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TWO WEIGHTED INEQUALITIES FOR MAXIMAL FUNCTIONS RELATED TO CESÀRO CONVERGENCE

A. L. BERNARDIS and F. J. MARTÍN-REYES

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Abstract

We characterize the pairs of weights (u, v) for which the maximal operator

$$M_\alpha^- f(x) = \sup_{R>0} R^{-1-\alpha} \int_{x-2R}^{x-R} |f(s)|(x-R-s)^\alpha \, ds, \quad -1 < \alpha < 0,$$

is of weak and restricted weak type (p, p) with respect to $u(x)\,dx$ and $v(x)\,dx$. As a consequence we obtain analogous results for

$$M_\alpha f(x) = \sup_{R>0} R^{-1-\alpha} \int_{R<|x-y|<2R} |f(y)|(|x-y|-R)^\alpha \, dy.$$

We apply the results to the study of the Cesàro-α convergence of singular integrals.

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1. Introduction

Let M_α be the maximal operator defined at a measurable function f on the real line by

$$M_\alpha f(x) = \sup_{R>0} \frac{1}{R^{1+\alpha}} \int_{R<|x-y|<2R} |f(y)|(|x-y|-R)^\alpha \, dy, \quad -1 < \alpha < 0.$$

This operator occurs in a natural way when one studies the Cesàro-α convergence of singular integrals [2]. Alternatively,

$$M_\alpha f(x) = \sup_{R>0} |f| * \varphi_R(x),$$

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where $\varphi_R(x) = R^{-1}\varphi(R^{-1}x)$ and $\varphi(s) = (|s|-1)^\alpha \chi_{(1,2)}(|s|)$. From this point of view, M_α is a particular case of the operator studied in [5]. It follows from [5, Theorem 1] that M_α is of restricted weak type $(1/(1+\alpha), 1/(1+\alpha))$ and that it is not of weak type $(1/(1+\alpha), 1/(1+\alpha))$ with respect to the Lebesgue measure when $\alpha < 0$ (notice that M_0 is equivalent to the Hardy-Littlewood maximal operator).

Weighted inequalities for M_α were studied in [2] and [3]. In [3] we obtained a characterization of weighted inequalities for a single weight. The doubling condition plays an essential role in the proof of this characterization; it was also the key reason why we were not able to study the two-weight case in [3].

In this paper we develop a different approach to the study of weighted inequalities for M_α which enables us to obtain a characterization of the two-weighted weak and restricted weak type inequalities for M_α. This new method consists of the study of one-sided versions of M_α

$$M_\alpha^- f(x) = \sup_{R>0} \frac{1}{R^{1+\alpha}} \int_{x-2R}^{x-R} |f(y)|(x-R-y)^\alpha \, dy$$

and

$$M_\alpha^+ f(x) = \sup_{R>0} \frac{1}{R^{1+\alpha}} \int_{x+R}^{x+2R} |f(y)|(y-x-R)^\alpha \, dy.$$

These operators are of interest because they naturally appear in the investigation of the Cesàro-α convergence of singular integrals with kernels supported in $(0, \infty)$ and in $(-\infty, 0)$.

The paper is organized as follows. In Section 2 we state and prove a characterization of two-weighted weak and restricted weak type inequalities for M_α^-, M_α^+ and M_α; in Section 3 we apply these results to the study of the existence of the singular integrals in the Cesàro-α sense.

Throughout the paper, u, v and w are weights, that is, positive measurable functions, $u(A)$ denotes the integral $\int_A u(s)\,ds$, p' denotes the conjugate exponent of p, $1 < p < \infty$, and the letter C means a positive constant that may change from one line to another.

2. Two-weighted inequalities

We start with the results for M_α^- (analogous results hold for M_α^+).

