\Phi_1(t) - \Phi_2(t) = i\pi\frac{t}{|t|}[e^{-a|t|} - e^{-b|t|}]/\log\left(\frac{b}{a}\right), \tag{11}$$

$$|\Phi_1(t) - \Phi_2(t)| < \pi/\log\left(\frac{b}{a}\right). \tag{12}$$

Since b/a may be arbitrarily large, we see that we can satisfy (5) for any $\epsilon > 0$ and still have (6) false for $\delta = 1$.

Statement of theorem for case (ii).

THEOREM 1. *Let a positive δ and a distribution function ϕ_1 be given. Then we can find $\epsilon > 0$, depending only on δ and ϕ_1, such that* (5) *implies* (6) *for all x and for all ϕ_2.*

Let $h_\eta(x)$ be the function defined by

$$h_\eta(x) = \max(0, 1 - |x/\eta|). \tag{13}$$

Then (4) gives

$$\int_{-\infty}^{\infty} h_\eta(x-w)\,d\phi(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{4\sin^2\frac{1}{2}\eta t}{\eta t^2}e^{itw}\,\Phi(t)\,dt, \tag{14}$$

both sides being absolutely convergent integrals. If ϵ is chosen so that (5) is satisfied, then (14) gives, for every η and w,

$$\left|\int_{-\infty}^{\infty} h_\eta(x-w)[d\phi_1(x) - d\phi_2(x)]\right| < \epsilon. \tag{15}$$

Since ϕ_1 is non-decreasing and (3) holds,

$$\phi_1(w) - \lim_{y\to w-0}\phi_1(y) = \lim_{\eta\to 0}\int h_\eta(x-w)\,d\phi_1(x), \tag{16}$$

the limits on both sides necessarily existing. Similarly (16) holds for ϕ_2. Therefore letting $\eta \to 0$ in (15), we have, for all w,

$$|(\phi_1(w) - \lim_{y\to w-0}\phi_1(y)) - (\phi_2(w) - \lim_{y\to w-0}\phi_2(y))| < \epsilon. \tag{17}$$

That is to say, at every point the discontinuities in ϕ_1 and ϕ_2 differ by at most ϵ. Another consequence of (15) is obtained by writing in turn $w+\eta, w+2\eta, \ldots, w+N\eta$ for w and adding the resulting inequalities. From the definition of $h_\eta(x)$,

$$\sum_{m=1}^{N} h_\eta(x - w - m\eta) = 1, \qquad w+\eta \leqslant x \leqslant w+N\eta,$$

and

$$0 \leqslant \sum_{m=1}^{N} h_\eta(x - w - m\eta) \leqslant 1,$$

$$w \leqslant x \leqslant w + \eta \text{ and } w + N\eta \leqslant x \leqslant w + (N+1)\eta.$$

Using the fact that ϕ_1 and ϕ_2 are non-decreasing, adding together (15) for these N values of w therefore gives

$$\int_{w}^{w+(N+1)\eta} d\phi_2(x) > \int_{w+\eta}^{w+N\eta} d\phi_1(x) - N\epsilon. \tag{18}$$

We write for brevity $\alpha = \frac{1}{4}\delta$. We can divide the whole line $(-\infty, +\infty)$ into a finite set of intervals $I_1, \ldots, I_m$ with the following properties. (i) Each I_n is closed on the left and open on the right. (ii) The total variation of $\phi_1(x)$ on I_n is less than α. Let L_n^1 and R_n^1 be the limits to which $\phi_1(x)$ tends as x tends to the left and right end-points within I_n. Similarly let L_n^2 and R_n^2 be the limits of ϕ_2. By (17) we have

$$R_n^2 - R_n^1 < L_{n+1}^2 - L_{n+1}^1 + \epsilon. \tag{19}$$

Now let λ be the length of the shortest I_n, let Λ be the combined length of $I_2, \ldots, I_{m-1}$, and let N be an integer greater than $(2\Lambda/\lambda)$. The choice of N and of the I_n depends only on δ and ϕ_1 and is independent of ϵ. Given any I_n with $1 < n < m$, we can choose two points x, x' inside I_n such that

$$x' - x > \tfrac{1}{2}\lambda. \tag{20}$$

Then we apply (18) with $w = x$, $w + \eta = x'$, giving

$$\phi_1(x') + \phi_2(x' + N\eta) > \phi_2(x) + \phi_1(x + N\eta) - N\epsilon. \tag{21}$$