THEOREM 2.1. *Let u and v be weights on $\mathbb{R}$ and let $-1 < \alpha < 0$. If $1 < p < \infty$, then the following are equivalent:*

(i) *M_α^- is of weak type (p, p) with respect to $u(x)\,dx$ and $v(x)\,dx$, that is, there exists C such that $u(\{M_\alpha^- f > \lambda\}) \le C\lambda^{-p} \int |f|^p v$, for all $\lambda > 0$ and all $f \in L^p(v)$.*

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(ii) (u, v) *satisfies* $A^{-}_{p,\alpha}$, *that is, there exists* C *such that for any three numbers* $a < b < c$,

$$\left(\int_b^c u(s)\,ds\right)^{1/p}\left(\int_a^b v^{1-p'}(s)(b-s)^{\alpha p'}\,ds\right)^{1/p'} \le C(c-a)^{1+\alpha}.$$

REMARK. Observe that if $\alpha < 0$ and $\operatorname{ess\,inf}_{x\in(a,b)} v^{1-p'}(x) > 0$ for some interval (a, b) then the two-weighted weak type (p, p) inequality is not possible for $1 < p \le 1/(1+\alpha)$ since (ii) does not hold in this case. However the operator M^{-}_{α} is of restricted weak type $(1/(1+\alpha), 1/(1+\alpha))$ with respect to the Lebesgue measure. Therefore it is interesting to study the restricted weak type inequalities for pairs of weights.

THEOREM 2.2. *Let* u *and* v *be weights on* $\mathbb{R}$ *and let* $-1 < \alpha < 0$. *If* $1 \le p < \infty$, *then the following are equivalent*:

(i) M^{-}_{α} *is of restricted weak type* (p, p) *with respect to* $u(x)\,dx$ *and* $v(x)\,dx$, *that is, there exists* C *such that* $u(\{x : M^{-}_{\alpha}\chi_E(x) > \lambda\}) \le C\lambda^{-p}v(E)$ *for all* $\lambda > 0$ *and all measurable* $E \subset \mathbb{R}$.

(ii) (u, v) *satisfies* $RA^{-}_{p,\alpha}$, *that is, there exists* C *such that for any three numbers* $a < b < c$ *and all measurable* $E \subset \mathbb{R}$

$$\left(\int_b^c u(s)\,ds\right)\left(\int_a^b \chi_E(s)(b-s)^{\alpha}ds\right)^{p} \le C(c-a)^{(1+\alpha)p}\int_a^b \chi_E(s)v(s)\,ds.$$

The corresponding results for M_{α} are obtained immediately from Theorem 2.1 and Theorem 2.2 and from the analogous ones for M^{+}_{α}.

Now we shall state the results for M_{α} which generalize the weak and restricted weak type inequalities from [3] to the two-weight case.

THEOREM 2.3. *Let* u *and* v *be weights on* $\mathbb{R}$ *and let* $-1 < \alpha < 0$. *If* $1 < p < \infty$, *then the following are equivalent*:

(i) M_{α} *is of weak type* (p, p) *with respect to* $u(x)\,dx$ *and* $v(x)\,dx$.

(ii) (u, v) *satisfies* $A_{p,\alpha}$, *that is, there exists* C *such that for any interval* I

$$\left(\int_I u(s)\,ds\right)^{1/p}\left(\int_{2I\setminus I} v^{1-p'}(s)d(s, I)^{\alpha p'}ds\right)^{1/p'} \le C|I|^{1+\alpha},$$

where $2I$ *is the interval with the same center and double length as* I *and* $d(s, I)$ *is the Euclidean distance from* s *to* I.

THEOREM 2.4. *Let* u *and* v *be weights on* $\mathbb{R}$ *and let* $-1 < \alpha < 0$. *If* $1 \le p < \infty$, *then the following are equivalent*:

(i) *M_α is of restricted weak type (p, p) with respect to $u(x)\,dx$ and $v(x)\,dx$.*

(ii) *(u, v) satisfies $RA_{p,\alpha}$, that is, there exists C such that for every interval I and all measurable $E \subset \mathbb{R}$*

$$\left(\int_I u(s)\,ds\right)\left(\int_{2I\setminus I} \chi_E(s) d(s, I)^\alpha ds\right)^p \le C|I|^{(1+\alpha)p} \int_{2I\setminus I} \chi_E(s) v(s)\,ds.$$

The proofs of Theorem 2.3 and Theorem 2.4 are omitted since they are immediate corollaries of the previous results.

In order to prove Theorem 2.1 and Theorem 2.2 we use a noncentred maximal operator which is pointwise equivalent to M_α^-. In what follows we define this operator and state the pointwise equivalence.

DEFINITION 2.5. For each $x \in \mathbb{R}$, let us consider the family of intervals $\mathscr{A}_x = \{(a, b) : b < x \text{ and } b - a \ge x - b\}$. We define the noncentred maximal operator N_α^- associated with M_α^- as

$$N_\alpha^- f(x) = \sup_{(a,b)\in\mathscr{A}_x} \frac{1}{(b-a)^{1+\alpha}} \int_a^b |f(s)|(b-s)^\alpha\,ds.$$

PROPOSITION 2.6. *Let $-1 < \alpha < 0$. There exists a constant C depending only on α such that $M_\alpha^- f \le N_\alpha^- f \le C M_\alpha^- f$, for all measurable functions f.*

PROOF. The first inequality is obvious. Let $(a, b) \in \mathscr{A}_x$, $R = x - a$ and let N be the natural number such that $x - 2^{-N}R \le b < x - 2^{-N-1}R$. Then

$$\begin{aligned}
&\int_a^b |f(s)|(b-s)^\alpha\,ds \\
&\quad= \sum_{i=0}^{N-1} \int_{x-R/2^i}^{x-R/2^{i+1}} |f(s)| \left(x - \frac{R}{2^{i+1}} - s\right)^\alpha \left(\frac{b-s}{x-(R/2^{i+1})-s}\right)^\alpha ds \\
&\qquad+ \int_{x-R/2^N}^b |f(s)|(b-s)^\alpha\,ds = \mathrm{I} + \mathrm{II}.
\end{aligned}$$

Since $(a, b) \in \mathscr{A}_x$,

$$\mathrm{II} \le \int_{x-2(x-b)}^b |f(s)|(b-s)^\alpha\,ds \le (x-b)^{1+\alpha} M_\alpha^- f(x) \le (b-a)^{1+\alpha} M_\alpha^- f(x).$$

On the other hand, since the function $s \to [(b-s)/(x - 2^{-i-1}R - s)]^\alpha$ is decreasing

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on $(x-2^{-i}R, x-2^{-i-1}R)$, $0 \le i \le N-1$,

$$I \le \left(\sum_{i=0}^{N-1}\left(b-\left(x-\frac{R}{2^i}\right)\right)^{\alpha}\frac{R}{2^{i+1}}\right)M_{\alpha}^{-}f(x)$$

$$\le M_{\alpha}^{-}f(x)\sum_{i=0}^{N-1}\int_{x-R/2^i}^{x-R/2^{i+1}}(b-s)^{\alpha}\,ds \le C(b-a)^{1+\alpha}M_{\alpha}^{-}f(x),$$

and we are done. □

PROOF OF THEOREM 2.1. By Proposition 2.6, (i) is equivalent to the weighted weak type (p,p) inequality for N_{α}^{-}. Let $a<b<c$ and let $\bar{a}<a$ be such that $b-\bar{a}=c-a$. If we consider the function $f(s)=v^{1-p'}(s)(b-s)^{\alpha(p'-1)}\chi_{(a,b)}(s)$, then for all $x\in(b,c)$

$$N_{\alpha}^{-}f(x) \ge \frac{1}{(b-\bar{a})^{1+\alpha}}\int_a^b v^{1-p'}(s)(b-s)^{\alpha p'}\,ds \equiv \lambda.$$

This means that $(b,c)\subset\{N_{\alpha}^{-}f \ge \lambda\}$. Then (ii) follows from (i) (with N_{α}^{-}) by a standard argument. □

The implication (ii) implies (i) follows from the following proposition and the fact that the maximal operator $M_u^{-}g(x)=\sup_{h<x}\left(\int_h^x |g|u/\int_h^x u\right)$ is of weak type $(1,1)$ with respect to the measure $u(x)\,dx$.