By the definition of N, the point $(x + N\eta)$ belongs to I_m and so

$$\phi_1(x + N\eta) > 1 - \alpha, \quad \phi_2(x' + N\eta) \leqslant 1.$$

Hence (21) becomes

$$\phi_1(x') > \phi_2(x) - N\epsilon - \alpha. \tag{22}$$

Again, applying (18) with $w = x - N\eta$, $w + \eta = x' - N\eta$,

$$\phi_2(x') + \phi_1(x' - N\eta) > \phi_1(x) + \phi_2(x - N\eta) - N\epsilon,$$

and since $(x' - N\eta)$ belongs to I_1 this becomes

$$\phi_2(x') > \phi_1(x) - N\epsilon - \alpha. \tag{23}$$

Let x' and x tend respectively to the right and left to the end-points of I_n. Then (22) and (23) give

$$L_n^2 \leqslant R_n^1 + N\epsilon + \alpha, \tag{24}$$

$$R_n^2 \geqslant L_n^1 - N\epsilon - \alpha. \tag{25}$$

These inequalities, (24) and (25), which have been proved for $1 < n < m$, are trivially true also for $n = 1$ and $n = m$.

Writing $n + 1$ for n in (24) and combining it with (19), we find

$$\begin{aligned} R_n^2 &< R_n^1 + R_{n+1}^1 - L_{n+1}^1 + (N+1)\,\epsilon + \alpha \\ &< R_n^1 + (N+1)\,\epsilon + 2\alpha. \end{aligned} \tag{26}$$

Similarly (25) combined with (19) gives

$$L_n^2 > L_n^1 - (N+1)\,\epsilon - 2\alpha. \tag{27}$$

Now R_n^2 and L_n^2 are the upper and lower bounds of ϕ_2 in I_n, and R_n^1 and L_n^1 differ by at most α. Therefore (26) and (27) imply

$$|\phi_2(x) - \phi_1(x)| < (N+1)\epsilon + 3\alpha = (N+1)\epsilon + \tfrac{3}{4}\delta \tag{28}$$

for all x in $(-\infty, +\infty)$. The choice of N depended only on δ and ϕ_1. Given δ and ϕ_1 we can choose ϵ to be any number less than $(\delta/(4(N+1)))$, and then (5) will imply (6). This proves the theorem.

Additional remarks. Another theorem can be derived from Theorem 1 by weakening both the hypothesis and the conclusion slightly. Let us define the distance between two distributions ϕ_1 and ϕ_2 by

$$||\phi_1 - \phi_2|| = \max\,(|\{\phi_1, \phi_2\}|,\ |\{\phi_2, \phi_1\}|), \tag{29}$$

where

$$\{\phi_1, \phi_2\} = \max_{x,x'}\,(\min\,(x' - x,\ \phi_1(x) - \phi_2(x'))). \tag{30}$$

This definition of the distance is equivalent to that given by Lévy [**1**, p. 47]. It is easy to see that $||\phi_1 - \phi_2||$ is the side of the largest square that can be inserted between the graphs $y = \phi_1(x)$ and $y = \phi_2(x)$ when these are plotted in cartesian coordinates in the usual way. Thus the convergence defined by $||\phi_2 - \phi_1|| \to 0$ is topologically weaker than uniform convergence of ϕ_2 to ϕ_1, but topologically stronger than point-wise convergence of ϕ_2 to ϕ_1. The modified form of Theorem 1 is

THEOREM 2. *Let δ and ϕ_1 be given. Then we can find $\epsilon > 0$ depending only on δ and ϕ_1, such that*

$$|\Phi_1(t) - \Phi_2(t)| < \epsilon \textit{ for all } t < \frac{1}{\epsilon} \tag{31}$$

implies

$$||\phi_2 - \phi_1|| < \delta. \tag{32}$$

The proof is similar to the proof of Theorem 1, only simpler. The counter-example given previously also shows that the weaker conclusion (32) does not follow from (5) with ϵ depending only on δ.

https://doi.org/10.4153/CJM-1953-062-x Published online by Cambridge University Press

The author is indebted to Dr. K. L. Chung for suggesting this problem to him, and for several stimulating discussions.

Reference

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https://doi.org/10.4153/CJM-1953-062-x Published online by Cambridge University Press