PROPOSITION 2.7. *Let $-1<\alpha<0$ and $p>1$. If (u,v) satisfies $A_{p,\alpha}^{-}$, then there exists $C>0$ such that for every measurable function f*

$$N_{\alpha}^{-}f \le C[M_u^{-}(|f|^p vu^{-1})]^{1/p}.$$

PROOF. Let $x\in\mathbb{R}$ and $(a,b)\in\mathscr{A}_x$. First, let us assume that $4\int_b^x u > \int_a^x u$. Since the pair (u,v) satisfies $A_{p,\alpha}^{-}$, we have

$$\int_a^b |f(s)|(b-s)^{\alpha}\,ds \le \left(\int_a^b |f|^p(s)v(s)\,ds\right)^{1/p}\left(\int_a^b v^{-p'/p}(s)(b-s)^{\alpha p'}\,ds\right)^{1/p'}$$

$$\le C\left(\int_a^x |f|^p(s)v(s)\,ds\right)^{1/p}\left(\int_b^x u(s)\,ds\right)^{-1/p}(x-a)^{1+\alpha}$$

$$\le C[M_u^{-}(|f|^p vu^{-1})]^{1/p}(x)(b-a)^{1+\alpha}.$$

Assume now that $4\int_b^x u \le \int_a^x u$. Let $\{x_i\}$ be the increasing sequence in $[a,x]$ defined by $x_0=a$ and $\int_{x_{i+1}}^x u=\int_{x_i}^{x_{i+1}} u=\frac{1}{2}\int_{x_i}^x u$. Let N be such that $x_N\le b<x_{N+1}$ (observe

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that $N \geq 2$). Then we have

$$\int_a^b |f(s)|(b-s)^\alpha\,ds = \sum_{i=0}^{N-2}\int_{x_i}^{x_{i+1}} |f(s)|(b-s)^\alpha\,ds + \int_{x_{N-1}}^b |f(s)|(b-s)^\alpha\,ds$$
$$= \mathrm{I} + \mathrm{II}.$$

By the $A^-_{p,\alpha}$ condition and the fact that $\int_{x_{N-1}}^x u \leq 4\int_b^x u$, we have

$$\begin{aligned}\mathrm{II} &\leq \left(\int_{x_{N-1}}^b |f|^p(s)v(s)\,ds\right)^{1/p}\left(\int_{x_{N-1}}^b v^{-p'/p}(s)(b-s)^{\alpha p'}\,ds\right)^{1/p'}\\ &\leq C[M_u^-(|f|^p vu^{-1})]^{1/p}(x)(b-a)^{1+\alpha}.\end{aligned}$$

On the other hand, since the function $s \to [(b-s)/(x_{i+1}-s)]^\alpha$ is decreasing in the interval (x_i, x_{i+1}), $0 \leq i \leq N-2$, we obtain

$$\begin{aligned}\int_{x_i}^{x_{i+1}} |f(s)|(b-s)^\alpha\,ds &\leq \left(\frac{b-x_i}{x_{i+1}-x_i}\right)^\alpha \int_{x_i}^{x_{i+1}} |f(s)|(x_{i+1}-s)^\alpha\,ds\\ &\leq \left(\frac{b-x_i}{x_{i+1}-x_i}\right)^\alpha \left(\int_{x_i}^{x_{i+1}} |f|^p(s)v(s)\,ds\right)^{1/p}\left(\int_{x_i}^{x_{i+1}} v^{-p'/p}(s)(x_{i+1}-s)^{\alpha p'}\,ds\right)^{1/p'}\\ &\leq C\left(\frac{b-x_i}{x_{i+1}-x_i}\right)^\alpha \left(\int_{x_i}^{x_{i+1}} |f|^p(s)v(s)\,ds\right)^{1/p}\left(\int_{x_{i+1}}^{x_{i+2}} u(s)\,ds\right)^{-1/p}(x_{i+2}-x_i)^{1+\alpha}\\ &\leq C(b-x_i)^\alpha(x_{i+2}-x_i)\left(\frac{\int_{x_i}^x |f|^p(s)v(s)\,ds}{\int_{x_i}^x u(s)\,ds}\right)^{1/p}\\ &\leq C[M_u^-(|f|^p vu^{-1})]^{1/p}(x)\int_{x_i}^{x_{i+2}}(b-s)^\alpha\,ds.\end{aligned}$$

Now, summing up in i, we get

$$\mathrm{I} \leq C[M_u^-(|f|^p vu^{-1})]^{1/p}(x)\int_a^{x_N}(b-s)^\alpha\,ds \leq C[M_u^-(|f|^p vu^{-1})]^{1/p}(x)(b-a)^{1+\alpha}.$$

Finally, putting together the estimates of I and II, we are done. □

PROOF OF THEOREM 2.2. The proof is similar to that of Theorem 2.1. We give just a sketch. First, (ii) follows from (i) on applying the standard argument to $\chi_{E\cap(a,b)}$. The converse follows from the fact that (ii) implies $N_\alpha^-\chi_E(x) \leq C[M_u^-(\chi_E v\, u^{-1})]^{1/p}(x)$, for some constant C independent of the measurable subset E. To prove the above inequality, let $x \in \mathbb{R}$, $(a,b) \in \mathscr{A}_x$ and assume first that $4\int_b^x u > \int_a^x u$. Since (u, v)

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satisfies $RA^-_{p,\alpha}$ we obtain

$$\begin{aligned}\int_a^b \chi_E(s)(b-s)^\alpha\,ds &\le C(x-a)^{1+\alpha}\left(\int_a^b \chi_E(s)v(s)\,ds\right)^{1/p}\left(\int_b^x u(s)\,ds\right)^{-1/p}\\ &\le C(b-a)^{1+\alpha}\left(\int_a^x \chi_E(s)v(s)\,ds\right)^{1/p}\left(\int_a^x u\right)^{-1/p}\\ &\le C(b-a)^{1+\alpha}[M_u^-(\chi_E v u^{-1})]^{1/p}(x).\end{aligned}$$

If $4\int_b^x u \le \int_a^x u$, we proceed as in the proof of Proposition 2.7. □

3. Singular integrals in the Cesàro sense

Let K be a Calderón-Zygmund kernel on $\mathbb{R}$, that is, a function $K \in L^1_{\mathrm{loc}}(\mathbb{R}\setminus\{0\})$ such that

(1) $|K(x)| \le C|x|^{-1}$, $|x| > 0$,
(2) $|K(x-y) - K(x)| \le C|y||x|^{-2}$, if $|x| > 2|y| > 0$,
(3) $\left|\int_{\epsilon<|x|<N} K(x)\,dx\right| \le C$ for all ϵ and all N with $0 < \epsilon < N$.

If the limit $\lim_{\epsilon\to 0^+}\int_{\epsilon<|y|<1} K(y)\,dy$ exists, then the principal-value singular integral

$$Tf(x) = \lim_{\epsilon\to 0^+}\int_{|x-y|>\epsilon} K(x-y)f(y)\,dy$$

exists for $f \in L^p(w dx)$ with w in the Muckenhoupt class A_p (see for instance [4]). When the kernel K has support in $(0,\infty)$ (or in $(-\infty,0)$), then, as proved in [1], the same result holds for a wider class of weights, more precisely for weights in the Sawyer class $A_p^- \equiv A^-_{p,0}$ ([7]).

Recently, in [2], we studied the existence in the Cesàro-α sense of the singular integral associated with K for $-1 < \alpha < 0$, that is, the existence of the limit

$$\lim_{\epsilon\to 0^+} T_{\epsilon,\alpha}f(x) = \lim_{\epsilon\to 0^+}\int_{|x-y|>\epsilon} f(y)K(x-y)\left(1-\frac{\epsilon}{|x-y|}\right)^\alpha dy,$$

in the setting of weighted L^p-spaces. The aim in this section is to obtain sharper results on singular integrals in the Cesàro-α sense for kernels with support in $(0,\infty)$ (or in $(-\infty,0)$). We shall show, using the results of Section 2, that, for these kernels, the results in [2] are true for a wider class of weights.

One of the key steps in [2] is the pointwise estimate from above of the maximal operator $T^*_\alpha f = \sup_{\epsilon>0}|T_{\epsilon,\alpha}f|$ by $C(M_\alpha f + T_0^* f)$. If the support of K is contained in $(0,\infty)$, then we can improve this estimate by replacing M_α with a smaller operator M_α^-.

PROPOSITION 3.1. *Let $-1 < \alpha < 0$ and let K be a Calderón-Zygmund kernel with support contained in $(0, \infty)$. If f is a measurable function such that $T_{\epsilon,\alpha} f(x)$ is defined for every $\epsilon > 0$, then there exists $C > 0$ independent of f such that*

$$T_\alpha^* f(x) \leq C\left[M_\alpha^- f(x) + T_0^* f(x)\right].$$

The proof is similar to that of [2, Proposition 2.5], and is therefore omitted. This proposition together with Theorem 2.1 in this paper and [1, Theorem 2.1] enables us to prove the following result.

THEOREM 3.2. *Let $-1 < \alpha < 0$ and let K be a Calderón-Zygmund kernel with support contained in $(0, \infty)$ such that the limit*

$$\lim_{\epsilon \to 0^+} \int_\epsilon^1 K(y) \left(1 - \frac{\epsilon}{y}\right)^\alpha dy$$

exists. Then the singular integral exists a.e. in the Cesàro-α sense if $f \in L^p(w\,dx)$ with $p(1+\alpha) > 1$ and $w \in A_{p,\alpha}^-$ (the pair (w, w) satisfies $A_{p,\alpha}^-$).

To prove the theorem we have to show first that the truncations $T_{\epsilon,\alpha} f$ are well defined for $f \in L^p(w dx)$, $w \in A_{p,\alpha}^-$, $p(1+\alpha) > 1$. This can be proved as in [2, Theorem 2.7]. The rest of the proof is a consequence of the following facts: (i) the existence of the limit $\lim_{\epsilon \to 0^+} T_{\epsilon,\alpha} f$ for f in a dense class and (ii) the weak type (p, p) boundedness with respect to $w(x)\,dx$ of the maximal operator T_α^*. The former is clear since $L^p(w dx) \cap L^p(dx)$ is dense in $L^p(w dx)$ and the convergence holds for $f \in L^p(w dx) \cap L^p(dx)$ by [2, Theorem 2.7]. The latter immediately follows from Proposition 3.1, Theorem 2.1, [1, Theorem 2.1] and the easy implication $w \in A_{p,\alpha}^- \Rightarrow w \in A_{p,0}^- \equiv A_p^-$.

REMARK. In particular, the result holds if w belongs to the Sawyer's class [7] $A_{p(1+\alpha)}^-$ since $A_{p(1+\alpha)}^- \subset A_{p,\alpha}^-$. This inclusion follows from $A_r^- \subset A_{p,\alpha}^-$, $1 < r < p(1+\alpha)$, which is true by Hölder's inequality and the implication $w \in A_{p(1+\alpha)}^- \Rightarrow w \in A_r^-$ for some $r < p(1+\alpha)$ (see [7] or [6]).

We do not know whether $A_{p(1+\alpha)}^-$ is equal to $A_{p,\alpha}^-$ for $\alpha < 0$ and $p > 1/(1+\alpha)$ but in the endpoint $p = 1/(1+\alpha)$ it is possible to see that $RA_{1/(1+\alpha),\alpha}^-$ equals the Sawyer's class A_1^-. The proof of this fact is similar to the proof of [3, Proposition 6.5]. Then, following the steps in the proof of [2, Theorem 2.7] and using the corresponding results in this paper and in [1] we have our next result.

THEOREM 3.3. *Let α and K be as in Theorem 3.2. If f belongs to the Lorentz space $L_{1/(1+\alpha),1}(\omega\,dx) = \{f : \int_0^\infty [\omega(\{x : |f(x)| > t\})]^{1+\alpha}\,dt < \infty\}$ and $\omega \in A_1^-$, then the singular integral exists a.e. in the Cesàro-α sense.*

EXAMPLE. Observe that the Calderón-Zygmund kernel

$$K(x) = \frac{1}{x}\frac{\sin(\log x)}{\log x}\chi_{(0,\infty)}(x)$$

given in [1] satisfies the condition in Theorem 3.2, that is,

$$\lim_{\epsilon\to 0^+}\int_\epsilon^1 K(y)\left(1-\frac{\epsilon}{y}\right)^\alpha dy$$

exists. In fact, for any $0 < \epsilon < 1/2$, if $\Omega(x) = \sin x/x$, then

$$\begin{aligned}\int_\epsilon^1 K(y)\left(1-\frac{\epsilon}{y}\right)^\alpha dy &= \int_\epsilon^1 \frac{\Omega(\log y)}{y}\left(1-\frac{\epsilon}{y}\right)^\alpha dy \\ &= \int_\epsilon^{2\epsilon}\cdots dy + \int_{2\epsilon}^1 \cdots dy = \mathrm{I}+\mathrm{II}.\end{aligned}$$

Applying the Hölder inequality to I with $p > 1/(1+\alpha)$ and changing variables we obtain

$$\begin{aligned}|\mathrm{I}| &\le \left(\int_\epsilon^{2\epsilon}\frac{|\Omega(\log y)|^p}{y}dy\right)^{1/p}\left(\int_\epsilon^{2\epsilon}\left(1-\frac{\epsilon}{y}\right)^{\alpha p'}\frac{1}{y}dy\right)^{1/p'} \\ &\le C\left(\int_{\log\epsilon}^{\log 2\epsilon}|\Omega(t)|^p\,dt\right)^{1/p} \le C\left(\int_{\log\epsilon}^{\log 2\epsilon}\frac{1}{|t|^p}\,dt\right)^{1/p}\end{aligned}$$

and therefore $\lim_{\epsilon\to 0^+}\mathrm{I} = 0$. On the other hand,

$$\mathrm{II} = \int_{2\epsilon}^1\frac{\Omega(\log y)}{y}\left[\left(1-\frac{\epsilon}{y}\right)^\alpha - 1\right]dy + \int_{2\epsilon}^1\frac{\Omega(\log y)}{y}dy = \mathrm{III}+\mathrm{IV}.$$

Clearly, by changing the variables, we see that $\lim_{\epsilon\to 0^+}\mathrm{IV}$ exists. In order to estimate III, we apply the mean value theorem to get

$$|\mathrm{III}| \le |\alpha|\int_{2\epsilon}^1|\Omega(\log y)|\left(1-\frac{\epsilon}{y}\right)^{\alpha-1}\frac{\epsilon}{y^2}dy.$$

Changing variables again, we obtain

$$|\mathrm{III}| \le |\alpha|\int_\epsilon^{1/2}|\Omega(\log(\epsilon/t))|\,(1-t)^{\alpha-1}\,dt.$$

Finally, $\lim_{\epsilon\to 0^+}\mathrm{III} = 0$, applying the dominated convergence theorem and the facts that Ω is bounded and $\lim_{\epsilon\to 0^+}\Omega(\log(\epsilon/t)) = 0$.

REMARK. If we do not assume anything about the support of K, then Theorem 3.2 is valid for weights w in $A_{p,\alpha}$. The proof is similar to the proof of [2, Theorem 2.7] but using Theorem 2.3 instead of [2, Theorem 2.6].

An analogous comment can be written about Theorem 3.3, that is, Theorem 3.3 is valid for Calderón-Zygmund kernels and weights w in the Muckenhoupt Class A_1 (in fact, notice that this result is contained in [2, Theorem 2.7]).

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IMAL
Güemes 3450
(3000) Santa Fe
Argentina
e-mail: bernard@ceride.gov.ar

Departamento de Análisis Matemático
Facultad de Ciencias
Universidad de Málaga
29071 Málaga
Spain
e-mail: martin@anamat.cie.uma.es